Continuous mappings with an infinite number of topologically critical points

by Cornel Pintea (Cluj-Napoca)

Abstract. We prove that the topological $\varphi$-category of a pair $(M,N)$ of topological manifolds is infinite if the algebraic $\varphi$-category of the pair of fundamental groups $(\pi_1(M), \pi_1(N))$ is infinite. Some immediate consequences of this fact are also pointed out.

1. Introduction. In this section we recall the notions of topologically regular point and topologically critical point of a continuous mapping and the topological $\varphi$-category of a pair of topological manifolds.

Let $M^m$, $N^n$ be topological manifolds and let $f : M \to N$ be a continuous map. For a given point $x_0 \in M$ consider a pair $(U, \varphi)$, $(V, \psi)$ of charts at $x_0$ and $f(x_0)$ respectively, satisfying the relation $f(U) \subseteq V$. Recall that the map $f_{\varphi \psi} : \varphi(U) \to \psi(V)$ defined by $f_{\varphi \psi} = \psi \circ f \circ \varphi^{-1}$ is the local representation of $f$ at $x_0$ with respect to the charts $(U, \varphi)$, $(V, \psi)$.

DEFINITION. The point $x_0 \in M$ is called a topologically regular point of $f$ if there exists a local representation $f_{\varphi \psi}$ of $f$ at $x_0$ such that for any $z = (z_1, \ldots, z^m) \in \varphi(U) \subseteq \mathbb{R}^m$,

\[
\begin{align*}
  f_{\varphi \psi}(z) &= \begin{cases} (z_1, \ldots, z^m, 0, \ldots, 0) & \text{if } m \leq n, \\ (z_1, \ldots, z^n) & \text{if } m \geq n. \end{cases}
\end{align*}
\]

Otherwise $x_0$ is called a topologically critical point of the map $f$.

Recall the following notations:

1) $R_{\text{top}}(f)$ is the set of all topologically regular points,
2) $C_{\text{top}}(f)$ is the set of all topologically critical points,
3) $B_{\text{top}}(f) = f(C_{\text{top}}(f))$ is the set of all topologically critical values of $f$.

1991 Mathematics Subject Classification: 57R70, 57S15, 57T20.
Key words and phrases: topologically critical points, covering mappings, $G$-manifolds.

[87]
Define also the topological $\varphi$-category of the pair $(M, N)$ as follows:
\[ \varphi_{top}(M, N) = \min\{|C_{top}(f)| : f \in C(M, N)\} \]
where $|A|$ denotes the cardinality of the set $A$. If $|C_{top}(f)|$ is infinite for all $f \in C(M, N)$, we write $\varphi_{top}(M, N) = \infty$.

If $M$, $N$ are differentiable manifolds and $f : M \to N$ is a differentiable mapping, then $R(f)$ and $C(f)$ denotes the set of all regular points of $f$ and the set of all critical points of $f$ respectively. (Regular and critical points are considered here in the usual sense, that is, they are defined by means of the rank of the tangent map.)

The $\varphi$-category of the pair $(M, N)$ is given by
\[ \varphi(M, N) = \min\{|C(f)| : f \in C^\infty(M, N)\}. \]
Again, $\varphi(M, N) = \infty$ if $|C(f)|$ is infinite for all $f \in C^\infty(M, N)$. A remarkable inequality which involves the $\varphi$-category of the pair $(M, \mathbb{R})$ is the following:
\[ \varphi(M, \mathbb{R}) \geq \text{cat}(M) \geq \text{cuplong}(M), \]
where $\text{cat}(M)$ denotes the Lusternik–Schnirelmann category of the manifold $M$ and $\text{cuplong}(M)$ denotes the cup-length of the manifold $M$ (see for instance [5, pp. 190–191]). Other results concerning the $\varphi$-category of the pair $(M, \mathbb{R})$ are obtained in [6]. For the equivariant (invariant) situation see also [2].

Remarks. 1) Let $M^m$, $N^n$ be topological manifolds such that $m \geq n$ and $f : M \to N$ be a continuous mapping. If a point $x_0 \in M$ is topologically regular, then there is an open neighbourhood $U$ of $x_0$ such that the restriction $f|_U : U \to N$ is open, that is, $f$ is locally open at $x_0$. If $m = n$, then $x_0 \in M$ is a topologically regular point if and only if $f$ is a local homeomorphism at $x_0$ (see [1, Proposition 1.3]).

2) Obviously $R_{top}(f)$ is an open subset of $M$, while $C_{top}(f)$ is closed, the two subsets being complementary to each other. A similar statement is true for $R(f)$ and $C(f)$ in the differentiable case.

3) If $M$, $N$ are differentiable manifolds and $f : M \to N$ is a differentiable mapping, then, according to the well-known Rank Theorem, the relation $R(f) \subseteq R_{top}(f)$ holds, or equivalently $C_{top}(f) \subseteq C(f)$. Therefore
\[ \varphi_{top}(M, N) \leq \varphi(M, N). \]

2. Preliminary results. We start by proving the following theorem:

THEOREM 2.1. Let $M^m$, $N^n$ be two connected topological manifolds such that $m \geq n \geq 2$. If $f : M \to N$ is a non-surjective closed and continuous mapping, then $f$ has infinitely many topologically critical points. In particular, if $M$ is compact and $N$ non-compact then $\varphi_{top}(M, N) = \infty$. 

Proof. Let us first prove that $f^{-1}(\partial \text{Im } f) \subseteq C_{\text{top}}(f)$. Indeed, otherwise there exists $x_0 \in f^{-1}(\partial \text{Im } f)$ such that $x_0 \in R_{\text{top}}(f)$. This means that $f$ is locally open around $x_0$ and therefore $x_0$ has an open neighbourhood $U$ such that $f(U) \rightarrow N$ is open, namely $f(U)$ is open. But this is a contradiction with the fact that $f(x_0) \in \partial \text{Im } f$. From the inclusion $f^{-1}(\partial \text{Im } f) \subseteq C_{\text{top}}(f)$ it follows that
\[
\partial \text{Im } f \subseteq B_{\text{top}}(f).
\]

Further on, we consider the following two cases:

**Case I.** $B_{\text{top}}(f) = \text{Im } f$. If the image of $f$ is finite, then the mapping $f$ is constant. This means that $C_{\text{top}}(f) = M$ and therefore $C_{\text{top}}(f)$ is infinite. Otherwise $B_{\text{top}}(f)$ is infinite, hence $C_{\text{top}}(f)$ is also infinite.

**Case II.** $\text{Im } f \setminus B_{\text{top}}(f) \neq \emptyset$. In this case we show that $N \setminus B_{\text{top}}(f)$ is not connected and therefore $B_{\text{top}}(f)$ is infinite. Because $\text{Im } f \setminus B_{\text{top}}(f) \neq \emptyset$ and $f$ is non-surjective we can consider $y \in \text{Im } f \setminus B_{\text{top}}(f)$ and $y' \in N \setminus \text{Im } f$.

Because $y \in \text{Im } f$ and $y' \in N \setminus \text{Im } f$ it follows that any continuous path joining $y$ to $y'$ intersects $\partial \text{Im } f$ and consequently the set $B_{\text{top}}(f)$. But since $y, y' \in N \setminus B_{\text{top}}(f)$, it follows that $N \setminus B_{\text{top}}(f)$ is not connected. 

Further on, the equivariant case will be briefly studied.

Let $G$ be a Lie group, $M$ a manifold and $\varphi : G \times M \rightarrow M$, $(g, x) \mapsto gx$, be a smooth action of $G$ on $M$. The triple $(G, M, \varphi)$ is called a $G$-manifold. The orbit of a point $x \in M$ will be denoted by $Gx$. If the action of $G$ on $M$ is free, recall that $M/G$ can be endowed with a differential structure such that the canonical projection $\pi_M : M \rightarrow M/G$ is a smooth $G$-bundle (see [3, Theorem 4.11, p. 186]). A function $f : M \rightarrow N$ between $G$-manifolds $M$ and $N$ is said to be $G$-equivariant if $f(gx) = gf(x)$ for all $g \in G$ and all $x \in M$. If $M$ and $N$ are two $G$-manifolds and $f : M \rightarrow N$ is $G$-equivariant, denote by $\tilde{f} : M/G \rightarrow N/G$ the function which makes the following diagram commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
M/G & \xrightarrow{\tilde{f}} & N/G
\end{array}
\]

Let $X$ be a differentiable manifold, $Y \subseteq X$ be a submanifold of $X$ and $l : Y \hookrightarrow X$ be the inclusion mapping. The subspace $(dl)_y(T_yY)$ of the tangent space $T_yX$ will be simply denoted by $T_yY$.

**Definition.** Let $f : M \rightarrow N$ be a differentiable mapping and $P$ be a submanifold of $N$. We say that $f$ intersects transversally the submanifold $P$ at $x \in M$ if either $f(x) \notin P$ or $(df)_x(T_xM) + T_{f(x)}P = T_{f(x)}N$.

We close this section with the following result:
Theorem 2.2. Let $G$ be a Lie group and $M, N$ be two $G$-manifolds such that the action of $G$ on $M$ and $N$ is free and $\dim M \geq \dim N$. Consider a $G$-equivariant map $f : M \to N$ and let $\tilde{f} : M/G \to N/G$ be its associated map defined above. For $x \in M$, the following assertions are equivalent:

(i) $x$ is a regular point of the function $f$;

(ii) $\pi_M(x)$ is a regular point of the function $\tilde{f}$;

(iii) $f$ intersects transversally the orbit $Gf(x)$ at $x$.

The proof of Theorem 2.2 is left to the reader.

3. The main result. In the first part of this section, the algebraic $\varphi$-category of a pair of groups is defined and studied. In the second part we prove the principal result of the paper.

For an abelian group $G$, the subset $t(G)$ of all elements of finite order forms a subgroup of $G$ called the torsion subgroup.

If $G, H$ are groups, then the algebraic $\varphi$-category of the pair $(G, H)$ is defined as follows

$$\varphi_{\text{alg}}(G, H) = \min\{[H : \text{Im } f] \mid f \in \text{Hom}(G, H)\}.$$

If $[H : \text{Im } f]$ is infinite for all $f \in \text{Hom}(G, H)$ we write $\varphi_{\text{alg}}(G, H) = \infty$.

Proposition 3.1. If $G, H$ are finitely generated abelian groups such that

$$\text{rank}[G/t(G)] < \text{rank}[H/t(H)]$$

then $\varphi_{\text{alg}}(G, H) = \infty$.

Proof. Let $f : G \to H$ be a group homomorphism. Because $f(t(G)) \subseteq t(H)$ there exists a group homomorphism $\tilde{f} : G/t(G) \to H/t(H)$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
G & \xrightarrow{f} & H \\
p_G & \downarrow & \downarrow p_H \\
G/t(G) & \xrightarrow{\tilde{f}} & H/t(H)
\end{array}$$

$p_G$ and $p_H$ being the canonical projections. Because $(H/t(H))/\text{Im } \tilde{f}$ is a finitely generated abelian group it follows, by the structure theorem, that

$$\frac{H/t(H)}{\text{Im } \tilde{f}} \cong \mathbb{Z}^n \oplus t\left(\frac{H/t(H)}{\text{Im } \tilde{f}}\right)$$

where $n = \text{rank}[H/t(H)]$ and $m = \text{rank}(\text{Im } \tilde{f}) \leq \text{rank}[G/t(G)]$. The remainder of the proof is obvious.

Corollary 3.2. If $G, H$ are free abelian groups such that $\text{rank } G < \text{rank } H < \infty$, then $\varphi_{\text{alg}}(G, H) = \infty$. 
The next theorem is the principal result of the paper.

**Theorem 3.3.** Let $M^m$, $N^n$ be compact connected topological manifolds such that $m \geq n \geq 2$. If $\varphi_{\text{alg}}(\pi_1(M), \pi_1(N)) = \infty$ then $\varphi_{\text{top}}(M, N) = \infty$.

**Proof.** Let $f : M \to N$ be a continuous mapping and $f_* : \pi_1(M) \to \pi_1(N)$ be the induced homomorphism. Because $\varphi_{\text{alg}}(\pi_1(M), \pi_1(N)) = \infty$ it follows that $[\pi_1(N) : \text{Im } f_*] = \infty$. On the other hand, using the theory of covering maps, there exists a covering map $p : \tilde{N} \to N$ such that $p_*([\pi_1(N) : \text{Im } f_*]) = \infty$. Because the number of sheets of the covering $p : \tilde{N} \to N$ is the index $[\pi_1(N) : \text{Im } f_*]$, it follows that $p : \tilde{N} \to N$ has an infinite number of sheets, that is, $\tilde{N}$ is a non-compact manifold. From the equality $p_*([\pi_1(N) : \text{Im } f_*]) = \infty$ it follows, using the lifting criterion, that there exists a mapping $\tilde{f} : M \to \tilde{N}$ such that $p \circ \tilde{f} = f$. But since $p$ is locally a homeomorphism it implies that $C_{\text{top}}(f) = C_{\text{top}}(\tilde{f})$, which together with the second part of Theorem 2.1 leads to the conclusion that $C_{\text{top}}(f)$ is infinite. \hfill \Box

**Corollary 3.4.** Let $M^m$, $N^n$ be compact connected topological manifolds such that $m \geq n \geq 2$. If $\pi_1(M)$ is finite and $\pi_1(N)$ is infinite, then $\varphi_{\text{top}}(M, N) = \infty$.

4. Applications. In this section some applications of Theorem 3.3 will be given.

**Proposition 4.1.** (i) If $m$, $n$, $k$ are natural numbers such that $1 < k < m$ and $k + n \geq m \geq 2$, then $\varphi_{\text{top}}(T^k \times S^n, T^m) = \infty$.

(ii) If $T_g$ is the connected sum of $g$ tori and $g < g'$, then $\varphi_{\text{top}}(T_g, T_{g'}) = \infty$.

(iii) If $P_g$ is the connected sum of $g$ projective planes and $g < g'$, then $\varphi_{\text{top}}(P_g, P_{g'}) = \infty$.

**Proof.** (i) follows easily from Theorem 3.3 by taking into account the fact that $\pi_1(T^k \times S^n) = \mathbb{Z} \times \cdots \times \mathbb{Z}$ and $\pi_1(T^m) = \mathbb{Z} \times \cdots \times \mathbb{Z}$.

(ii) We show that $\varphi_{\text{alg}}(\pi_1(T_g), \pi_1(T_{g'})) = \infty$. Let $f : \pi_1(T_g) \to \pi_1(T_{g'})$ be a group homomorphism. Because $f([\pi_1(T_g), \pi_1(T_g)]) \subseteq [\pi_1(T_{g'}), \pi_1(T_{g'})]$, $f$ induces a group homomorphism

$$[f] : \pi_1(T_g)/[\pi_1(T_g), \pi_1(T_g)] \to \pi_1(T_{g'})/[\pi_1(T_{g'}), \pi_1(T_{g'})]$$

which makes the following diagram commutative:

$$\begin{array}{cccc}
\pi_1(T_g) & \xrightarrow{f} & \pi_1(T_{g'}) \\
\downarrow{p_g} & & \downarrow{p_{g'}} \\
\pi_1(T_g)/[\pi_1(T_g), \pi_1(T_g)] & \xrightarrow{[f]} & \pi_1(T_{g'})/[\pi_1(T_{g'}), \pi_1(T_{g'})]
\end{array}$$
where \( p_g, p_g' \) are the canonical projections. Taking into account the fact that the groups \( \pi_1(T_g)/[\pi_1(T_g), \pi_1(T_g)] \) and \( \pi_1(T_{g'}/[\pi_1(T_{g'}), \pi_1(T_{g'})] \) are free abelian groups of rank \( 2g \) and \( 2g' \) respectively (see [4, p. 135]), by Corollary 3.2, we see that
\[
\pi_1(T_{g'}/[\pi_1(T_{g'}), \pi_1(T_{g'})]) \rightarrow \text{Im}[f]
\]
is an infinite group. The remainder of the proof is obvious.

(iii) The proof is similar to that of (ii). ■

**Proposition 4.2.** Let \( M^n, N^n \) be compact connected differentiable manifolds such that \( m \geq n \geq 3 \) and \( G \) be a compact connected Lie group acting freely on both manifolds. If \( \pi_1(M) \) is finite and \( \varphi_{\text{alg}}(\pi_1(G), \pi_1(N)) = \infty \), then any equivariant mapping \( f : M \rightarrow N \) has an infinite number of critical orbits.

**Proof.** Because \( f : M \rightarrow N \) is a \( G \)-equivariant mapping, it induces a differentiable mapping \( \tilde{f} : M/G \rightarrow N/G \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
pM \downarrow & & \downarrow pN \\
M/G & \xrightarrow{\tilde{f}} & N/G
\end{array}
\]

It is enough to show that \( \tilde{f} \) has an infinite number of critical points. For this purpose it is enough to show \( \varphi_{\text{alg}}(\pi_1(M/G), \pi_1(N/G)) = \infty \). Consider the exact homotopy sequences
\[
\ldots \rightarrow \pi_q(G) \xrightarrow{i_q} \pi_q(M) \xrightarrow{(p_M)_q} \pi_q(M/G) \rightarrow \pi_{q-1}(G) \rightarrow \ldots
\]
\[
\ldots \rightarrow \pi_q(G) \xrightarrow{j_q} \pi_q(N) \xrightarrow{(p_N)_q} \pi_q(N/G) \rightarrow \pi_{q-1}(G) \rightarrow \ldots
\]
of the fibrations \( G \hookrightarrow M \rightarrow M/G \) and \( G \xrightarrow{j} N \rightarrow N/G \). Taking \( q = 1 \) it follows, using the connectedness of \( G \), that
\[\pi_1(M/G) \cong \pi_1(M)/\text{Im} i_1, \quad \pi_1(N/G) \cong \pi_1(N)/\text{Im} j_1.\]
Because \( \pi_1(M) \) is finite, \( \pi_1(M)/\text{Im} i_1 \cong \pi_1(M/G) \) is finite. The hypothesis \( \varphi_{\text{alg}}(\pi_1(G), \pi_1(N)) = \infty \) implies that \( \pi_1(N)/\text{Im} j_1 \cong \pi_1(N/G) \) is infinite. Therefore, by Corollary 3.4, \( \varphi_{\text{alg}}(\pi_1(M/G), \pi_1(N/G)) = \infty \). ■

**Example.** Let \( m, n, a_1, \ldots, a_m \) be natural numbers such that \( 2n \geq m \geq 3 \) and \((a_1, \ldots, a_m) = 1\). Consider the actions of \( S^1 \) on \( S^{2n+1} \) and \( T^m \) given by
\[
S^1 \times S^{2n+1} \rightarrow S^{2n+1}, \quad (z, (z_1, \ldots, z_n)) \mapsto (zz_1, \ldots, z_m),
\]
\[
S^1 \times T^m \rightarrow T^m, \quad (z, (z_1, \ldots, z_n)) \mapsto (z^{a_1}z_1, \ldots, z^{a_m}z_m).
\]
The above two actions are obviously free and the conditions of Proposition 4.2 are satisfied. Therefore, any $S^1$-equivariant mapping $f : S^{2n+1} \to T^m$ has an infinite number of critical orbits.

References