ON SOME PROBLEMS RELATED TO
FUNDAMENTAL CYCLE SETS OF A GRAPH:
RESEARCH NOTES

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1. Introduction

The paper is concerned with some structural features of a fundamental cycle set
graph and mutual connections between the adjacent fundamental cycle graphs.
It seems that making use of the properties described in this paper we shall be able
to improve the methods for solving some extremal problems related to cycles of
a graph.

Before getting into the details, we shall begin with a brief summary of some
definitions and notations which will be used in this paper. Other terms not defined
here can be found in [3].

Let $G = (V(G), E(G))$ be a simple graph, i.e., without loops and multiple edges.
In what follows, the set of all the edges and the set of all the vertices of a graph $F$
are denoted by $E(F)$ and $V(F)$, respectively. We consider in this paper labelled graphs
only. Two graphs $G_1$ and $G_2$ satisfy the relation $G_1 = G_2$ if they are the isomorphic
labelled graphs. A simple path from $v$ to $w$ is a sequence of distinct vertices and edges
leading from $v$ to $w$. A closed simple path is a cycle. With every graph $G$ we can
associate the vector space of all cycles and unions of edge-disjoint cycles called the
cycle space of $G$. A cycle basis of $G$ is defined as a basis for the cycle space of $G$
which consists entirely of cycles. There are special cycle bases of a graph which can
be derived from spanning trees of $G$ (in the sequel, a spanning tree of $G$ will
be called simply a tree of $G$). Let $t$ be a tree of $G$. Then, the set of cycles $c(t)$ obtained
by inserting each of the remaining edges of $G$ into $t$ is a fundamental cycle set of $G$
with respect to $t$. Two cycle bases $c_1$ and $c_2$ satisfy the relation $c_1 = c_2$ if there
exists a one-to-one correspondence $v$: $c_1 \leftrightarrow c_2$ between the cycles of $c_1$ and the
cycles of $c_2$, where the elements in $c_1$ and $c_2$ are considered as labelled graphs which
are cycles. Paper [9] contains some necessary and sufficient conditions for a cycle
basis to be a fundamental cycle set.

Two trees $t_1$ and $t_2$ of a graph $G$ are said to be adjacent if there exist edges
e₁ ∈ 𝑡₁ − 𝑡₂ and e₂ ∈ 𝑡₂ − 𝑡₁ such 𝑡₂ = 𝑡₁ ⊕ e₂. The tree graph T of a connected graph G is defined as a graph in which each vertex corresponds to a tree of G and two vertices of T are adjacent if they correspond to adjacent trees of G.

Let c be a cycle basis of a graph G. The intersection graph B(G, c) of c over the set of edges of G is called a cycle graph of G with respect to c. If c is a fundamental cycle set, then B(G, c) is called a fundamental cycle graph. Some necessary conditions for a graph to be a cycle graph or a fundamental cycle graph were presented in [8].

The length of a cycle basis c = \{f_j\} is defined as follows

\[ |c| = \sum_{j \in E} |f_j|, \]

where |f_j| denotes the number of edges in f_j.

The problem of finding a minimum cycle basis and a minimum fundamental cycle set of a graph has been considered in [4, 6].

Enumeration of all the cycles of a graph using the vector space approach, finding a minimum cycle basis and a minimum fundamental cycle set of a graph, and finding the longest cycle of a graph are three problems related to cycle bases of a graph which still have some open question, see [2], [7]; [4], [5], and [2], [9], respectively.

It can easily be shown that there exists a one-to-one correspondence between cycles of a graph G and connected induced subgraphs of B(G, c), where c is a cycle basis of G, but unfortunately this correspondence is not necessarily onto. Thus the following questions arise

**Problem 1.1.** What cycle basis c of a graph G is chosen so that B(G, c) has a minimum number of connected induced subgraphs.

**Problem 1.2.** Let G be a given graph. Does there exist a cycle basis c of G such that there exists an one-to-one correspondence between the family of all cycles of G and the family of all connected subgraphs of B(G, c)?

Another problem related to a cycle basis of a graph appeared in [2].

**Problem 1.3.** Let G be a given graph. It is impossible to find a cycle basis c of G such that for every cycle f of G, the basic cycles that comprise f can be ordered in such a way that all ring sums of the consecutive subsequences of the basic cycles are cycles.

The main purpose of this paper is to clarify some structural features of fundamental cycle set graphs (Section 3) and adjacent fundamental cycle graphs (Section 4).

2. Conjectures and counterexamples

One of the first questions appearing to someone who has introduced a new concept is how this new concept is related to other notions introduced so far. Usually some conjectures are formulated as the result of such an investigation.

In this section we present some counterexamples for the conjectures which are concerned with the concepts defined in Section 1 and with other notions related to trees and cycle bases.

**Fig. 2.1**

It is easy to find a graph G such that its minimum cycle basis does not minimize the number of the (induced) connected subgraphs in the cycle graph of G (7), (8).

**Conjecture 2.1** (7). The cycle graph of a graph G with the minimum number of edges has the minimum number of the induced connected subgraphs among all the cycle graphs of G.

We can show only that in general this conjecture is not valid, i.e., if we take into consideration Θ_n, the class of all graphs with n vertices, then the number of the induced connected subgraphs of a graph is not an increasing function of the number of edges of the graph. For instance, two graphs shown in Fig. 2.1 belong to Θ_4. F has four edges and G has five edges but F has 15 induced connected subgraphs and G has only 13 ones.

**Fig. 2.2**

It can be easily shown that graphs F and G are not the cycle graphs of the same graph so they do not form a counterexample for conjecture 2.1.

Paper [9] contains the counterexamples for the following conjecture of Dixon and Goodman (see [2]): for any cycle basis c of a given graph G and for every cycle f of G it is possible to order the basic cycles which comprise f in such a way that all ring sums of the consecutive subsequences of the basic cycles are cycles.

Papers [4], [6] which deal with the problem of finding a minimum cycle basis contain some examples showing that the extremum trees of a graph with respect
to a properly defined weight function are not good approximations of the tree which generates the minimum fundamental cycle set. It has also been shown that a local neighbourhood search optimization algorithm fails to find an optimal solution to the minimum fundamental cycle set problem.

A central tree has been introduced by Deo [1] as the best starting tree for generating the rest of the trees in a graph. A central tree is a tree $t_0$ in $G$ such that $r(t_0) \leq r(t)$ for every tree $t_i$ in $G$, where $I$ is the complement of $t$ (i.e., $\cup I = G$) and $r(F)$ is the rank function of a graph (i.e., $r(F) = v(F) - p(F)$, where $v$ and $p$ denote the number of vertices in $F$ and the number of connected subgraphs in $F$, resp.). Unfortunately, as it is illustrated in Fig. 2.2, in general a central tree does not generate a minimum fundamental cycle set. The minimum fundamental cycle set of the graph $G$ is generated by non-central tree $t_1$ and the central tree $t_2$ of $G$ generates the fundamental cycle set which is not minimal.

3. Adjacency of fundamental cycle sets of a graph

A local neighbourhood search type algorithm for finding a minimum fundamental cycle set of a graph has been presented in [4]. The algorithm starts with a fundamental cycle set corresponding to an arbitrary tree of a graph and then at every step the minimum fundamental cycle set among those which can be generated by the trees adjacent to the tree generating the current fundamental cycle set is chosen as a new, improved solution.

It is obvious that the adjacency of trees of a graph induces somehow the adjacency of fundamental cycle sets but we must be aware of the fact that even non-adjacent trees can generate the same fundamental cycle set.

Figure 3.1 shows a graph $F$ and four of its trees. Tree $t_4$ is adjacent to $t_2$ and $c(t_1) = c(t_2)$, $t_3$ is adjacent to $t_2$ and $c(t_1) \neq c(t_3)$, and despite $t_4$ is not adjacent to $t_2$ we have $c(t_3) = c(t_4)$.

Let $t_1$ and $t_2$ be two adjacent trees of a graph $G$, i.e., there exist two edges $e_1 \in t_1 - t_2$ and $e_2 \in t_2 - t_1$ such that $t_2 = t_1 - e_1 \cup e_2$, and let $f_i$ denote the fundamental cycle in $c(t_i)$ such that $e_i \notin f_i$, i.e., $e_i$ belongs to the fundamental cycle of $t_i$ with respect to the chord $e_i$ and $e_i$ belongs to the fundamental cutset of $t_i$ with respect to the tree edge $e_i$. Then, the fundamental cycle set $c(t_2)$ of tree $t_2$ is of the form

$$(1) \quad c(t_2) = f_1 \cup f_2 \cup \{e_1 \notin f_2\} \cup \{e_2 \notin f_1\}$$

where the third set on the right-hand side consists of the new fundamental cycles, i.e., which do not belong to $c(t_1)$ and the second one is the subset of $c(t_1)$ containing fundamental cycles which do not contain the tree edge $e_1$.

Let $c_1$ and $c_2$ be two cycle bases of a graph $G$ which are not necessarily fundamental cycle sets. Then, $c_1$ and $c_2$ are said to be adjacent if there exist cycles $f \in c_1 - c_2$ and $g \in c_2 - c_1$ such that $c_2 = c_1 - f \cup g$. Now, we may ask which fundamental cycle sets generated by the adjacent trees are adjacent. The following theorem gives the answer to this question.

Theorem 3.1. Two fundamental cycle sets $c(t_1)$ and $c(t_2)$ generated by the adjacent trees $t_1$ and $t_2$ are adjacent if and only if $e_1$ belongs to exactly two fundamental cycles of $c(t_1)$.

Proof. The theorem follows directly from (1).

Figure 3.2 shows a graph and its two adjacent trees which generate the fundamental cycle sets that are not adjacent cycle bases.

As it was illustrated in Fig. 3.1, the same fundamental cycle set may be generated by different and even non-adjacent trees.

Let $I(c)$ denote the set of edges of a graph which belong to at least two cycles of a cycle basis $c$. To find all trees of a graph which generate the same fundamental cycle set we shall prove the following lemma.
LEMMA 3.1. Let  be a tree of a graph  such that  and  and . Then

\[ I(c(t_1)) = I(c(t_2)) \]
which each vertex corresponds to a fundamental cycle set of G and each edge corresponds to a pair of tree-adjacent fundamental cycle sets of G. The rest of this section is intended to clarify some structural features of the fundamental cycle set graph and some relations between the tree graph T and the fundamental cycle set graph F of a graph G.

Let $\{e_1, e_2, \ldots, e_n\}$ be a subset of edges of a graph G. Then, following the idea of Kishi and Kajitani [5] we can define a subset of trees of G

$$T = \left[ \begin{array}{c} e_1, e_2, \ldots, e_n \\ x_1, x_2, \ldots, x_n \end{array} \right],$$

where $x_i = 0$ or 1 (i = 1, 2, ..., n) as the collection of the trees which contain $e_i$ if $x_i = 1$ but do not if $x_i = 0$. Let $t \in T$; then all vertices of the tree graph which correspond to trees in

$$T = \left[ \begin{array}{c} I(c(t)) \\ 1 \end{array} \right]$$

are condensed into one vertex in the fundamental cycle set graph, since all trees which contain edges $I(c(t))$ generate the same fundamental cycle set.

Let $F_c$ denote the local subgraph of F with respect to a fundamental cycle set $c$ which is the subgraph defined by the collection of all the fundamental cycle sets whose distance from $c$ is 1. Note that a subset of trees of the form (2) may contain also trees which do not belong to $T_c$, the local subgraph of the tree graph T with respect to $t$.

In order to describe the structure of local subgraphs F we proceed similarly as it has been done in the case of local subgraphs of T, see [5]. First, we define $\tau$- and $\gamma$-sets.

Let $c_0$ be a fundamental cycle set of a graph G and $x_i$ be an edge which does not belong to $I(c_0)$. Then, the subset of fundamental cycle sets $\tau_1(x_i)$ is defined as the collection of all the fundamental cycle sets $c_1$ and whose distance from $c_0$ is 1. Similarly, the subset of fundamental cycle sets $\gamma_1(x_i)$ for $a_i \in I(c_0)$ is defined as the collection of all the fundamental cycle sets whose $I$-sets do not contain $a_i$ and whose distance from $c_0$ is 1.

It is easily shown that any $\tau$- or $\gamma$-set of a fundamental cycle set $c_0$ forms the complete subgraph of the local subgraph $F_{c_0}$ of F with respect to $c_0$. Moreover, applying Lemma 3.2, it can be shown that

$$\tau_1(x_i) = \gamma_1(x_i) = \emptyset,$$

whether the fundamental cutset of a tree which contains $I(c_0)$ with respect to $a_i$ contains $x_i$ or not, where $t_0$ can be obtained from a tree $t_0$ which generates $c_0$ by removing $a_i$ and adding $x_i$.

Let us consider a graph G shown in Fig. 3.3 and its tree $t = \{a_1, a_2, a_3, a_4, x_1, x_2, x_3, x_4\}$. In this case we have $I(c(t)) = \{a_1, a_2, a_3, a_4\}$ and notice that $\gamma_1(a_1) = \gamma_1(a_2) = \gamma_1(a_3) = \gamma_1(a_4) = \gamma_1(x_1) = \gamma_1(x_2) = \gamma_1(x_3) = \gamma_1(x_4)$.

The following theorem is a counterpart of Theorem 4 in [5] dealing with the decomposition of a local subgraph.

Theorem 3.2. Let $c$ be a fundamental cycle set of G, let $I(c) = \{a_1, a_2, \ldots, a_q\}$ denote the set of edges of G which belong to at least two fundamental cycles of $c$, and let $\{x_1, x_2, \ldots, x_k\}$ be the set of edges of G which do not belong to $I(c)$. Then the sets of all the vertices and the edges of the local subgraph $F_c$ can be partitioned as follows:

$$V(F_c) = \bigcup_{i=1}^q V(t_i(a_i)),$$

$$E(F_c) = \bigcup_{i=1}^q E(t_i(a_i)) \cup \bigcup_{j=1}^q E(y_j(a_j),$$

where

$$V(t_i(a_i)) \cap V(t_j(a_j)) = \emptyset \quad \text{or} \quad V(t_i(a_i)) = V(t_j(a_j)) \quad (i \neq k),$$

$$E(t_i(a_i)) \cap E(t_j(a_j)) = \emptyset \quad \text{or} \quad E(t_i(a_i)) = E(t_j(a_j)) \quad (i \neq k),$$

whether $x_1$ and $x_2$ are contained in the same fundamental cycle set or not,

$$V(y_j(a_j)) \cap V(y_k(a_k)) = \emptyset \quad \text{or} \quad V(y_j(a_j)) = V(y_k(a_k)) \quad (i \neq k),$$

$$E(y_j(a_j)) \cap E(y_k(a_k)) = \emptyset \quad \text{or} \quad E(y_j(a_j)) = E(y_k(a_k)) \quad (i \neq k),$$

whether $a_i$ and $a_k$ belong to the same subset of fundamental cycles of $c$ or not, and

$$E(t_i(a_i)) \cap E(y_j(a_j)) = \emptyset.$$

The proof of the theorem can be easily derived from the preceding consideration so it is omitted.

It is well known that a tree graph contains a hamiltonian cycle. First, the existence has been proved by induction and then the investigation of some topological features of a tree graph led to two procedures for generating a hamiltonian cycle in a tree graph, see [5]. We conjecture that a fundamental cycle set graph is also hamiltonian.
In fact, the existence of a Hamiltonian cycle in a fundamental cycle set graph of a graph is a little use to our problems presented in Section 1, since we are more interested in finding a special cycle basis or a special fundamental cycle set of a graph than in enumerating all cycle bases or fundamental cycle sets of a graph.

Example 3.3. Consider the graph $G$ shown in Fig. 3.4 (a) (see also [5]). The tree graph of $G$ has 12 vertices and the fundamental cycle set graph $F$ of $G$ has only 4 cones. Figure 3.4 (b) shows one of the Hamiltonian cycles of the tree graph of $G$ obtained by the procedure presented by Kishi and Kajitani (Different figures in vertices correspond to different fundamental cycle sets of $G$ generated by trees associated with vertices.) Figure 3.4 (c) is self-explained.

One can easily show that we must take into consideration the labelled fundamental cycle graph $B(G, e(G))$ to be able to transform it into a fundamental cycle graph of $G$ with respect to the tree-adjacent fundamental cycle set $e(G)$. Let $r_{ij}$ and $s_{ij}$ denote the labels of vertices and edges of $B(G, e(G))$, respectively, defined as follows: $r_{ij} = r(f_i) = \{f_i\}$ for the vertex corresponding to cycle $f_i$ and $s_{ij} = s(f_i, f_j) = \{f_i \cap f_j\}$ for the edge corresponding to a non-empty intersection of two cycles (be careful since labels $s_{ij}$ can be derived from the vertex labels, we introduced them for the sake of simplicity).

In the rest of this section, we describe $B(G, e(G))$ in terms of the labelled fundamental cycle graph $B(G, e(G))$, where $t_e$ and $t_f$ are two adjacent trees of $G$.

Let us notice that $e(G) - e(t_e) = N_1$ so that only those vertices of $B(G, e(G))$ change their labels and neighbors.

The following steps constitute an algorithm for transforming $B(G, e(G))$ into $B(G, e(G_2))$. $r_{ij}$, $s_{ij}$ denote the labels of vertices and edges of the latter graph.

Step 1. Labeling of vertices.

$$r_i = \{g_i\} = \{f_i \cap f_i\} = f_i - f_i f_i = r_i \cup s_{ii} = r_i \cup s_{ii}$$

for every vertex corresponding to the fundamental cycle in $N_1$ (see Step 2 (a)), and $r_j = r_j$ otherwise.

Step 2. Labeling of edges.

(a) $\cap f_i \neq \emptyset$ and $g_i \cap f_i \neq \emptyset$ (for $g_i \in N_2$) and other fundamental cycles of $B(G, e(G))$ remain the same, $f_i$ does not change its neighbors. We must modify only the labels of edges connecting $f_i$ with $N_1$.

$$s_{ij} = g_i \cap f_i = f_i - (f_i \cap f_i) = r_i - s_{ii}.$$

(b) Edges connecting vertices in $N_1$. Let us consider two fundamental cycles $g_i$ and $g_j$ ($g_i, g_j \in N_2$). We have

$$g_i \cap f_i = f_i \cup g_i \cap f_i = f_i \cup (f_i \cap f_i) = f_i \cup (f_i \cap f_i) = f_i \cup (f_i \cap f_i).$$

Notice that $f_i \cap g_i \neq f_i \cup f_i$, since cycle basis $e(G)$ is a fundamental cycle set, i.e., every cycle in $e(G)$ contains an edge which does not belong to other cycles in $e(G)$.
(see [9]). Therefore \( g_i \cap g_j \neq \emptyset \), i.e., there exists the edge in \( B(G, c(t_{ij})) \) between \( g_i \) and \( g_j \) and it has the following label

\[
s'_{ij} = s_{ij} - r_i \cup r_j - (r_i \cup r_j) = s_{ij} - r_i - r_j.
\]

(c) Edges between \( N_1 \) and \( N_2 \). If \( g_i \in N_1 \) and \( g_j \in N_2 \), then

\[
s'_{ij} = g_i \cap g_j = f_i \ominus f_i \cap f_j = f_i \ominus f_i \cap f_j = f_i \ominus f_j.
\]

and since \( f_i \ominus f_j = \emptyset \) for \( f_j \in N_2 \), we obtain

\[
s'_{ij} = f_i \ominus f_j = s_{ij}.
\]

(d) Edges between \( N_1 \) and \( N_2 \). Let \( g_i \in N_1 \) and \( g_j \in N_2 \). In this case we have

\[
s'_{ij} = g_i \cap g_j = f_i \ominus f_i \cap f_j = f_i \ominus f_i \cap f_i = f_i,
\]

and we shall show that \( s'_{ij} = \emptyset \) if and only if \( f_i \cap f_j = f_i \cap f_i \). Suppose that \( s_{ij} = \emptyset \), i.e., \( f_i \cap f_j = f_i \cap f_i = \emptyset \). Hence \( f_i = f_i \ominus f_i \cap f_i = \emptyset \), and therefore \( f_i \cap f_j = f_i \cap f_i = \emptyset \). Thus, \( f_i \cap f_j = f_i \cap f_i \). On the other hand, if \( f_i \cap f_j = f_i \cap f_i \), then \( f_i \cap f_i = f_i \cap f_i = \emptyset \) and \( f_i \cap f_i = f_i \cap f_i = \emptyset \) so that \( s_{ij} = \emptyset \).

Let us suppose that \( f_i \cap f_j = \emptyset \); then \( s_{ij} = f_i \ominus f_i \cap f_j = f_i \cap f_i = s_{ij} \neq \emptyset \), so that a new edge between \( g_i \) and \( g_j \) appears in \( B(G, c(t_{ij})) \).

In the opposite case, i.e., if \( f_i \cap f_j \neq \emptyset \) we have: if \( s_{ij} = s_{ij} \), then \( s'_{ij} = \emptyset \) and

\[
s'_{ij} = f_i \ominus f_i \cap f_j = \left( (f_i \cap f_i) - f_i \cap f_i \right) \cap f_j = (f_i \cap f_i) \cap f_i - f_i \cap f_i
\]

\[
= f_i \ominus f_i \cap f_j \cap f_i = s_{ij} \cup s_{ij} = s_{ij} \neq \emptyset
\]

if \( s_{ij} = s_{ij} \).

The procedure presented above describes the elementary transformation of a fundamental cycle graph into a cycle graph corresponding to a tree-adjacent fundamental cycle set. We hope that the method can be generalized to produce a fundamental cycle graph which corresponds to a fundamental cycle set having special properties required by an extremal problem considered related to a cycle basis of a graph.


An efficient algorithm for enumerating all cycles of a planar graph based on the cycle vector space approach has been presented in M. M. Syslo, *An efficient cycle vector space algorithm for listing all cycles of planar graph*, SIAM J. Computing 10 (1981)."