A note on the Hasse principle. Addenda

by

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In [2] we have discussed a method of giving counter-examples to the Hasse principle for systems of many quadratic forms in many variables. Here we would like to give some more details, making Lemma 10 in [2] and its proof more precise, and also to correct Example 2 there.

1. Let $D$ be a quaternion division algebra over a field $k$ of characteristic $\neq 2$, $\Phi$ be a nondegenerate skew-hermitian form of rank $n$ w.r. to the standard involution $J$ of $D$. Further we keep the notation in [2]. Denote by $M(\Phi) = m(GU(\Phi)(k))$ the group of multiplicators of $\Phi$. Let $GU^+(\Phi) = SU(\Phi) \cdot G_m$ (the almost direct product, where $SU(\Phi)$ is the special unitary $k$-group of $\Phi$) and let $M^+(\Phi) = m(GU^+(\Phi)(k))$. The correct formulation of Lemma 10 in [2] is as follows.

**Lemma 10.** Let $k$ be any field of characteristic $\neq 2$ and $D$ be a non-trivial quaternion division algebra. For any natural number $n$, there is a skew-hermitian form $\Phi$ w.r. to $J$ of rank $n$ such that

$$PGU(\Phi)(k) \neq PGU(\Phi)_0(k).$$

**Proof.** It is well known and easy to see that for any skew-hermitian form $\Phi$, the group $GU^+(\Phi)$ is just the connected component of the group $GU(\Phi)$. Since $D$ is non-trivial, $U(\Phi)(k) = SU(\Phi)(k)$, and $PGU(\Phi)(k) \neq PGU(\Phi)_0(k)$ is equivalent to $M(\Phi) \neq M^+(\Phi)$.

Let $g \in GU(\Phi)(k)$, $\lambda = \text{Nrd}(g)$. Then we know that, if $n = \text{rank} \Phi$, $\text{Nrd}(g) = \lambda^2$ or $-\lambda^2$, and that $g \in GU^+(\Phi)(k)$ (i.e. $\lambda \in M^+(\Phi)$) if and only if $\text{Nrd}(g) = \lambda^2$. Now assume that $(1, i, j, ij)$ is a canonical basis of $D$ over $k$, $i^2 = \theta$, $j^2 = \eta$, $ij = -ji$, $\theta$ and $\eta$ belong to $k^*$. We consider in $D$ the equation

$$X^2 - i \cdot X = \lambda \cdot i.$$

For $x = x_0 + x_1 i + x_2 j + x_3 ij$, it follows by easy calculations (see [2], Example 1 for details) that this equation is equivalent to the following two systems of equations:
\begin{align*}
\begin{cases}
  x_0 = x_1 = 0, \\
x_2 = x_3 = 0,
\end{cases} & \quad \begin{cases}
  \eta x_2 - \theta x_3 = \lambda; \\
x_0 - 3\theta x_1 = \lambda.
\end{cases}
\end{align*}

Since $D$ is non-trivial, these two systems never have solutions simultaneously.

Now we take $\lambda = \eta x_2^2 - \theta x_3^2$ for some $x_2$ and $x_3$ (not all zero). Then the equation

$$X^T \cdot i \cdot X = \lambda \cdot i$$

has a solution $X$ with $N_{\mathbb{R}}(X) = -\lambda$.

For any odd natural number $n$ we put $\Phi = \text{diag} (i, \ldots, i)$ ($n$ entries $i$). Then for $X$ above we put $Y = \text{diag} (X, \ldots, X)$. Then $Y$ is a similitude of $\Phi$ and clearly $Y \notin \text{GU}^+(\Phi)(k)$.

If $n$ is even then using similar arguments to the above, it is not hard to choose skew-quaternions $\alpha, \beta \in D$ such that for the form $\Phi_0 = \text{diag}(\alpha, \beta)$, we have $M(\Phi_0) \neq M^+(\Phi_0)$. We then put $\Phi = \text{diag}(\alpha, \beta, \ldots, \beta)$ ($n-1$ entries $\beta$) and we have again $M(\Phi) \neq M^+(\Phi)$.

2. From the above it follows that Example 2 in [2] is not correct. Here we should take, for the two-dimensional case, the form $\Phi_0$ of the previous part, and for the three-dimensional case, the form $\Phi = \text{diag}(i, i, i)$ in order to write the systems of quadratic forms of small size, after having made some complicated computations.

3. Denote by $M'(\Phi) = \bigcap_v (M(\Phi_v) \cap k^*)$ the group of all elements of $k$ which are multipliers of the form $\Phi$ locally everywhere, where $k$ is assumed to be a global field of characteristic $\neq 2$ and $v$ runs over all valuations of $k$.

Then by [2], if $M(\Phi) \neq M^+(\Phi)$, then Card $(M'(\Phi)/M(\Phi)) = 2^{s-2}$. Therefore, we can describe briefly our general method as follows:

(a) Choose the algebra $D$ and the form $\Phi$ s.t. $s \geq 2$ and $M(\Phi) \neq M^+(\Phi)$.

(b) Choose $\lambda \in M'(\Phi)/M(\Phi)$.

(c) Write the system according to the value $\lambda$ obtained.

Practically, after having done step (a), to do step (b), we can proceed as follows. Take the decomposition of $M'(\Phi)$ into cosets modulo $M^+(\Phi)$:

$$M'(\Phi) = \bigcup_i M^+(\Phi) : \lambda_i.$$ 

Since $M'(\Phi) \neq M(\Phi)$, there are $\lambda_i$ such that $\lambda_i \notin M(\Phi)$. From this we can write the system of quadratic forms as required.

I would like to thank B. E. Kuniavski for valuable comments on the results of [2]. In particular, he has pointed out that our examples provide a lower bound $N = O(r)$, where the Hasse principle fails for systems of $r$ quadratic forms in $N$ variables. It is still an open question if it is so if $N = O(r^2)$ (cf. [1], Problem 9, pp. 106–107).