where $J(i)$ is an index set depending on $i$. If we define a function $f$ by specifying that on $R_g$

$$f(y_i) = b_i \quad (j \in J(i), \ i = 1, \ldots, r)$$

and by allowing $f$ to be arbitrary elsewhere, then $f$ will satisfy (9.12).

References


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On the abstract theory of primes I
by
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Introduction. 1. For a semi-group $\mathcal{G}$ (with respect to multiplication) of real numbers $a \geq 1$ satisfying some given asymptotical laws of distribution Beurling [2] investigated the asymptotical distribution of the generators $b$ of $\mathcal{G}$. He proved a general theorem which, applied to the semi-group of natural integers, gives the prime number theorem of Hadamard and Vallee-Poussin (1). Forman and Shapiro [11] divided the numbers $a$ into classes $H_i (1 \leq i \leq h)$ forming a group $\mathcal{K}$ (for any $a \in H_i$ and $a' \in H_j$ we have $aa' \in H_k$ where $k$ depends only on $i$ and $j$) and satisfying

$$\sum_{\sigma, a' \in H_1} 1 = \alpha_1 a + O(a^{1-\varepsilon})$$

with some positive constants $\alpha_1, \beta (\beta \leq 1)$. They proved that under those circumstances the numbers $\pi(a, H)$ of the generators $b \leq a$, $b \in H_i$ are asymptotically the same for all the classes $H_i$ forming a sub-group $K_i$ of $\mathcal{K}$, whereas the number of the remaining generators $a \leq a$ (if $K_i \neq K$) has a smaller order of magnitude as $a \to \infty$ (1). As special cases of this abstract theorem we may deduce the asymptotical laws for primes in arithmetical progressions or prime ideals in ideal classes.

The aim of all the work in the abstract theory of primes up to now has been the proof of the asymptotical law for $\pi(a, H)$. In a short note [9] I have mentioned that in the abstract scheme used by Forman and Shapiro one can treat the smallest prime problem for different progressions simultaneously. For this purpose it is necessary to change the

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(1) Other writers after Beurling (as Nyman [18], Erdős [3]) either start from different conditions or use, instead of the analytical method of the zeta function, the elementary method of A. Selberg.

(2) This is an intentionally simplified description. Actually Forman and Shapiro start from a free Abelian group $G$ on a countable number of generators and use a homomorphism into positive rationals such that the images of the generators are all integral. The distribution of the generators in the classes $H$ of a semi-group is also the subject of a recent work of A. Zaimov [1], who replaces the remaining term in (1) by $O(a/\log a)$. 
scheme as follows. Firstly we introduce a parameter \( D > D_0 > 2 \) which corresponds to the difference in the case of arithmetical progressions and which may increase indefinitely. We suppose that
\[
1 < h < D^3,
\]
where \( c_0 \) is any positive constant. Secondly we take for granted that the coefficients \( a_i \) in (1) do not depend on \( i \). Otherwise in some of the classes there might be no generators \(^{(2)} \) and thus we could not prove the existence of a generator \( b = B^{(0)} \), \( b \in H \), for every \( H \), which is our aim. A proof of the existence of a generator \( b \in H \) is usually based on the preponderance of the residue at \( s = 1 \) in the sum of all residues of a certain expression containing the logarithmic derivatives of all \( L \)-functions \( \zeta(n, \sigma) \) with characters \( \chi \) of the group \( K \) (cf. [17], p. 134). If there are in (1) at least two different coefficients \( a_i \), then, as will be proved in the Appendix (§ 28), \( s = 1 \) is a pole for at least two functions \( \zeta'/\zeta \) and in some cases the dominating term in the residue sum disappears, giving no evidence of the existence of a generator in the corresponding class \( H \). In all the applications of any importance (they will be given in a continuation of this paper) the equality condition \( a_1 = \ldots = a_{\ell} \) for the coefficients \( a_i \) in (1) will be satisfied.

According to these changes we replace (1) by
\[
\sum_{a \neq \text{null}} 1 = ax + O(D^3x^{1-\varepsilon}), \quad a = D^3,
\]
where the constants \( l, c_0, \theta \) do not depend on \( i \) \((0 < l < 1, \ 0 < c_0 < 1, \ 0 < \theta < 1) \). Next we introduce another parameter \( q \) such that \(^{(3)}\)
\[
1 < q < D^\theta
\]
and we suppose that if \( q > 1 \), then the numbers \( a \in \mathcal{A} \) and \( x \) in (3) are exclusively terms of the progression
\[
1, \ g, \ g^2, \ g^3, \ldots
\]
It is understood that all further constants (including those of the symbols \( O \) and \( \ll \)) may depend on \( c_0, c_1, \theta, l \).

Denoting by \( R(a) \) the remainder term in (3) we have for \( q > 1 \) and for any number \( x \) of the progression (5):
\[
\sum_{a \neq \text{null}} 1 = ax + R(a) - R(x/g), \quad a = \theta(1 - q^{-1}), \quad R(x) \ll D^3x^{1-\varepsilon},
\]
whence
\[
\sum_{q \neq \text{null}} \frac{R(q^h) - R(q^h-1)}{q^h} = -\frac{R(x)}{x} + \sum_{q \neq \text{null}} R(q^h) \frac{1}{q^h} - \frac{1}{q^{h-1}} = -\frac{R(x)}{x} + \frac{1}{q} \sum_{q \neq \text{null}} R(q^h)
\]
\[
\ll D^\theta x^{-\varepsilon} + (1 - q^{-1}) D^3 x^{-\varepsilon} \sum_{q \neq \text{null}} q^{-1} < D^\theta \left( x^{-\varepsilon} + (g x^{-\varepsilon} \sum_{q \neq \text{null}} q^{-1} \frac{1 - q^{-1}}{1 - q^{-1}}) \right) < D^\theta x^{-\varepsilon},
\]
\[
\sum_{q \neq \text{null}} \frac{1}{q} = a_1 \sum_{q \neq \text{null}} 1 + \sum_{q \neq \text{null}} \frac{R(q^h) - R(q^h-1)}{q^h}
\]
\[
= a_1 \left( 1 + \frac{\log x}{\log q} \right) + \sum_{q \neq \text{null}} \frac{1}{q} - \sum_{q \neq \text{null}} \frac{1}{q^h} \frac{R(q^h) - R(q^h-1)}{q^h}
\]
\[
= 1 - \frac{1}{\log q} \log x + \log x + O(D^\theta x^{-\varepsilon}).
\]

The last estimate for the left-hand side of (7) holds as well if \( q = 1 \) (now for any \( x > 1 \)), but then the factor \( (1 - q^{-1}) \log q \) becomes unity. In this case (7) could be proved more conveniently by the use of Abel's identity (17), p. 317).

For even \( h \) let \( K_h \) denote any sub-group of the group \( K \) with the index 2. Then we have, by (7),
\[
\lim_{x \to \infty} \left( \sum_{a \neq \text{null}} \frac{1}{a} - \sum_{a \neq \text{null}} \frac{1}{a} \right) = C_h = C_h(D, q).
\]

With the use of this notation the principal result of the present paper may be formulated as the following

**Theorem.** (i) If \( \theta > 1 \), then there is a positive constant \( c \) such that for any \( x > 1 \) and any \( H \) in the interval \( (x, xD^\theta) \) there is at least one generator of \( H \). For an odd class-number \( h \) the conclusion holds as well in the case of \( \theta = 1 \).

(ii) Let \( h \) be even and \( \theta < 1 \). If there is a constant \( c_0 > 0 \) such that in (8) for any \( \alpha \)
\[
C_h > D^{-\alpha},
\]
then the conclusion of (i) holds (with the constant \( c \) depending also on \( c_0 \)).
Corollary. For appropriate constants \( c_1, c_2 > 0 \) and any \( x > D^3 \), we have

\[
\pi(x, H_i) > x/(i!4^i) \log x.
\]  
(10)

If \( x > \exp(D^3) \) (with a suitable \( c_2 > 0 \)), then by known methods one can prove the asymptotical formula for \( \pi(x, H_i) \), which is a stronger result. Nevertheless from that formula one cannot deduce the estimate for the least generator \( b \in H_i \),

\[
\min b_{H_i} < D^\epsilon
\]  
(11)

the proof of which is our present task. Since the proof of the existence of a generator \( b \in H_i \) in the interval \( (x, xD^\epsilon) \) is only a little more complicated, instead of (11) we shall prove the assertion of the theorem. It follows from part (i) of the theorem that Linnik’s estimate (11) for the least prime of arithmetical progression \( Du + a \ (u = 0, 1, \ldots; \ (D, d) = 1 \) (see [15]) can be proved without using the functional equation of Dirichlet L-functions and their existence in the whole plane.

If \( q < 1 \), \( x \to \infty \) and other restrictions are imposed, then the theorem holds for intervals \( (x, xD^\epsilon) \), where \( \epsilon \) denotes an arbitrary small positive constant. For a sketch of the proof see §§ 25-27.

In the present theorem we are concerned with a linear distribution problem in a semi-group. An analogous theorem on two-dimensional distribution will be proved in the continuation of this paper. From that theorem we will deduce an estimate for the smallest prime representable by a primitive binary quadratic form \( \psi(u, v) \) if the point \( (u, v) \) is bound to a given angle in the plane of the variables. From the one-dimensional theorem of the present paper we can deduce among others an upper bound for the height of a polynomial \( \psi(x) \mod\psi(x) \) irreducible in a fixed algebraic field. These results have been announced in [9], [10].

2. In this paper we shall use the analytical method of the zeta function with a complex variable, since the elementary method at present does not give a satisfactory estimate for the least prime \( \equiv d \mod D \). The method is in outline the same as that used by Linnik [15], its chief weapon being an estimate for the density of zeros of all \( L \)-functions in the neighbourhood of \( s = 1 \) (cf. the fundamental Lemmas 10 and 25 of the present paper). A variant of this method, due to Roudski [see (18) or (17), Ch. X], was used in my previous papers [9]-[18], where the least prime problem was solved for classes of ideals in any algebraic field. In these papers for the proof of the main auxiliary theorems a general lemma (6), p. 133) was used together with an upper bound for the number of ideals of any class having a special property (7, Lemma 7, proved by Selberg’s sieve method). In Linnik’s, Roudski’s and my own papers these theorems were proved by a method which essentially depends on the distribution of the small primes (see, for example, [6], p. 133 and [7], p. 238, where the sum over the primes \( \sum_p \log p \) has to be small enough for a sufficiently large \( \varepsilon \). The same method cannot be used for the proof of the present theorem, since (3) does not give the necessary information about the distribution of the small numbers \( a \in G \). But the method would work if we added a new condition (a rather unnatural one):

\[
\sum_{0 < n < x} \log b < x
\]  
for \( 1 \leq x \leq D^3 \).

To avoid the necessity of any new condition, for the proof of the main lemmas in this paper we shall follow Turán’s method, which was used by Turán [22] and Knappoewski [13] in proving the corresponding theorems for Dirichlet \( L \)-functions.

It seems probable that the truth of (11) does not depend on the distribution of the large numbers \( x > D^{100} \) of \( G \); that is to say, one might hope to prove (11) using (3) merely for \( x \leq D^3 \) with some sufficiently large \( \varepsilon < 1 \). However, this restriction excludes the use of the method of \( L \)-functions, the only known method by which (11) can be proved.

All the constants used in this paper (with at most one exception of the constant \( \epsilon \)) are positive. The values of \( c_1, c_2, c_3, \ldots \) remain as fixed in § 1, whereas other constants (generally denoted by \( e, c, \ldots \)) retain the meaning only throughout the same paragraph. More important constants and constants of some general lemmas are denoted by \( A, B, C \).

The complex variable is generally denoted by \( s = a + it \), \( t = \infty \), but in a few cases we use \( w \) as well. By \( \dagger \) \( a \) (‘\( \dagger \)’ divides \( a \)) we mean that \( d, a \) are numbers of \( G \) and there is a number \( a' \in G \) such that \( a = da' \); the maximal \( d \) which divides \( a \) and \( a' \) will be denoted by \( (a, d') \).

Some details in the proofs may differ according to whether \( q = 1 \) or \( q > 1 \). In order to treat both cases simultaneously we shall use the following factors:

\[
\theta = \begin{cases} \frac{(1 - q^{-1})/\log q}{1} & \text{if } q > 1, \\ 0 & \text{if } q = 1 \end{cases}, \quad \varphi = \begin{cases} 1 & \text{if } q > 1 + D^{-1}, \\ 0 & \text{otherwise}; \end{cases}
\]

\[
\gamma = \begin{cases} 1 & \text{if } q > 1, \\ 0 & \text{if } q = 1. \end{cases}
\]

The functions \( \zeta(s, \chi) \) and their zeros near the line \( \sigma = 1 \).

3. Lemma 1. The function

\[
\zeta(s, H) = \sum_{n \in H} n^{-s} \quad (\sigma > 1)
\]
is regular in \( \sigma > 1 - \theta \) except for a simple pole at \( s = 1 \) with residue \( a \) if \( q = 1 \), or simple poles at \( s = 1 + 2k\pi i/\log q \) \( (k = 0, \pm 1, \ldots) \) with residues \( \theta a \) if \( q > 1 \). In the latter case \( \zeta(s, H) \) is a periodic function, its period being \( 2\pi i/\log q \).

We have uniformly in the strip \( G(1 - \theta + \eta \leq \sigma \leq 2) \) \( (0 < \eta < \frac{1}{2} \theta) \)

\[
\zeta(s, H) = \begin{cases} 
 1 - a^{-1} + O(D^\eta \gamma^{-1}|s|) & \text{if } q = 1, \\
 1 - q^{-1} + O(D^\eta \gamma^{-1}(1 + \sigma|s|)) & \text{if } q > 1 .
\end{cases}
\]

(12)

Proof. First let \( q = 1 \) and let \( \zeta(s) \) be the zeta-function of Riemann.

By (3) in \( \sigma > 1 \)

\[
\zeta(s, H) = \sum_{a \leq H} a^{-s} = \int_1^\infty \frac{R(s, x)}{\sigma^{s-1}} \, dx 
= \frac{\pi}{\sin \pi \sigma} \int_1^\infty \frac{R(s, x)}{\sigma^{s-1}} \, dx \quad (R(s, x) \ll a + D^\sigma x^{-\theta}).
\]

In \( G \) the last integral may be represented by the sum

\[
\sum_{n=1}^\infty g_n(s) \quad \text{with} \quad g_n(s) = \int_1^\infty \frac{R(s, x)}{\sigma^{s-1}} \, dx \ll \eta^{-1} D^\sigma \gamma^{-(n+1)}.
\]

whence the regularity and (12) follows.

If \( q > 1 \), then, by (9), in \( \sigma > 1 \)

\[
\zeta(s, H) = a_0 \sum_{a \leq H} \frac{1}{q^{\sigma(a)}} + (1 - q^{-s}) \sum_{q | a} \frac{E(q^s)}{q^{\sigma}}.
\]

Since the general term of the last series in \( G \) satisfies \( \ll D^\sigma q^{-\sigma} \), the function represented by that series is regular and \( \ll D^\sigma (1 - q^{-s}) \). This proves (12) for \( s < 1/\log q \). The periodicity of \( \zeta(s, H) \) in \( \sigma > 1 \) follows from the representation

\[
\zeta(s, H) = d_0 + (1 - q^{-s}) + d_1 q^{-s} + \ldots, \quad \text{where} \quad d_0 = \sum_{q^k \leq H} 1,
\]

and further it holds by analytic continuation.

4. Further let \( \chi \) denote the characters of the group \( K \), \( \chi_0 \) being the principal character, and let \( \chi(a) = \chi(H) \) for all \( a \in H \). Write

\[
(13) \quad \zeta(s, \chi) = \sum_{H} \chi(H) \zeta(s, H) = \sum_{a \in H} \chi(a) a^{-s} \quad (\sigma > 1).
\]

Since

\[
\sum_{H} \chi(H) = \begin{cases} 
 1 & \text{if } \chi = \chi_0, \\
 0 & \text{otherwise}
\end{cases}
\]

(see [12], §10), by Lemma 1 the function \( \zeta(s, \chi) \) is regular in \( \sigma > 1 - \theta \), except for a simple pole of \( \zeta(s, \chi) \) at \( s = 1 \) with residue \( ha \) if \( q = 1 \) or simple poles at \( s = 1 + 2k\pi i/\log q \) \( (k = 0, \pm 1, \ldots) \) with residues \( \theta a \) if \( q > 1 \). In the latter case all the functions \( \zeta(s, \chi) \) are periodic, the period being \( 2\pi i/\log q \). By (13), (12) and (2) we have uniformly in \( 1 - \theta + \eta \leq \sigma \leq 2 \)

\[
(14) \quad \zeta(s, \chi) = \begin{cases} 
 c_0 (a-1)^{-1} + O(D^\eta \gamma^{-1}|s|) & \text{if } q = 1, \\
 c_0 \frac{1 - q^{-1}}{1 - q^{-s}} + O(D^\eta \gamma^{-1}(1 + \sigma|s|)) & \text{if } q > 1, 
\end{cases}
\]

where

\[
\begin{align*}
 c_0 &= \begin{cases} 
 1 & \text{if } \chi = \chi_0, \\
 0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

(\( \mathbb{S} \) being a semi-group with the generators \( b \), in the half-plane \( \sigma > 1 \) we have

\[
\zeta(s, \chi) = \prod (1 - \chi(b) b^{-s})^{-1},
\]

whence there are no zeros of \( \zeta(s, \chi) \) in \( E \).

Let \( \mu(a) = (-1)^a \) if \( a \) is a product of \( \tau \) different generators, and \( = 0 \) if \( b^\tau \mid a \). Further let

\[
\Lambda(a) = \begin{cases} 
 \log b & \text{if } a = b^\tau \ (n \geq 1), \\
 0 & \text{otherwise.}
\end{cases}
\]

From the product-form of \( \zeta(s, \chi) \) we deduce

\[
1/\zeta(s, \chi) = \sum_a \chi(a) \mu(a) a^{-s}, \quad (\sigma > 1).
\]

(16)

5. Lemma 2. Let \( R \) be the rectangle whose vertices are \( -1 + \pi i/\log q \), \( 2 + \pi i/\log q \) and let

\[
g(s) = \frac{\log q}{1 - q^{-s}} - \frac{1}{s-1}.
\]
Then in $R$

$$g(s) = \frac{\log q}{1 - q^{-1}} + o(\log q).$$

Proof. On the horizontal sides of $R$ \(|g(s)| = (1 + q^1)^{-1} \log q + O(1)\), on the left vertical side \(|g(s)| \leq (q^1 - 1)^{-1} \log q + O(1)\), and on the right vertical side \(|g(s)| \leq (1 - q^{-1})^{-1} \log q + O(1)\). Hence the lemma follows by the maximum modulus principle.

Corollary. In $R$

$$\log q = \frac{1}{q - 1} + O(\log q).$$

Since, for any $q > 1$, $1 - q^{-1} \leq \log q$, we have, by (15), (16), for all $\eta \in (0, 1]$ and $q \geq 1$

$$\frac{1}{q - 1} \leq \frac{1}{q - 1} \leq \log q.$$

Let $f(s)$ be a regular function and let $|f(s)| \leq e^{\rho_1}$ in $|s| < \frac{1}{2} r$. Then by Landau's lemma (see (23), III, § 39)

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s} < \frac{M}{r} \left( \frac{|s| - \frac{1}{2} r}{|s| - \frac{1}{2} r} \right),$$

where $\rho$ runs through the zeros of $f(s)$ in $|s| < \frac{1}{2} r$. Taking $r = \theta$, $a_n = 1 + \eta + \xi_n$, $\eta = 1/(1 + \rho_2)$ and using (15), (18) we deduce that in $|s| - \frac{1}{2} r \\leq \frac{1}{2} r$

$$\zeta(s) = \frac{(s-1)^{-1}}{\log q},$$

where

$$\frac{1}{q - 1} \leq \frac{1}{q - 1} + O(\log q).$$

From this estimate and (19) by arguments used in [5], § 11 we can deduce the inequality

$$|\zeta'(s)| \leq \frac{1}{q - 1} \leq 1 + c_0 \log D$$

and $c_0$ is small enough.

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Taking $s = \sigma > 1$ we have in (19) $\rho \geq 0$. Hence for all positive $r < 1$

$$|\zeta'(1 + r, \chi_0)| \leq 1 + c_0 \log D.$$

From (19), (21) we can deduce

$$\rho < \log (1 + \rho_2]\eta_0)(\frac{1}{\sigma} \log (1 + \rho_2]\eta_0)(\frac{1}{\sigma} \log (1 + \rho_2]\eta_0) \leq \rho < \frac{1}{2} \theta_0 - 1 / D(1 + \rho_2]\eta_0)$$

where $\rho = \rho(r, \chi, t)$ denotes the number of zeros of $\zeta(s, \chi)$ in $|s| < \frac{1}{2} r < \frac{1}{2} r$.

Proof. Let $C_0, C_1, C_2$ denote circles having a common centre at $s_0 = 1 + \theta + \eta_0 \xi_0$ ($\eta_0 = 1 / (1 + \rho_2]\eta_0)$, the radii being $2\theta_0, \theta_0, \theta_0$, respectively. Further let $\zeta(s)$ denote the number of zeros of $\zeta(s, \chi)$ in $C_0$. If $\zeta(s) = \frac{1}{2} r$, then by (15) in $C$

$$|\zeta(s, \chi)| \leq \exp(|c_0 \log (1 + \rho_2]\eta_0)|.$$

By Jensen's theorem (24), § 3.61 and (18)

$$\int_0^{\frac{1}{2} r} \frac{v(x)}{x} dx = \frac{1}{2} \int_0^{\frac{1}{2} r} \int_0^{\frac{1}{2} r} \log \zeta(s_0 + 2 \theta_0 \xi, \chi) d\theta d\phi - \log \zeta(s_0, \chi),$$

whence

$$c_0 \log (1 + \rho_2]\eta_0) \leq c_0 \log (1 + \rho_2]\eta_0)_0,$$

and thus

$$\frac{1}{\sigma} \log (1 + \rho_2]\eta_0) \leq \frac{1}{\sigma} \log (1 + \rho_2]\eta_0) \leq \frac{1}{\sigma} \log (1 + \rho_2]\eta_0).$$

Observing that $R$ can be covered by $\frac{1}{2} r$ circles $C$ having their centres on the line $\sigma = 1 + \theta + \eta$, we get the lemma for $\chi \neq \chi_0$.

If $\chi = \chi_0$, then we use the function $(s - 1)\zeta(s, \chi_0)$ or $(1 + q^{-1})\zeta(s, \chi_0)$ (according as $q = 1$ or $q > 1$) and similar arguments.

7. Lemma. If $q = 1$, then for appropriate $c$ no function $\zeta(s, \chi)$ has a zero in the region $\sigma > 1 - c / \log D(1 + \theta_0)$, $|\theta_0| \geq 3$. 

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The proof is based on the inequality
\[ -3\zeta'/\zeta(a_0, x_0) - 4 \Re \zeta'/\zeta(a_0 + 2i\gamma, x) - \Re \zeta'/\zeta(a_0 + 2i\gamma, x^2) \geq 0 \quad (a_0 > 1) \]
(cf. [5], §12). In \( \sigma \geq 1 - \frac{1}{4}\log 4 + \eta \) \((\eta = 1/(D + |A|))\) we have by (19)
\[ \zeta'/\zeta(s, x) = \sum_{|\nu| < \sqrt{|A|}} (s - \nu)^{-1} + O(\log D(1 + |\eta|)) \quad (s = 1 + \eta + i\theta). \]

Let \( \gamma = \beta + i\nu \) \((|\nu| \geq 3)\) be a zero of \( \zeta(s, \chi) \). Since \( \Re \sum(s - \nu)^{-1} \geq 0 \) in \( \sigma > 1 \), we have
\[ \Re \zeta'/\zeta(a_0 + \nu, x) \geq (a_0 - \beta)^{-1} - a_0 \log D|\nu|, \]
\[ \Re \zeta'/\zeta(a_0 + 2\nu, x^2) \geq -a_0 \log D|\nu|. \]
Take for \( a_0 \) any number satisfying (20). Then, by (23),
\[ \frac{15}{4}: \frac{1}{a_0 - 1} \geq 4 - \frac{1}{a_0 - 1} - \frac{1}{a_0 - 1} - a_0 \log D|\nu|. \]

Putting
\[ a_0 - 1 = 1/\log D|\nu|, \quad 1 - \beta = 1/\log D|\nu| \]
and dividing through by \( A \log D|\nu| \), we get the inequality
\[ \frac{15}{4} \geq \frac{4}{1 + 1/A} - \frac{a_0}{A} \]
in which \( A \) cannot increase indefinitely. Since \( \beta = 1 - 1/\log D|\nu| \), the lemma follows.

**Lemma 5.** For appropriate \( \epsilon \) no function \( \zeta(s, \chi) \) with a complex \( \chi \) has a zero in the region
\[ \sigma > 1 - \epsilon/\log D, \quad |\nu| < 5 \quad \text{or} \quad \sigma > 1 - \epsilon/\log D(1 + \log |\nu|) \]
(according as \( q = 1 \) or \( q > 1 \)).

\( \chi \) not being the principal character, the proof proceeds along the same lines as that of the previous lemma.

**Lemma 6.** For appropriate \( \epsilon \) no function \( \zeta(s, \chi) \) with a real \( \chi \neq x_0 \) has a zero in the region
\[ \sigma > 1 - \epsilon/\log D, \quad 0 < |\nu| \leq \delta \]
or
\[ \sigma > 1 - \epsilon/\log D(1 + \log |\nu|), \quad 0 < |\nu| \leq \pi/\log q \]
(according as \( q = 1 \) or \( q > 1 \)).

Proof. First let \( q = 1 \) and let \( \zeta(s, \chi) \) have a zero \( \gamma = \beta + i\nu \) with
\[ 0 < \gamma < 1/\log D, \quad 0 < 1 - \beta < 1/\log D \quad (B < 1). \]
Then it has another zero \( \gamma' = \beta - i\nu \). Writing
\[ a_0 = a_0 + i\nu, \quad a_0 = 1 + 1/\log D \quad (A < 1) \]
we have, by (19),
\[ \Re \zeta'/\zeta(a_0, \chi) \geq \frac{1}{a_0 - \beta} + \frac{a_0 - \beta}{(a_0 - \beta)^2 + 4\nu^2} - a_0 \log D. \]
By (16), (20)
\[ \Re \zeta'/\zeta(a_0, \chi) \leq |\zeta'/\zeta(a_0, \chi_0)| \leq \frac{1}{2(a_0 - 1)} \]
and thus
\[ \frac{1}{a_0 - \beta} + \frac{a_0 - \beta}{(a_0 - \beta)^2 + 4\nu^2} < a_0 \log D + \frac{1}{a_0 - 1}, \]
whence
\[ \frac{a_0 - \beta}{(a_0 - \beta)^2 + 4\nu^2} < a_0 \log D + \frac{1}{a_0 - 1} + \frac{1}{a_0 - 1} \leq a_0 \log D + \frac{A}{B} \log D. \]
Since
\[ \frac{1/\log D}{a_0} \leq \frac{A}{B} \leq \frac{1/\log D}{1/\log D}, \]
we have
\[ \frac{1/\log D}{a_0} \leq \left( \frac{1/\log D}{(\log D)^2} + \frac{4}{(\log D)^3} \right) \left( \frac{1}{2} A + \frac{A^2}{B} \right) \log D, \]
whence
\[ \frac{1}{A} \leq \left( \frac{1}{2} A + \frac{A^2}{B} \right) \left( \frac{1}{A} \left( \frac{1}{2} A + \frac{A^2}{B} \right) \right). \]
This is impossible for any fixed \( A \) (with \( 1/4 \geq a_0 \)) and all large \( B \), since the right-hand side tends to \( 1/2 A \) as \( B \to \infty \). This proves the lemma in the case we have been dealing with.

Now let us suppose that \( \zeta(s, \chi) \) has a zero \( \beta + i\nu \) with \( \gamma \geq 1/\log D \).

Since \( \chi = x_0 \), by (25)
\[ \begin{array}{c}
-3\zeta'/\zeta(a_0, x_0) - 4 \Re \zeta'/\zeta(a_0 + i\nu, x) - \Re \zeta'/\zeta(a_0 + 2i\nu, x_0) \geq 0 \\
(a_0 > 1).
\end{array} \]
The estimates for the first two terms of (25) may be acquired as in the proof of Lemma 5. By (19)
\[\text{re}^{c'}(\zeta(s) + 2iy) = - \text{re}^{c_0 + 2iy - 1} + \text{re}^{\frac{1}{s_0 + 2iy - \varepsilon} + O(\log D) > - c_0 \log D}\]
(since \(\text{re}^{\sum} \geq 0\) and \(|\text{re}(s_0 + 2iy - 1)| < |s_0 - 1| + 2|y| < \frac{1}{2} \log D\) and we may proceed as in Lemma 5.

Now let us consider the case of \(q > 1\). If \(\pi/\log q > 2\), then the necessary inequalities for the zeros in \(1 \leq |t| \leq \pi/\log q - 1\) are obtainable as in the proof of Lemma 4. In the remaining parts \(0 < |t| < 1\) and \(\pi/\log q - 1 < |t| < \pi/\log q\) we use the function \(Z(s) = \zeta(s - \pi/\log q, \chi)\) and the same arguments as for \(q = 1\) of the present lemma but changing at most \(D \to D(1 + \pi/\log q)\). Consider that the zeros of \(Z(s)\) are conjugate, since the coefficients of the Dirichlet series of \(Z(s)\) are real. The same proof holds if \(\pi/\log q < 2\).

**Lemma 7.** Let \(\chi\) be a real character \(\neq \chi_0\). For appropriate \(c'\) there is at most one real zero \(\beta' > 1 - c'/\log D\) and if \(q > 1\) at most one zero \(\beta'' = \beta'' + iv/\log q\) with \(\beta'' > 1 - c'/\log D(1 + \pi/\log q)\).

**Proof.** Let \(\beta\) and \(\beta'\) be two real zeros of \(\zeta(s, \chi)\) such that \(\beta' \geq \beta\). By the arguments used in the proof of (24)
\[\frac{1}{s_0 - \beta} + \frac{1}{s_0 - \beta'} < c_0 \log D + \frac{5}{4} (\sigma_0 - 1)^{-1},\]
whence for a sufficiently large \(A < 1\)
\[2l(s_0 - \beta) < c_0 \log D + \frac{1}{4} A \log D < \frac{1}{4} A \log D,\]
\[s_0 - \beta > 4/3 A \log D, \quad \beta < s_0 - 4/3 A \log D = 1 - c'/\log D.\]

This proves the statement concerning the real zeros. The same method of proof may be used for the zeros on the line \(t = \pi/\log q\).

**9. Lemma 8.** For appropriate \(c\) the function \(\zeta(s, \chi_0)\) has no zero in the region
\[\sigma > 1 - c/\log D, \quad 0 < |t| < 5\]
or
\[\sigma > 1 - c/\log D(1 + \pi/|t|), \quad 0 < |t| < \pi/\log q\]
(according as \(q = 1\) or \(q > 1\)).

**Proof.** Writing
\[G(s) = \begin{cases} (s-1)\zeta(s, \chi_0) & \text{ if } q = 1, \\ (1-q^{-1})\zeta(s, \chi_0) & \text{ if } q > 1, \end{cases}\]
\[s_0 = 1 + 1/A \log D, \quad \sigma = \sigma_0 + iv\]
and using (19) and (22), by the arguments of [5], § 18 we can prove that for a sufficiently large \(A < 1\) and a small \(\gamma < 1/\log D\)
\[\text{re}^{c}(\zeta(s_0)) < \frac{1}{2} (s_0 - 1)^{-1}.\]

Let us suppose that \(\zeta(s, \chi_0)\) has a zero \(\beta + iv\) with \(0 < \gamma < 1/\log D\); then it has another zero \(\beta' - iv\). By (19) and Lemma 3
\[G'(\zeta(s)) = \sum_{\eta \neq \zeta^*} (s_0 - \eta)^{-1} + O(\log D)\]
whence
\[\text{re}^{c}(\zeta(s_0)) \geq \frac{1}{\sigma_0 - \beta} + \frac{\sigma_0 - \beta}{(\sigma_0 - \beta)^2 - 4s_0^2} - c_0 \log D,\]
and using (26) we may go on as in Lemma 6. This proves the required result concerning zeros near the line \(t = 0\). If \(D\) is replaced by \(D(1 + \pi/\log q)\), the same method may be used for zeros near the line \(t = \pi/\log q\).

Now suppose that \(q = 1\), \(\gamma > 1/\log D\) and \(\alpha_0\) is any number for which (20) holds. Then by (19)
\[\text{re}^{c}(\zeta(s_0 + 2iy, \chi_0)) \geq (s_0 - \beta)^{-1} - c_0 \log D,\]
\[\text{re}^{c}(\zeta(s_0 + 2iy, \chi_0)) = - c_0 \log D.\]

Using (20) and (25) with \(\chi = \chi_0\) and the arguments of Lemma 5 we prove the remaining part of the present lemma for \(q = 1\). The method of proof works as well for the zeros off the lines \(t = 0\), \(t = \pi/\log q\) in the case of \(q > 1\).

**Lemma 9.** For appropriate \(c\) the function \(\zeta(s, \chi_0)\) has at most one real zero \(\beta' > 1 - c'/\log D\) and if \(q > 1\) at most one zero \(\beta'' + iv/\log q\) with \(\beta'' > 1 - c'/\log D(1 + \pi/\log q)\).

This may be proved by the arguments of Lemma 7.

**10. Lemma 10.** For appropriate \(c\) at most one of the \(\zeta(s, \chi)\) functions \(\zeta(s, \chi)\) with a real character has a zero \(\beta > 1 - c'/\log D\) and, if \(q > 1\), at most one of them has a zero \(\beta + iv/\log q\) with \(\beta > 1 - c'/\log D(1 + \pi/\log q)\).

**Proof.** First let \(q = 1\) and let the two functions \(\zeta(s, \chi_0)\) and \(\zeta(s, \chi_0)\) with two real and different characters \(\chi_1, \chi_0\) have the real zeros
\[ \beta_i > 1 - \theta/\delta \text{ and } \beta_j > 1 - \theta/\delta, \text{ respectively. Further let } \chi_1 \neq \chi_2, \chi_3 \neq \chi_2. \]

Taking \( a_0 = 1 + \frac{1}{A \log D} \) with a sufficiently large \( A \ll 1 \) we have, by (19) and (29),

\[
\begin{align*}
\zeta' / \zeta(a_1, x_1) &> (c_0 - \beta_j)^{-1} - c_1 \log D, \\
\zeta' / \zeta(a_0, x_0) &> (c_0 - \beta_j)^{-1} - c_2 \log D, \\
\zeta' / \zeta(a_1, x_1) &> -c_1 \log D, \\
\zeta' / \zeta(a_0, x_0) &> -c_2 \log D.
\end{align*}
\]

Since the sum of the left-hand sides in this set of inequalities is \( \leq 0 \), by (16), we deduce

\[
(c_0 - \beta_j)^{-1} + (c_0 - \beta_j)^{-1} < c_1 \log D + c_2 (c_0 - 1)^{-1}
\]

and may go on as in the proof of Lemma 7.

Now let \( x_0 = x_1 \), the other premises remaining unchanged. Then instead of \( \zeta(t, x_1) \) we use the function \( G(a) \) of \( \xi \). By (19)

\[
G' / G(a_0) > (c_0 - \beta_j)^{-1} - c_1 \log D
\]

and

\[
\zeta' / \zeta(a_0, x_1) > (c_0 - \beta_j)^{-1} - c_1 \log D.
\]

Since, by (16),

\[
\zeta' / \zeta(a_0, x_1) + \zeta' / \zeta(a_0, x_0) < 0,
\]

we deduce

\[
(c_0 - \beta_j)^{-1} + (c_0 - \beta_j)^{-1} < (c_0 - 1)^{-1} + c_1 \log D
\]

and may go on as before.

The same arguments may be used in case of \( q > 1 \) if, dealing with the zeros on \( t = \pi / \log q \), we replace \( D \) by \( D(1 + \pi / \log q) \).

The results embodied in Lemmas 4-10 may be summarized by the following

**FUNDAMENTAL LEMMA 11.** For appropriate \( c \) in the region

\[
\sigma > 1 - c \log D (1 + \log |t|) (\sigma > 1 - \theta/\delta)
\]

there is zero of \( \zeta(t, x) \) in case of a complex \( x \). For at most one real \( x \) there may be in (27) a simple real zero \( \beta' \leq 1 \) and, if \( q > 1 \), the zeros \( \beta' = 2k\pi / \log q \) \( (k = 0, \pm 1, \ldots) \). Besides, if \( q > 1 \), for at most one real \( x \) there may be in (27) the set of simple zeros \( \beta' = 2k\pi / \log q \) \( (k = 0, \pm 1, \ldots) \) with \( \Re \beta' < 1 \), \( \Im \beta' = \pi / \log q \).

The zeros \( \beta', \beta' = 2k\pi / \log q \) and \( \beta' = 2k\pi / \log q \) (if existing) will be called the exceptional zeros of \( \zeta(t, x) \).

**An upper bound for the number of generators.**

**Lemma 12.** Let \( \alpha_m \) \( (m = 1, 2, \ldots, N) \)

be a set of numbers \( \alpha \in \mathbb{G} \) such that for any \( \alpha \in \mathbb{G} \)

\[
\sum_{\alpha \in \mathbb{G}} 1 = N f(d) + R_d
\]

where \( f(d) \) is a multiplicative function \( \neq 0 \); that is to say, \( f(a_0, a_1) = f(a_1) f(a_0) \) whenever \( (a_0, a_1) = 1 \). Further let \( N_s \) denote the number of those numbers \( \alpha \)

of (28) which are not divisible in \( \mathbb{G} \) by any generator \( b \triangleq a \). Write

\[
F(a) = \sum_{d \mid a} \mu(d) \left( \frac{a}{d} \right), \quad S_s = \sum_{d \nmid a} \frac{\mu^2(d)}{F(d)}, \quad S_d(a) = \sum_{d \nmid a} \frac{\mu^2(d)}{F^2(d)},
\]

\[
\lambda_a = \begin{cases} \mu(a) \left( 1 - \frac{1}{\phi(b)} \right)^{-1} S_s(a)/S_d(a) & \text{if } a \triangleq a, \\ 0 & \text{otherwise}. \end{cases}
\]

Then

\[
N_s < N |S_s| + \sum_{a \triangleq \alpha \in \mathbb{G}} |\lambda_a| \lambda_a R_{a_{\triangleq a}(a_{\triangleq a})}.
\]

This may be proved by the sieve method of A. Selberg [19]. See [7], § 3.

**Lemma 13.** In the previous lemma let (28) be all the numbers \( a \triangleq a \)

of any class \( H \) (as defined in § 1). If \( a \triangleq a \), \( x \triangleq D \) and \( c_1 \) is large enough, then the main term in (31) satisfies

\[
N/|S_s| \ll x/\delta \log x
\]

(this with the \( \delta \) defined in § 2).

**Proof.** By (3)

\[
N = \alpha x + O(D^\theta x^{-\delta}).
\]

Considering that the classes \( H \) form a group, we deduce that for any \( d \in \mathbb{G} \) the number of numbers \( a \triangleq a \) such that \( a \in H \) and \( d \mid a \), by (3) is

\[
\frac{\alpha x}{d} + O\left( D^\theta \frac{x^{1-\delta}}{d} \right) = \frac{N}{d} + O\left( D^\theta \frac{x^{1-\delta}}{d} \right)
\]

Hence we have (29) with

\[
f(d) = d, \quad R_d < D^\theta \frac{x^{1-\delta}}{d}.
\]
Now by (30)

$$S_s = \sum_{n \leq x} \mu^2(a) \frac{1}{n} \prod_{\nu=1}^{m} \left( 1 - \frac{1}{b^\nu} - \frac{1}{b^{2\nu}} - \ldots \right) = \sum_{n \leq x} \sum_{a \in \mathcal{O}} \frac{1}{a}.$$

where (a) stands for the set of numbers a' ∈ \mathcal{O} such that the product of all different generators (*) of any a' is ≤ a. Hence

$$S_s > \sum_{n \leq x} \frac{1}{a} > \sum_{y \leq x} \frac{1}{a}.$$

If η is large enough, then by (7) the last sum is > \frac{1}{2} h \delta n \log^r x > c_1 \delta \log x.
Since N < 2ζ, by (32), we get the required result.

13. **Lemma 14.** Let W denote the remaining term in (31) and let in Lemma 15

$$z^s = e^{s} \log^r x \quad \text{where} \quad e_4 = e_4 - e_4 + \max(e_4, l);$$

then

$$W < \delta \log x.$$

**Proof.** For any a we have |a| < 1 (see [7], (38)), whence by (31), (33)

$$W < D^r x^{1 - \delta} \sum_{a_1 < a_2 < x} \left( \frac{a_1}{a_2} \right)^{1 - \delta},$$

and the desired estimate would follow from

$$\sum_{a_1 < x} \left( \frac{a_1}{a_2} \right)^{1 - \delta} < \frac{z^s}{h \log^r x}.$$

Using (3), we prove first the estimates

$$\sum_{a_1 < x} a^{1 - \delta} < \delta x^{1 - \delta} < D^{a},$$

whence

$$\sum_{a_1 < x} \left( \frac{a_1}{a_2} \right)^{1 - \delta} \leq \sum_{a_1 < x} a^{1 - \delta} < (h \delta)^2 x^{2 \delta}.$$

By (37)

$$S_s = \sum_{a_1 < a_2 < x} \left( \frac{a_1}{a_2} \right)^{1 - \delta} < \frac{(h \delta)^2 x^{2 \delta}}{a}.$$
By (39) and by the definition of $A(a)$, (40) would clearly follow from
\[ \sum_{x \leq \theta_1} \log b + \sum_{x > \theta_1} \log b + \ldots < \sigma/\theta_1. \]
Since the number of terms on the left is $< \log \theta \log \theta < D^u \log \theta u$, none of them exceeding $(\log \theta + \log D' \theta^{(1-\eta_0)}) \log \theta u$, the truth of (43) is evident
(if $\sigma_1$ is large enough).
(41) and (42) may be deduced from (40) by partial summation (cf. (7)).

**A general lemma.** 15. In this section the main weapon of proof will be the following inequality of Turán:

Let $z_j (j = 1, 2, \ldots, n; n < N)$ be a set of complex numbers with $\max |z_j| \geq 1$ and let $m \geq 0$. Then there is a constant $c (0 < c < 24)$ such that for appropriate integer
\[ v \in \{ m+1, m+2 \} \] we have
\[ |z_1^m z_2^m \cdots z_n^m| > \left( \frac{N}{\sigma(m+n)} \right)^N. \]

By [21], Sata, this is true for $c = 96 \zeta^2$. An improvement of $c$ and a simplification of the proof are given in [20].

**Lemma 16.** Let $k, A, D, \lambda, \tau_0$ denote unbounded parameters, $\lambda$ being an integer $\geq 2$,
\[ A > 2/\theta_1, \quad D > 2, \quad A_k \geq D, \quad 0 < \epsilon < \lambda (\theta_1/4B) \log D \]
and $B > 1$ is a sufficiently large constant, $0 < \theta_0 \leq 1$, $D \leq \epsilon \leq \tau_0 < D$. Further let $F(s)$ be meromorphic in $\sigma > 1 - \epsilon \theta_0$ with simple poles $\nu = \beta + i \tau$ which lie in $\sigma < 1$. Denoting by $m_\nu$ the residue of $F(s)$ at $s = \nu$ we take for granted that for any real $\tau_0$
\[ |F(s) - \sum_{\nu_1 < \nu < \nu_2} \frac{m_\nu}{s - \nu} | < c^\epsilon \log D (1 + |\nu_1|) \quad (|s - \nu_1| < \theta_0/2) \]
and
\[ \sum_{\nu_1 < \nu < \nu_2} (1 + |m_\nu|) < r \log D (1 + |\nu_0|) \quad (\nu_0' \log D (1 + |\nu_0|) \leq r \leq \theta_0), \]
supposing that the constant in $<$ (as other constants during the proof) may depend on $\epsilon, \epsilon', \theta_0$. For any real $\tau$ let
\[ J(\tau, k, A) = - \frac{1}{2\pi i} \int_{s = \epsilon}^{s = \epsilon + i \theta_0} \frac{e^{\nu(s-\epsilon)} - e^{\nu(s+\epsilon)}}{2\pi i} F(1 + s + i \tau) \, ds \]
and let $D_0 \geq 1$ denote the square $(1 - E_k \log D \leq \leq \sigma \leq 1, |1 - \epsilon| \leq E_k \log D)$.

(i) If for all $\nu \in Q_0$ the residues $m_\nu$ are positive integers and if there is
at least one $\nu \in Q_1$, then for appropriate $C_1 \geq 1$ and for any $C \geq 1$, $\epsilon \leq \lambda (C + C_1 \lambda)$ such that for any $\tau$ of the interval $I [\tau - \epsilon, \tau + \epsilon] \leq \lambda (2\log D)$
\[ |J(\tau, k, \epsilon, \lambda - \log D)| > \exp (-C_1 \lambda), \]
where the constant $C_1$ may depend on $C, C_1$.

(ii) For all $\nu \in Q_2$ let the residues $m_\nu$ be positive integers except for at most $\leq E_k \log D (1 + |\nu|) \leq \sigma \leq \theta_0/3, \theta_0 + 1/3 \log D (1 + |\nu|)$ (with a suitable constant $C_1 \geq 1$) such that for any selected $\tau \leq D$ the distance between every point $s = \sigma + i \tau L$ and the nearest pole of $F(1 + s + i \tau L)$ exceeds $1/C_1 \log D (1 + |\nu|)$ (cf. [83], § 8) whence by (46), (47) on the line $F(1 + s + i \tau \epsilon < \log D (1 + |\nu|)$). Besides the length of the piece of $L$ between any two of its points $\sigma + \epsilon L, \sigma + \epsilon' (\epsilon + 1)$ does not exceed 2.

Writing
\[ g(s) = e^{\nu(s-\epsilon)} - e^{\nu(s+\epsilon)} \quad 2\pi i \]
and moving in (48) the path of integration to the line $L$ we deduce
\[ J(\tau, k, A) = - \frac{1}{2\pi i} \int_{s = \epsilon}^{s = \epsilon + i \theta_0} \frac{e^{\nu(s-\epsilon)} - e^{\nu(s+\epsilon)}}{2\pi i} F(1 + s + i \tau) \, ds, \]
where $G$ denotes the region on the right of $L$.

Since $k \geq 1, A_k \geq D$, $A > 2/\theta_1, \theta_1$ on $L$ we have by (50), $g(s)^k < D^{-1k} \log D (1 + |\nu|)^{-k}$ whence the integral in (51) is $< D^{-1k} \log D$.

Now let us suppose that there is a pole $\nu \in Q_0$ of $F(s)$ and let $\tau_0$ be any selected $\tau_1 L$. Suppose that for a fixed $C \geq C_1$ and for all integers $k \geq \max (2, C_0), C_1 + C_1 \lambda$ we have
\[ |J(\tau, k, \epsilon, \lambda - \log D)| \leq \exp (-B_k \lambda), \]
From this (49) will be proved by arriving at a contradiction.
First let us estimate that part $S_1$ (say) of the sum in (51) which corresponds to the region $G \cap C$ with $t \geq \tau_1 + E \log D$. The sum $\sum_{m_j}$ over all $m_j$ of the rectangle

$$R_0(1 - \frac{\varepsilon}{2} \leq \varepsilon < 1, \; \tau_0 + t_0 \leq t \leq \tau_0 + t_0 + E \log D)$$

by (47) does not exceed

$$< \left( \delta + \frac{E}{\log D} \right) \log D(1 + t_0).$$

Let $G'$ be the region in the strip $\tau + t_0 \leq t \leq \tau_0 + t_0 + E \log D$ between $\varepsilon = 1$ and the displacement $L'$ of the line $L$ by 1 in the direction of the positive real axis. The part of the sum in (51) (with $\tau = \tau_1$, $A = \lambda^{-1} \log D$) corresponding to $G'$ satisfies

$$\sum_{m_j} e^{-\frac{A}{D}(1 - \frac{\varepsilon}{2})}$$

$$< \left( \frac{A e^{-\frac{A}{D}}}{\log D} \right) \log D(1 + t_0)$$

Let $G'$ be the region in the strip $\tau + t_0 \leq t \leq \tau_0 + t_0 + E \log D$ between $\varepsilon = 1$ and the displacement $L'$ of the line $L$ by 1 in the direction of the positive real axis. The part of the sum in (51) (with $\tau = \tau_1$, $A = \lambda^{-1} \log D$) corresponding to $G'$ satisfies

$$\sum_{m_j} e^{-\frac{A}{D}(1 - \frac{\varepsilon}{2})}$$

$$< \left( \frac{A e^{-\frac{A}{D}}}{\log D} \right) \log D(1 + t_0)$$

$$< \left( \frac{A e^{-\frac{A}{D}}}{\log D} \right) \log D(1 + t_0)$$

$$< \frac{E}{\log D} \left( \frac{E}{\log D} \right)$$

Summing over all $t_n = nE/2\log D$ ($n = 1, 2, \ldots$) we get

$$S_1 < \frac{E}{\log D} \sum_{n=1}^{\infty} \frac{1}{(2\log D)^n}$$

$$< \frac{E}{\log D} \left( \frac{E}{2\log D} \right)$$

The same estimate holds for the part $S_1$ of the sum in (51) which corresponds to the region $G \cap C$ with $t \leq \tau_0 - E \log D$.

Let $G_t$ denote the remaining part of $G$ without $Q_R$. The corresponding part of the sum in (51) (with $\tau = \tau_1$, $A = \lambda^{-1} \log D$) satisfies

$$\ll \sum_{m_j} e^{-\frac{A}{D}(1 - \frac{\varepsilon}{2})}$$

This being smaller than the right-hand side of (33), we have by (31) and (32)

$$\sum_{m_j} m_j g(\varepsilon, -1 - \varepsilon) \ll e^{-\frac{B}{E} + E \left( \frac{1}{2} \log E \right)^{-2}}$$

whence, counting every $g \in Q_E$ in the sequel $m_j$ times, we obtain

$$\sum_{m_j} \frac{g(\varepsilon, -1 - \varepsilon)}{g(\varepsilon, 0 - 1 - \varepsilon)} < \sum_{m_j} \frac{g(\varepsilon, 0 - 1 - \varepsilon)}{g(\varepsilon, 0 - 1 - \varepsilon)} < e^{-\frac{B}{E} + E \left( \frac{1}{2} \log E \right)^{-2}}.$$

Let $R'$ denote the rectangle $(-1, 0) \leq t \leq \tau_0, || \leq \frac{A}{2} \log D$ and let $A = \lambda^{-1} \log D$. Writing $s_t = \eta - t_0 - t_0, s_t, s_t'$ we have

$$|g(s_t)| = e^{-1} \min_{m_j} \left| 1 - \frac{e^{\frac{A}{2} t_0}}{2A} \right| > e^{-1},$$

since on the boundary of $R'$ the function $(1 - e^{\frac{A}{2} t_0})/2A$ is in modulus $> e^{-1}$ and has no zeros $e^{R}$. Hence

$$|g(s_t)| > e^{-1},$$

denoting

$$s_t = s_t - s_t'$$

we have, by (54),

$$\sum_{m_j} g(s_t) \ll e^{-\frac{B}{E} + E \left( \frac{1}{2} \log E \right)^{-2}}.$$

By (47) there is a constant $C_1$ such that

$$N = C_1 E,$$

the number of the poles $g \in Q_E$ does not exceed $N$. Take

$$m = [C_1].$$

Let us apply Turán’s inequality (45) to the numbers $s_t, m, N$. By (44)
there is an integer \( r > C \lambda \), \( r < C \lambda + N \) for which (45) holds and further we use \( k = r \). Then by (57) and (58) the restrictions imposed on \( k \) in the present lemma are satisfied with \( C_1 = C \lambda E \). Observing that

\[
\frac{N}{m + N} \geq \frac{1}{1 + O(C_1)},
\]

we have, by (56), (45)

\[
\left( \frac{1}{e^{\left( 1 + C_1 \right)} \log \left( 1 + C_1 \right)} \right)^{-\mathcal{O}(1)} \leq e^{-\left( B - 10 \lambda - 10 \lambda^2 \right)/2} E \left( \frac{1}{2} e^\lambda \right)^{C_1}.
\]

The logarithms of the reciprocal values of these numbers are, respectively,

\[
\begin{align*}
U &= C_1 E \log (1 + C_1), \\
U_1 &= (B - 3C - 3C_1 E) \lambda, \\
U_2 &= C_1 \log E / 2e^\lambda - \log E \geq \frac{1}{2} C_1 \log E / 2e^\lambda
\end{align*}
\]

(supposing \( E \) large enough). Now we take \( B = 4C \). Then the orders of magnitude for \( U_1 \) and \( U_2 \) are largely that of \( U \) as \( E \to \infty \). Hence for a sufficiently large \( E < 1 \) the right-hand side of (59) is smaller than the left-hand side, which provides the desired contradiction.

16. Let us now consider case (ii) of the present lemma. By the same arguments as those used before we get instead of (56)

\[
\sum_{\sigma_0, \sigma_1 \leq 1} \sum_{k = 1 \atop \sigma_0, \sigma_1 \leq 1} \left( \frac{\sigma_0}{\sigma_1} \right)^k \leq e^{-B_1 + \log (1 + E) / 2}.
\]

The number of the points \( t_0 \in [0, 1] \) is at least one \( \tau_1 \in I \) such that the distance between \( \tau_1 \) and the nearest \( t_0 \) exceeds \( 1/C_1 \log D \) (for appropriate \( C_1 \ll 1 \)). Further let \( \tau_1 \) be any fixed point having this property.

By (55) and (59)

\[
\begin{align*}
g(t_0 - \tau_1) & = \frac{\rho(t_0 - \tau_1)}{\rho(\delta / \log D + i(t_0 - \tau_1))}, \\
\rho(t_0 - \tau_1) & = \frac{\rho(\delta / \log D + i(t_0 - \tau_1))}{\rho(\delta / \log D + i(\delta / \log D))}, \\
g(s) & = \frac{e^{\delta s} - e^{-\delta s}}{2\delta s} \text{ (} A = \delta^{-1} \log D). \end{align*}
\]

From the estimate of \( g(s) \) in the previous paragraph we can deduce

\[
\left| \frac{1}{\rho(\delta / \log D + i(\delta / \log D))} \right| < e^\delta.
\]

For any selected \( t_0 \) and arbitrary \( \delta \in (0, \delta] \) let

\[
\varepsilon = -\delta, \quad \varepsilon = 2A\delta, \quad t = 2A(t_0 - \tau_1).
\]

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\[
\begin{align*}
|g(s)| & \leq \frac{1 - e^{-\delta t_0}}{1 - e^{-\delta t_1}} \leq \\
& \leq \frac{\left( 1 - e^{-\delta t_1} \right) / 2 + 4e^\delta}{\left( 1 - e^{-\delta t_1} \right) / 2 + 4e^\delta} \leq \frac{\sin t_1 / e^{t_1}}{\sin t_1 / e^{t_1}} + \frac{2e}{\sin \left( 2t_1 \right)} \leq \frac{2e}{\sin \left( 2t_1 \right)} \leq 3,
\end{align*}
\]

\[
|g'(s)| = \left| \frac{1}{2} \left( e^{\delta s} - e^{-\delta s} \right) - \frac{e^{\delta s} - e^{-\delta s}}{2\delta s} \right| \leq \frac{2}{2\delta s} \lesssim \frac{1}{\left| t_0 - \tau_1 \right| \lesssim 3C_1 \log D},
\]

whence

\[
\sum_{\sigma_0, \sigma_1 \leq 1} \left( \frac{\sigma_0}{\sigma_1} \right)^k \leq e^{B_1 + k \lambda} \log D \leq k e^{B_1 + k \lambda},
\]

\[
\sum_{\sigma_0, \sigma_1 \leq 1} \left( \frac{\sigma_0}{\sigma_1} \right)^k \leq e^{B_1 + k \lambda} \log (1 + E) / 2 + 2eC_B e^{-B_1 \lambda}.
\]

Now taking \( B_1 = 7C + 3C_1 \) and using (60), we deduce

\[
\sum_{\sigma_0, \sigma_1 \leq 1} \left( \frac{\sigma_0}{\sigma_1} \right)^k \leq e^{-B_1 + k \lambda} + e^{B_1 + k \lambda} \log (1 + E) / 2 + 2eC_B e^{-B_1 \lambda}
\]

and may proceed as before: Taking the logarithms of the reciprocal numbers we obtain next to \( U_1, U_2 \), one more term \( U_3 = 2C_1 \log C > C_1 \) with a higher order of magnitude than that of \( U \), whence a contradiction.

A density lemma. 17. The constants of this paragraph do not depend on any constants used previously.

**Lemma 17.** Let \( k \) and \( A > 0 \) denote unbounded parameters, \( k \) being an even \( > 2 \), and let

\[
R(a) = \frac{1}{2\pi i} \int_{-10a}^{10a} \left( \frac{e^{\lambda x} - e^{-\lambda x}}{2\lambda x} \right)^k e^{-\lambda s} \lambda^s ds \text{ (} a \geq 1\).
\]

Then for appropriate constant \( c > 1 \)

\[
R(a) \leq e^{ck} / (\pi A), \quad \text{if} \quad e^{ck} \leq a < e^{ck^2},
\]

\[
R(a) = 0 \quad \text{otherwise}.
\]

**Proof.** Using the binomial expansion we get from the integrand

\[
\left( e^{\lambda x} - e^{-\lambda x} \right)^k e^{-\lambda s} = e^{\lambda x} e^{-\lambda s} \sum_{n=0}^{k} \binom{k}{n} \left( e^{\lambda x} - e^{-\lambda x} \right)^n e^{-\lambda s}
\]

a finite number of terms \( e^{\lambda x} e^{-\lambda s} \) (say). If \( a \geq e^{ck^2} \), then all the exponents \( k \) are \( \leq 0 \), whence \( R(a) = 0 \), since

\[
\int_{-10a}^{10a} e^{ck} e^{-\lambda s} ds = 0 \quad (k \leq 0).
\]

(4) Using more intricate arguments Turán (22), Lemma 1) gives a sharper estimate of \( R(a) \).
If \( a \leq a^\delta \), then all the exponents \( k \) are \( \geq 0 \) and we obtain the desired result by moving the path of integration to \( \sigma = -1 \) and using the identity
\[
\int_{c-i\infty}^{c+i\infty} e^{s x} \, ds = 0 \quad (x > 0).
\]
Now let \( \delta^\delta < a < e^\delta \). Then there are \( \leq (1+1)^k \leq e^\delta \) terms \( e^{\delta k} \) with \( k > 0 \); for these we move the contour of integration to \( \sigma = -1 \).

Since
\[
\text{Res}_{s=1} \frac{(2\pi)^{s-1}}{(k-1)! (2\pi)^{s-1}} \frac{1}{(k-1)! (2\pi)^{s-1}} < \frac{1}{2} \left( \frac{3}{k} \right)^{k^{1/2}} \frac{1}{2} \leq e^{\delta^2},
\]
(61) follows by the theorem on residues (the contribution of the terms with \( x < 0 \) being zero).

15. By (19) and (22) any of the functions \( F(s) = \zeta(s, \chi) \), with \( \chi \neq \chi_0 \), satisfies the conditions of Lemma 16(i). For any real \( \tau \) let
\[
J_2(x, k, A) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx} - e^{sx}}{2\pi x} \left( -e^{x/2} - e^{-x/2} \right) \frac{(e^{x/2} - e^{-x/2})}{2} \, ds,
\]
where \( A > 1 \) and \( k \) is an integer \( \geq 2 \).

**Lemma 18.** If \( c_1 \) is the constant of Lemma 15 and \( e^{\delta^2} > D^2 \), then
\[
\sum_{k} |J_2(x, k, A)|^2 < e^{\delta^2}.
\]

**Proof.** By the definition of \( R(a) \) with the aid of (62), (16), (41), (61) and
\[
\sum_{k} \chi(k) = \begin{cases} h & \text{if } H \text{ is the principal class } H_1, \\ 0 & \text{otherwise} \end{cases}
\]
(cf. [12], § 10), we deduce
\[
\sum_{k} |J_2(x, k, A)|^2 = \sum_{k} \sum_{\chi} \begin{cases} \chi(a) R(a) & \text{by (44)}, \\ \sum_{e, l} \chi(e) \chi(l) & \text{(say)}, \\ \sum_{e, l} \chi(e) \chi(l) & \text{by (47)}, \end{cases}
\]
\[
\sum_{k} \sum_{\chi} \chi(k) = \begin{cases} \delta & \text{if } k \text{ is principal,} \\ 0 & \text{otherwise} \end{cases}
\]
\[
\sum_{k} |J_2(x, k, A)|^2 < e^{\delta^2} A^{-1} \sum_{k} \sum_{e, l} \chi(e) \chi(l) \chi(k) < e^{\delta^2} A^{-1} \sum_{k} \sum_{e, l} \chi(e) \chi(l) \chi(k) < e^{\delta^2} A^{-1} \sum_{k} \sum_{e, l} \chi(e) \chi(l) \chi(k) < e^{\delta^2} A^{-1}.
\]

**Corollary.** For any selected numbers \( \tau, \lambda \) satisfying the conditions of Lemma 16 (where \( \chi' \) is now equal to the constant \( \chi \) of Lemma 11) let \( \nu = \nu(\tau, \lambda) \) be the number of the functions \( \zeta(s, \chi) \) (\( \chi \neq \chi_0 \)) having at least one zero in the square \( Q \) \( (1-\lambda)/\log D \leq \sigma \leq 1, \quad |t-\tau| \leq 2(\log D) \). Then for at least \( \nu \tau \lambda \) of them (49) holds with the same \( k = k_1 < c_1 \lambda \).

**Proof.** (49) and (63)
\[
|\nu(\tau, \lambda) e^{-\lambda^2/2}| < \sum_{\chi} |J_2(\tau, k_1, \chi, 1/\lambda \log D)|^2 < e^{\delta^2},
\]
whence
\[
\nu < e^{\delta^2}.
\]

From (65) we get the following lemma, analogous to the Linnik-Rodosski density lemma:

**Fundamental Lemma 19.** Let \( N_1 \) denote the number of the zeros of the function \( \zeta(s, \chi) \) in the rectangle \( R_1(2-\lambda)/\log D \leq \sigma \leq 1, \quad |t| \leq \lambda/\log D \). Then for appropriate \( c_1, c_2 \)
\[
N_1 < e^{\delta^2} \quad (c \leq \lambda \leq c_2 \log D).
\]

**Proof.** Considering that \( R_1 \) may be covered by \( \leq e^\delta \) squares \( Q \) (corresponding to \( \tau = n\lambda/\log D, \quad n = 0, \pm 1, \ldots \)), we deduce that the number of the functions \( \zeta(s, \chi) \) (\( \chi \neq \chi_0 \)) having at least one zero in \( R_1 \) is \( < e^{\delta+\lambda} \). It increases at most by unity if we remove the restriction \( \chi \neq \chi_0 \). Hence we get (66) observing that the number of zeros \( \zeta(s, \chi) \) for any function \( \zeta(s, \chi) \) is \( < \delta^2 \) by (22).

The lemma and the proof hold as well for rectangles \( (1-\lambda)/\log D \leq \sigma \leq 1, \quad |t-\tau| \leq \lambda/\log D \) with \( |t| \leq D \).

**On the real exceptional zero.** 19. In this section we suppose that some of the functions \( \zeta(s, \chi) \) have the real exceptional zero
\[
(67) \quad \beta' = 1 - \delta, \quad 0 < \delta < c_1/\log D
\]
with arbitrary small \( c_1 \). By Lemma 10 there is at most one such character. It is real, and in what follows it will be denoted by \( \chi' \).

**Lemma 20.** If \( \chi' = \chi_0 \), then for appropriate \( c_1 \) we have \( \delta > D^{-1} \).

**Proof.** Writing
\[
G(s) = \varphi(s) \zeta(s, \chi_0) \quad \text{where} \quad \varphi(s) = \begin{cases} \frac{s-1}{2} & \text{if } q = 1, \\ (1-q^{-s})/(1-q^{-1}) & \text{if } q > 1, \end{cases}
\]
with
we have, by (15), \( G(s) < D^\delta \) in \( |s-1| < r \) \((0 < r < \frac{1}{2} \theta)\). Hence in \( |s-1| < \frac{1}{r} \)

\[
G'(s) = \frac{1}{2\pi i} \int_{\sigma-z=r} \frac{G(w)}{(w-s)^2} dw < r^{-1} D^\delta.
\]

By \( \delta \alpha = \text{Res} \zeta(s, \chi_0) = \lim_{s \to 1} (s-1)G(s)/\varphi(s) = \delta G(1), \)

whence, by (3), \( G(1) = \delta \alpha > D^{-\delta} \), and thus \( D^{-\delta} < G(1)-G(1') = \delta G(1) (\delta < \delta < 1) \). Now using (68) (with \( r = \frac{1}{2} \theta \)) we obtain the required result.

**Corollary.** If \( h \) (the class number of \( \mathbb{Q} \)) is odd, then \( \delta > D^{-\delta} \).

Proof. In the case of an odd \( h \) the single real character being \( \chi_0 \), we have \( \chi' = \chi_0 \).

**Lemma 21.** If \( \delta > \frac{1}{2} \), then for appropriate \( c \) we have \( \delta > D^{-c} \).

Proof. We may suppose that \( \chi' \neq \chi_0 \) (otherwise we have the desired result by Lemma 20). The function

\[
g(a) = \sum_{d \leq a} \chi(d)
\]

has the multiplicative property \( g(aa') = g(a)g(a') \) whenever \((a, a') = 1\), and for any generator \( b \)

\[
g(b^k) \geq \begin{cases} 0 & \text{always,} \\ 1 & \text{if } b \text{ even} \end{cases}
\]

(cf. [5], p. 104), whence

\[
g(a^k) \geq 1.
\]

Further we have

\[
\zeta(s, \chi) = \sum_{a} g(a) a^{-s} \quad (s > 1)
\]

and thus for any \( y > 0 \)

\[
\sum_{a} g(a) e^{-a} = \frac{1}{2\pi i} \int_{1-i y}^{1+i y} e^{-s} T(s, \chi) \zeta(s, \chi_0) ds
\]

(cf. [17], p. 380). If \( y = D^{-B} \) (with a sufficiently large \( B < 1 \)), by (3) the left-hand side of (69) is

\[
\geq e^{-1} \sum_{a \leq D^B} \frac{1}{a} = e^{-1} \sum_{a \leq D^B} \frac{1}{a} > \frac{1}{2} e^{-1} \delta \alpha D^{B+1}.
\]

Moving the path of integration in (69) to the line \( L(\sigma = a, \chi) = \sigma = 1 - \delta + \log D \) we pass over a pole or poles of the integrand with the sum of residues

\[
\sum_{a \leq D^B} \left( 1 + \frac{2k\pi}{\log q} \right) \exp \left( -\frac{\pi}{2} \frac{2k\pi}{\log q} \right) < \zeta(1, \chi) D^{-\delta} \alpha
\]

since, by the asymptotic estimate of \( J(s) \) (cf. [17], p. 395) and by (4)

\[
\sum_{a \leq D^B} \left( 1 + \frac{2k\pi}{\log q} \right) \exp \left( -\frac{\pi}{2} \frac{2k\pi}{\log q} \right) < \log q < \log D.
\]

(For \( \log q \leq 1 \), the estimate of the exponential sum is evident. If \( \log q > 1 \), then writing \( 2x \log q = e \) we prove the estimate \( e^{-1} \) separately for each of the two parts of the sum corresponding to \( ek < e^s \) and \( ek > e^s \), respectively.)

By (15)

\[
\sum_{a \leq D^{B+1}} \left( 1 + \frac{2k\pi}{\log q} \right) < \log q < \log D.
\]

For a sufficiently large \( B < 1 \) this has a smaller order of magnitude than (70). Hence (71) is not less than a half of (70), whence

\[
\zeta(1, \chi) > D^{-\delta} \alpha.
\]

By the arguments of (68) we can prove that \( \zeta(s, \chi) < D^{\delta/2} (\chi \neq \chi_0) \) in \( |s-1| < \frac{1}{2} \delta \). This combined with (72) gives the desired estimate of \( \delta \).

**Lemma 22.** If \( \delta > D^{-\delta} \).

Proof. We may suppose (as in the previous proof) that \( \chi' \neq \chi_0 \). Writing \( f(a) = \sum_{a} \chi(a) a^{-s} \) we have

\[
f(a) = \zeta(\sigma, \chi') \quad \text{for all } a > 1.
\]

Using (3), we can prove by partial summation that the series \( f(a) \) converges uniformly in \( 1 - \delta + \epsilon < \sigma < 2 \) \((0 < \epsilon < \delta)\), whence \( f(a) \) is a continuous function \( (1) \) in that interval. Let in \( (8) \) \( E \) be the set of all the classes \( H \) with \( \chi'(H) = 1 \); then

\[
f(1) = \sum_{a \leq D^B} \chi(a) = \lim_{n \to \infty} \sum_{a \leq D^B} \chi(a) = \sum_{a \leq D^B} \chi(a) = 0.
\]

(1) By the arguments of (14), p. 157, one can prove that for any \( \chi \neq \chi_0 \) the series \( \sum \chi(a) a^{-s} \) converges uniformly in \( 1 - \delta + \epsilon < \sigma < 2 \), its sum being the function (13). However, in the present case we can do well with a weaker relation.
Considering that the regular function \( \zeta(s, x) \) is continuous at \( s = 1 \) and taking limits in (73) as \( \sigma \to 1 + 0 \) we deduce

\[
\zeta(1, x) = f(1).
\]

Hence by (74) and (9)\( ^4 \), \( \zeta(1, x) > D^{-\delta} \) and we may go on as in the previous lemma.

The results proved in the present paragraph may be summarized by the following

**Lemma 25.** Under such circumstances as stated in the theorem of §1 for appropriate \( c \) we have \( \delta > D^{-1} \).

**20. Lemma 24.** There are constants \( c, c' \) such that, if \( \delta < c' \log D \) and \( \gamma_0 = \beta_0 + i\gamma_0 \) is a zero in \( 1 - 2 \leq \sigma < 1, \left| \tau \right| \leq D \) of any function \( \zeta(s, x) \), different from the exceptional zeros of Lemma 11, then

\[
\beta_0 < 1 - \frac{\epsilon}{\log D} \log \frac{\epsilon c}{\delta \log D}.
\]

**Proof.** Let

\[
\beta_0 = 1 - \frac{\epsilon}{\log D}.
\]

It will be sufficient to prove the lemma merely for \( \lambda < c_1 \delta \log D \) with some small constant \( c_1 < 1/16 \). If, in fact, \( \lambda > c_1 \delta \log D \), then, since by Lemma 23, \( \delta > D^{-1} \), we have \( \log(\epsilon c/\delta \log D) < c_1 \log D \), whence (75) holds for any \( \epsilon < c_1 \delta / \lambda \).

In this paragraph we shall deal with the case \( x_0 \neq x' \). Let us consider that the function (\( ^7 \))

\[
\Phi(s) = \begin{cases} \zeta'(s, x) + \zeta'(s, x') & \text{if } x = x_0, \\ \zeta'(s, x) + \zeta'(s, x') & \text{if } x \neq x_0 \end{cases}
\]

(\( ^7 \) In order to show the necessity of (0) (or some equivalent condition) let us consider that \( \zeta(s, x) \) has no zeros in \( \sigma > 1 \) and \( \zeta(s, x) \to 1 \) as \( \sigma \to \infty \). Hence for any real \( x \) we have \( \zeta(1, x) > 0 \) and thus, by (74), \( \gamma_0 > 0 \). If \( \gamma_0 \) or \( \zeta(1, x) \) vanishes, then in some of the classes \( A \) there may be no generators, which does not suit our purpose.

The situation will be shown more clearly at the end of this paper. For the present let us quote to quote the following example from [11], p. 617: If \( \Theta \) denotes the semi-group of natural integers \( n = p_1 p_2 \ldots \) (all \( p_i \) distinct primes) divided into the classes \( H_1, H_2 \) according as \( \sum a_i \) is even or odd, then \( \zeta(1, x) = \zeta(1/2)z(x) \) and thus \( \zeta(1, x) = 0 \); i.e., there are no generators in \( H_1 \).

(\( ^7 \) Using some additional arguments we can base the proof exclusively on the second form of \( \Phi(s) \) in (76). In that case \( \gamma_0 \) denotes any pole of \( \Phi(s) \) subject to the same restrictions (cf. [13], § 2 or [18], pp. 342-343).
being multiplicative and \( \geq 0 \), we have

\[
\sum_{a \in \mathbb{Z}^+} \frac{g(a)}{a} \sum_{d \mid a, d > B} \frac{1}{d} \leq \sum_{a \in \mathbb{Z}^+} \frac{g(a)}{a} \sum_{d \mid a, d > B} \frac{1}{d} \leq \sum_{a \in \mathbb{Z}^+} \frac{g(a)}{a}.
\]

Writing

\[
U_1 = \sum_{a > B} \frac{g(a)}{a} \exp(-CD^{-2}a) \left( \exp(-\frac{1}{2}D^{-N}a) - \exp(-\frac{1}{2}D^{-N}a) \right),
\]

\[
U_3 = \sum_{a > B} \frac{g(a)}{a} \exp(-\frac{1}{2}D^{-N}a) \left( \exp(-\frac{1}{2}D^{-N}a) - \exp(-\frac{1}{2}D^{-N}a) \right).
\]

we have

\[
U_1 = \sum_{a > B} \frac{g(a)}{a} e^{-\frac{1}{2}D^{-N}a} (1 - e^{-\frac{1}{2}D^{-N}a}) \geq \frac{1}{8} \sum_{a > B} \frac{g(a)}{a},
\]

\[
U_3 \geq 0.
\]

Further we use the identity

\[
\sum_{a} \frac{g(a)}{a} = \frac{1}{2\pi i} \int_{\gamma-\log y}^{\gamma+\log y} y^{1-s} \Gamma(s-1) \zeta(s, \chi) \zeta(s, \chi) ds
\]

(cf. [17], p. 380) with \( y = D^{-1/2}, y = D^{-1} \) and move the path of integration to the line \( \sigma = 1 - \frac{1}{2} + 1/\log D \). The integrand has a double pole at \( s = 1 \) with the residue

\[
E + (c - \log y) \mu
\]

where

\[
\mu = \zeta(1, \chi) \Gamma(\sigma) - \lim_{s \to 1} (\Gamma(s) - e^{-1}),
\]

\[
E = \lim_{s \to 1} \zeta(s, \chi) \zeta(s, \chi) - \mu (s-1)^{-1},
\]

and, if \( q \geq 1 \), there are simple poles at \( s = 1 + \frac{2k \pi i}{\log q} \) \( (k = \pm 1, \pm 2, \ldots) \) with residues

\[
y^{2k \pi i/\log q} \Gamma \left( \frac{2k \pi i}{\log q} \right).
\]

We have

\[
\sum_{a \in \mathbb{Z}^+} \left| \frac{\Gamma \left( \frac{2k \pi i}{\log q} \right)}{a} \right| \leq \sum_{a \in \mathbb{Z}^+} \left( \frac{2k \pi i}{\log q} \right)^{-1/2} \exp \left( -\frac{\pi k \pi i}{2 \log q} \right) \ll (\log q)^{1/2} \sum_{k = \log q} \exp \left( -\frac{\pi k \pi i}{2 \log q} \right) \ll \log q.
\]

From these estimates we deduce by subtraction

\[
U_1 + U_3 = \mu \left( 3B \log D + O(\log q) \right) + O(D^{-\frac{1}{2}+\varepsilon_1}).
\]

By \( \S 9 \), \( \mu > D^{-\varepsilon_1} \). Hence, if \( B \) is large enough, then the remaining term in (82) is in modulus less than a third of the principal term. And, since \( U_3 \geq 0 \), we get the inequality

\[
\mu \left( 4B \log D + c_{\varepsilon_1} \log q \right) > U_1,
\]

whence, by (81), (80) and (78),

\[
\mu > c_{\varepsilon_1} \log D \sum_{a \in \mathbb{Z}^+} \frac{g(a)}{a} \sum_{a \in \mathbb{Z}^+} \frac{1}{a} \log D \sum_{a \in \mathbb{Z}^+} \frac{g(a)}{a}.
\]

By (79)

\[
\sum_{a} g(a)a^{-\rho} \exp(-aD^{-1/2}) \exp(-aD^{-1/2}) = \frac{1}{2\pi i} \int_{-\log q}^{2\pi i} \Gamma(s) \zeta(s, \chi) \zeta(s, \chi) \zeta(s, \chi) ds
\]

(cf. [17], p. 380). Moving the path of integration to the line \( \sigma = 1 - \frac{1}{2} + 1/\log D \) we infer that the right-hand side of (84) is

\[
D \log \mu \left( \Gamma(\delta) + \sum_{k = 1}^\infty \sum_{k = \log q} \Gamma \left( \delta + \frac{2k \pi i}{\log q} \right) \right) + R, \quad R \leq D^{-\frac{1}{2}+\varepsilon_1}.
\]

Observing that, by (4), (67), \( \delta \) is of lower order of magnitude than \( 1/\log q \), we have \( \sum R \ll \log q \), whence the expression in brackets is equivalent to \( \Gamma(\delta) \). For a sufficiently large \( B < 1 \) the remaining term \( R \) is in modulus \( \ll 1 \) and thus the principal term exceeds \( \frac{1}{2} \) (since the left-hand side of (84) is \( > 1 \)). Obviously

\[
\sum_{a} g(a)a^{-\rho} \exp(-aD^{-1/2}) - R < c_{\varepsilon_1} \sum_{a \in \mathbb{Z}^+} g(a) a^{-1}
\]

and thus

\[
\mu/a < c_{\varepsilon_1} \sum_{a \in \mathbb{Z}^+} g(a)/a
\]

Hence, by (83),

\[
c_{\varepsilon_1} \sum_{a \in \mathbb{Z}^+} g(a)/a > \mu/a > \delta \log D \sum_{a \in \mathbb{Z}^+} g(a)/a.
\]
whence \( c_3 > e^{-\phi_4} |\log D| \). This proves the lemma in the case of \( \lambda_0 \neq \lambda_0' \).

21. Now let us suppose that \( \lambda_0 = \lambda_0' \). Then instead of (76) we use the function \( F(s) = \zeta(s) / \zeta(s, \chi) \). Arguing as before (but in the case of \( \chi = \lambda_0 \) using Lemma 16 (ii)) we get the inequality

\[
\sum_{p^h < c_3 \leq D^h} 1 / b > e^{-\phi_4}.
\]

Hence

\[
\sum_{p^h < c_3 \leq D^h} 1 / u > e^{-\phi_4},
\]

where \( u \) denotes those numbers of \( \mathfrak{G} \) the generators of which are all \( > D^h \). We have

\[
\sum_{s < D^h} \sum_{p^h < c_3 \leq D^h} \frac{1}{a} \leq \sum_{s < D^h} \sum_{p^h < c_3 \leq D^h} \frac{1}{a).
\]

Write

\[
U_1 = \sum_{s < D^h} a_{-1} \left( \exp(-aD^{-\delta}) - \exp(-aD^{-\lambda_0}) \right),
\]

\[
U_2 = \sum_{s < D^h} a_{-1} \left( \exp(-aD^{-\delta}) - \exp(-aD^{-\lambda_0}) \right).
\]

Then

\[
U_1 > \frac{1}{\delta} \sum_{p^h < c_3 \leq D^h} a_{-1}, \quad U_2 \geq 0
\]

(cf. (81)) and

\[
U_1 + U_2 = \mu_4 (3B \log D + O(\log |\log D|)) + O(D^{-\delta_0 + \epsilon_1}),
\]

where

\[
\mu_4 = \text{Res}_{s=1} \zeta(s, \lambda_0) > D^{-\delta_0}.
\]

For a large \( B < 1 \) the remaining term being unimportant, we deduce

\[
\mu_4 (3B \log D + O(\log |\log D|)) > U_1,
\]

whence, by (87), (86), (85)

\[
\mu_4 > \frac{c_4}{\log D} \sum_{p^h < c_3 \leq D^h} \frac{1}{a} > \frac{c_4}{\log D} \sum_{a < D^h} \frac{1}{a} \sum_{p^h < c_3 \leq D^h} \frac{1}{a} \cdot \sum_{s < D^h} \frac{1}{a}.
\]

By the arguments of (84)

\[
\sum_a a^{-\phi} \exp\left( -aD^{-\lambda_0} \right) = \frac{1}{2\pi i} \int_{1-\epsilon_4}^{1+\epsilon_4} D^{s\lambda_0 - \phi} \Gamma(s) G(s, \chi) ds
\]

\[
= D^{s\lambda_0} \mu_4 \left( \int_{1}^{\infty} (1 + \delta \log q) + R, \quad R < D^{-\lambda_0 + \epsilon_4},
\]

\[
\sum_a a^{-\phi} \exp(-aD^{-\lambda_0}) - R < c_3 \sum_{a < D^h} a^{-1}.
\]

Hence, by (89), (88)

\[
c_4 \sum_{s < D^h} \frac{1}{a} > \frac{c_4}{\delta \log D} \sum_{a < D^h} \frac{1}{a}
\]

and we may go on as in § 20.

A simple consequence of Lemma 24 (cf. [6], pp. 146, 147) is the following

**Fundamental Lemma 25.** Let \( \delta \) be defined by (67). For appropriate constant \( A \) and

\[
\delta_0 = \begin{cases} \frac{A}{\log D} & \text{if } \delta \leq A / \log D, \\ \frac{A / \log D}{\delta \log D} & \text{otherwise (4)}, \end{cases}
\]

\[
\lambda_0 = A \log \frac{\varepsilon A}{\delta \log D} - \zeta(A, \chi|\log D|
\]

there are in \( (1 - \lambda_0 \log D \leq \sigma \leq 1, |\tau| \leq D) \) no zeros other than those of the function \( \tilde{\zeta}(s, \chi) \) than at most the exceptional ones of Lemma 11.

**Proof of the theorem.** 22. Let \( L_2 \) denote a broken line in the strip \( 1 - \frac{1}{2} \delta - 1 / |\log D| (1 + \delta / |\log D|) < \sigma < 1 - \frac{1}{2} \delta (|\log D| = 1) \) such that (i) for any \( s = \sigma + it \in L_2 \) we have \( \zeta'(s)/\zeta(s, \chi) < \log^3 D (1 + \delta / |\log D|) \) and (ii) that the length of the piece of \( L_2 \) between any two of its points \( s + it, s' + it + 1 \) is < 2 (cf. § 12). Write

\[
g = \frac{1}{2} \delta \leq \frac{1}{2}, \quad c_1 = 1 - g.
\]

From the identity

\[
\sum_{a} A(a) \frac{f(a)}{a^{1-\delta}} \exp\left( -\frac{1}{4y} \log^2 a / |a| \right) = \frac{\sqrt{\pi}}{2y} \int_{2^2x}^{1+\epsilon_2} \sum_{a} f(a) a^{2-\delta} \exp(-\phi_3 a) \exp(-\phi_3 a) ds
\]

\[
(s > 1, y > 0)
\]

(4) That is to say, when the exceptional zero \( \beta' \) does not exist.
Let $I_B$ denote the integration $B$ times repeated with respect to $\eta$, the range of integration being $(\eta, \eta + 1)$. By (93), (92)

$$I_B \Phi \leq \frac{2}{y \log^2D} \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

$$= \frac{2}{y \log^2D} \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

$$\leq \frac{2}{y \log^2D} \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

where $\eta, \eta'$ runs through the zeros of $\zeta(s, \chi)$ to the right of the line $L_0$. If $1 < \eta < \log D$, then the last term does not exceed

$$\leq \frac{2}{\log^2D} \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

where $\eta, \eta'$ runs through the zeros of $\zeta(s, \chi)$ to the right of the line $L_0$. If $1 < \eta < \log D$, then the last term does not exceed

$$\leq \frac{2}{\log^2D} \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

Let $\eta$ and $\eta'$ be the exceptional zeros of the functions $\zeta(s, \chi)$, $\zeta(s, \chi')$ with real characters $\chi, \chi'$ (see Lemma 11) and let $\eta$ be a typical zero of $\prod_{\zeta(s, \chi)}$. Further we use the following notation:

$$\delta' = 1 - \delta', \quad \delta'' = 1 - \delta''', \quad \eta = 1 - \delta + \gamma\eta$$

$$E' = \begin{cases} 1 & \text{if } \delta' \text{ exists,} \\ 0 & \text{otherwise} \end{cases}, \quad E'' = \begin{cases} 1 & \text{if } \delta'' \text{ exists,} \\ 0 & \text{otherwise} \end{cases}$$

$$S = 1 + \delta, \quad S' = 1 + \delta', \quad S'' = 1 + \delta''$$

In the last sum, $\eta$ runs over the regular (or non-exceptional) zeros of $\zeta(s, \chi)$ in the region $G_2$ (say) to the right of the line $L_0$.

Let $\Phi = \Phi(s, y, H)$ denote the left-hand side of (90). For $y > 1$ using $s = s' = s'' + \gamma\eta$ (an integer $\geq 0$) we have by (91), (92)

$$\Phi = 2Vy \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

$$\leq 2Vy \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

$$\left( S' - S'' \right) + \left( \frac{y}{\log^2D} \right)$$

$$\leq 2Vy \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

$$\leq 2Vy \sum_{\eta, \eta' \in G} \exp \left( -\frac{1}{2\eta} \frac{\log \eta}{\log D} \right)$$

$$\left( S' - S'' + \gamma\eta \log D \right)$$

Write $s = s' + \gamma\eta\log D$, $y = \eta\log D$ ($\eta \geq \eta_0 > 2$).
23. First let us suppose that \( E'' = 0 \) and \( E' = 1 \). Then, by (91),

\[
S = 1 - \chi'(H) \frac{a^{-\alpha' - \beta'} - 1}{z \log q} + \epsilon \sum_{k \leq \eta} e^{\frac{2 \pi k}{\log q} \log \frac{1}{z \log q} - \chi'(H) 2 \frac{2 \pi k}{\log q} \log \frac{1}{z \log q}},
\]

Denoting by \( V_1, V_2 \) the parts of the last sum with \( |k| \leq \log q \) and \( |k| > \log q \), respectively, (11), we have for a sufficiently large \( \eta_0 \leq 1 \)

\[
|V_1| < \delta \sum_{k > \log q} \frac{2 \pi k}{\log q} \log D < \frac{4 \delta e^{-\eta_0 \log D \log q}}{1 - e^{-\eta_0 \log D \log q}}
\]

(\( \delta > D^{-4} \), by Lemma 23), whence

\[
|I_2 V_2| < \frac{1}{2} (1 - e^{-\eta_0 \log D})
\]

(2) Performing the operation \( I_2 V_2 \) for any selected \( k \) (0 \( \leq k \leq \eta_0 \)) we get 2\( \eta \) differences, the typical first term being

\[
I_2 V_2 = \frac{\eta}{4 g \log q \log D} \sum_{k \leq \log q} \frac{2 \pi k}{\log q} \log D
\]

Subtracting the corresponding second term we get a quantity of the same order of magnitude multiplied by a factor

\[
\frac{4 \delta e^{-\eta_0 \log D \log q}}{2 g \log q} < 1 - e^{-\eta_0 \log D \log q}
\]

(say). We may suppose that \( B \geq 2 \). Then, if \( \chi'(H) = 1 \),

\[
|I_2 S| > \frac{1}{2} (1 - e^{-\eta_0 \log D})
\]

(Another proof of (99) will be given at the end of \( \S 27 \). If \( \chi'(H) = -1 \), then instead of (97) we use \( |I_2 V_1| < \frac{1}{4} \delta \). Hence, by (96) and (97)

\[
|I_2 S| > \frac{1}{2} (1 - e^{-\eta_0 \log D})
\]

Now let us denote the right-hand side of (96) by \( 1 + U_1 \). If \( E'' = 1 \), then by (91)

\[
S = 1 + U_1 + \epsilon e^{\log q \eta_0} U_1,
\]

where \( U_1 \) stands for the appropriate term. Consider that

\[
\epsilon e^{\log q \eta_0} = \begin{cases} 1 & \text{if } \eta = q^{\alpha} \\ -1 & \text{if } \eta = q^{\alpha + 1} \end{cases}
\]

(as integer). By what has been proved

\[
|\text{re} I_2 (1 + U_1)| > \frac{1}{2} (1 - e^{-\eta_0 \log D})
\]

(\( \eta_0 \) and \( \eta \) are large enough. Now imposing the restriction that \( \eta \) runs merely over the powers \( \eta = q^\alpha \) with a properly fixed parity of \( \eta \) we find that the real parts of \( I_2 (1 + U_1) \) and \( I_2 e^{-2 \pi it \log D} U_1 \) agree in sign. Then, by (100),

\[
|\text{re} I_2 S| > \frac{1}{2} (1 - e^{-\eta_0 \log D})
\]

whence (99) again.

If \( \beta' \) does not exist, then a stronger inequality holds. Using the number \( \delta \) of Lemma 25 we have in any case

\[
|I_2 S| > \frac{1}{2} (1 - e^{-\eta_0 \log D}) > \frac{1}{2} (1 - e^{-2 \pi it \log D}) \geq \frac{g A \log D}{2 A}
\]

(supposing \( \eta_0 > 2 |A| \)). If \( \eta_0 \) is large enough, then \( T_1 \) taken together with the remaining term of (94) is in modulus \( < \frac{1}{8} g (\eta_0 |A|) \log D \) (since \( \delta_0 > D^{-4} \), by Lemmas 25 and 23) and so is \( T_1 + T_2 \):

\[
T_1 + T_2 < \frac{1}{2} g (\eta_0 |A|) \log D
\]

Hence, if \( U \) denotes the left-hand side of (94),

\[
U > \frac{1}{8} g (\eta_0 |A|) \log D.
\]

24. Now we introduce the number

\[
s = xe^{2 \pi i \gamma} \in A^{1+\epsilon}
\]

and divide the sum \( \Phi \) on the left of (90) into four sums:

\[
\Phi = S_1 + \sum_{\eta \leq \log q} \sum_{\eta \leq |a|} \frac{1}{\eta} \log \frac{1}{\eta} \sum_{a < \log q} \exp \left( \frac{1}{2 \eta} \log \frac{1}{\eta} \right) + S_2 + \ldots
\]
where
\[ S_0 = h \sum_{x \geq M \text{ odd}} \frac{f(x)}{x} \left( -\frac{\log^2 b/x}{4y} \right), \quad S_1 = h \sum_{x \geq M \text{ odd}} \frac{A(a)}{a^n} \exp \left( -\frac{\log^2 a/x}{4yt} \right), \]
\[ S_2 = h \sum_{x \geq M \text{ odd}} \frac{A(a)}{a^n} \exp \left( -\frac{\log^2 a/x}{4yt} \right). \]
By (43)
\[ 6S_1 < \int \frac{\log f(x)}{x} \left( -\frac{\log^2 t/x}{4y} - \frac{\log t/x}{2yt} \right) dt \]
\[ = \int \exp \left( -\frac{\log^2 t/x}{4y} + \frac{\log t/x}{2yt} - \frac{\log t}{t} \right) dt \]
\[ < 2 \int \exp \left( -\frac{\log^2 t/x}{4y} + \frac{\log t/x}{2yt} - \frac{\log t}{t} \right) dt \]
\[ = 2 \exp \left( -\frac{\log^2 x}{4y} + \frac{\log x}{2yt} \right) = 2e^{-\frac{\log x}{4y}} = 2e^{-\frac{1}{4y}}. \]
From (103), (3) and (42) we deduce
\[ S_0 < \sum_{x \geq M \text{ even}} f(x) = \sum_{x \geq M \text{ odd}} + \sum_{x \geq M \text{ even}} < aD^s + \theta^{-1} - \theta^s \]
\[ < 6^{-1} \theta^{-1} \log^2 x. \]
Observing that \( x = 1 - 2\theta \), we have for any selected \( \theta \)
\[ \sum_{x \geq M \text{ odd}} \frac{\log f(x)}{x} \left( -\frac{\log^2 b/x}{4y} \right) < \sum_{x \geq M \text{ odd}} \theta^{-1} - \theta^s \]
\[ < 1 \int \frac{\log f(x)}{x} \left( -\frac{\log^2 t/x}{4y} - \frac{\log t}{t} \right) dt + \theta^s < \theta^s, \]
whence, if \( \eta_s < 1 \) is large enough,
\[ \eta_s < \theta^{-1} \log^2 x. \]
Now by (102), (104), (105), (106)
\[ \eta_s \sum_{x \geq M \text{ odd}} \frac{f(x)}{x} \left( -\frac{\log^2 b/x}{4y} \right) = \Phi \left( -S_0 + S_1 + \eta_s S_2 \right) \]
\[ > \Phi - \theta^{-1} (c_0 \theta^2 \log^2 + c_0 \theta \log^2 - \theta^s - \theta^s) \]
\[ > \Phi - 2 \eta_s \theta^{-1} \theta^2 \log^2 (c_0 \theta^2 \log^2 + c_0 \theta \log^2 - \theta^s - \theta^s) \theta^{-1}. \]

Hence, by (101)
\[ (107) \quad \frac{1}{2} \sum_{\substack{x > 1 \text{ odd} \ \theta x \text{ even} \ \theta^s \text{ odd}}} \frac{\log f(x)}{x} \left( -\frac{\log^2 b/x}{4y} \right) \]
\[ > \frac{1}{2} \sum_{\substack{x > 1 \text{ odd} \ \theta x \text{ even} \ \theta^s \text{ odd}}} \frac{\log f(x)}{x} \left( -\frac{\log^2 b/x}{4y} \right) \]
\[ > \Phi - 2 \eta_s \theta^{-1} \theta^2 \log^2 (c_0 \theta^2 \log^2 + c_0 \theta \log^2 - \theta^s - \theta^s) \theta^{-1}. \]
if \( \eta_s \eta_s \) is large enough. Since \( x < \theta^{D^{(2)} + \theta} \), the theorem follows for \( q = 1 \).
If \( q > 1 \), then from (107) we can deduce the existence of a generator \( \theta \) of \( (x, \theta) \) in the interval \( (\theta, \theta) \) (\( \theta < \theta^2 + \theta \)) merely for all \( x \equiv \theta \) with a positive restriction that \( n \) runs over the integers with a fixed parity (cf. § 23). But then increasing \( n \) at most by \( \log q/\log D \) we get the theorem for all \( x \).
Considering that for any \( x > D^s \) (with a sufficiently large \( \theta \)) the left-hand side of (107) does not exceed
\[ \pi(x, \theta) D^s \log x, \]
we deduce (10).

**Improvement of the theorem for \( q < 1 \), \( x \to \infty \).** By the use of some additional conditions the theorem of § 1 may be improved as follows.

For arbitrary small \( \epsilon, \theta ' \) and all \( D > D_0(\epsilon) \), \( x \to \infty \), let
\[ (108) \quad \Phi > \theta^{-1} \epsilon, \]
\[ (109) \quad \sum_{\substack{x > 1 \text{ odd} \ \theta x \text{ even} \ \theta^s \text{ odd}}} \frac{\log f(x)}{x} \left( -\frac{\log^2 b/x}{4y} \right) \]
and let (9) hold for any \( \theta > 0, D > D_0(\epsilon) \). If \( q < 1 \), then there are constants \( c, c' \) (depending on \( \theta, c_1, 1, \epsilon, \theta, q \) such that for any positive \( x \leq \theta (\text{if } q = 1) \), \( x \geq \theta (\text{if } q > 1) \)) and all \( x > x_0 = D^2 \log^2 (x, \theta) \) there is a generator \( \theta b \) at \( \theta b \) in the interval \( (\theta, \theta) \). Then if \( x > x_0 D^s \), we have \( \pi(x, \theta) > \theta \theta^s \log x \).

We begin by proving that with regard to the real exceptional zero \( \beta' = 1 - \delta' \) the additional conditions imply the estimate
\[ (110) \quad \delta' > \theta^{-1} \epsilon, \quad (D > D_0(\epsilon)) \]
where \( x_0 \) (and \( x_0 \) in the sequel) stands for an arbitrarily small positive constant.
Let first \( \epsilon' > x_0 \). Then by (9)
\[ (111) \quad \xi(1, \epsilon') > 2 \theta^{-1} \epsilon, \]
(21) If \( q > 1 \) and \( \epsilon' \) is large enough, then taking \( x < \theta/2 \) we get \( \xi > x_0 \), which is necessary in (114).
(cf. the proof of Lemma 22). By (109)

\[ (112) \quad \zeta(1 + 2s_1, \xi) \leq c_1(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}} \, ds \leq c_1(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}}, \]

which implies

\[ (113) \quad \left| \zeta(1 + 2s_1 + \iota \chi') \right| \leq c_1(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}}. \]

Since by (12)

\[ \zeta(1 - \theta/2 + \iota \chi', \chi') \ll D^2(1 + |t|), \]

using (113) and (5), Lemma 3, we deduce

\[ |G(s)| \leq c_2(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}}, \]

whence in \(|s-1| \leq \xi_1\),

\[ |G'(s)| \leq c_3(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}}. \]

(cf. (88)). Now using (111) and arguing as in Lemma 20 we get (110) for \(\chi' \neq \chi_0\).

If \(\chi' = \chi_0\), then we use the function \(G(s)\) of §19, which is regular in \(s > 1 - \theta\) and, by (13), (112), satisfies

\[ |G(1 + 2s_1 + \iota \chi)| \leq c_0(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}}, \quad G(1 - \theta/2 + \iota \chi) \ll D^2(1 + |t|). \]

By [5], Lemma 3,

\[ |G(s)| \leq c_2(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}}, \]

whence in \(|s-1| \leq \xi_1\),

\[ |G'(s)| \leq c_3(\xi)\sum_{2}^{\infty} \frac{x^{1+s_1}}{x^{1+s_2}}. \]

and observing that by (108) \(G(1) > D^{-\theta}\), we may go on as in Lemma 20.

26. Now let us choose an integer \(B \geq \xi_1 + 2, \xi_1\) being the constant of Lemma 19, and let

\[ x = D^s, \quad \xi > 0; \]

\[ y = \eta x^{-1} \log D, \quad \eta_0 \leq \eta \leq \eta_0 + B < 1, \quad 1 \leq s = r(\xi) \leq \min(\varepsilon_0, \log D) \]

where \(\eta_0, C \ll 1\) are large enough. Then the remaining term in (94) is \(\ll D^{-\eta_0}\) (cf. [4], p. 301). By the arguments used in §22 (but now denoting by \(G_1\) the part \(|t| > \log D\) of \(G\), and changing \(G_1, G_2\) accordingly) we deduce that

\[ I_{B} \ll e^{-\eta_0 \log \eta_0 - y \log y} D^{-\eta_0} \]

(cf. [4], pp. 302, 303). Supposing \(\eta \leq 1\), in the next paragraph we shall prove that for a sufficiently large \(\eta_0 \leq 1\) and for all \(\xi \geq \eta_0\)

\[ (114) \quad \frac{1}{2}(1 - e^{-\eta_0}) \]

whence (cf. [4], pp. 304, 305)

\[ \frac{1}{6} \log \frac{D}{4\pi}, \quad \eta_0 \geq \frac{1}{6} \log \frac{D}{4\pi} \frac{12A \eta}{4\pi} \]

Using the number \(x = \eta x^{-1} \log D\), we divide the sum (99) into four partial sums as in §24. By the arguments used there

\[ S_2 + S_3 + hS_2 < c \eta x^{-1} \log D \]

whence

\[ I_B \frac{1}{21} \frac{1}{y} \sum_{x < n < y} \frac{\log n}{1 - \psi(n) / \psi(y)} \exp \left( \frac{-1}{4y} \log^2 \frac{\psi(n)}{\psi(y)} \right) \geq \frac{1}{21} \frac{1}{y} \sum_{x < n < y} \frac{\log n}{1 - \psi(n) / \psi(y)} \exp \left( \frac{-1}{4y} \log^2 \frac{\psi(n)}{\psi(y)} \right) \geq \frac{1}{12} \log \frac{D}{4\pi} \frac{12A \eta}{4\pi} \]

and going on as in [4], pp. 308, 309, we prove the required results.

27. In this paragraph we shall be concerned with the proof of (114) when \(\eta_0 > 1\), \(\psi' = 0\), \(\psi' = 1\) and \(\chi'(H) = 1\) (the other cases may be treated as in §23). Then by (91)

\[ S = U_1 - U_2, \]

where

\[ U_1 = 1 + \sum_{x \leq n < y} \frac{1}{1 - \psi(n) / \psi(y)} \]

\[ U_2 = 1 + \sum_{x \leq n < y} \frac{1}{1 - \psi(n) / \psi(y)} \]

are evidently real. Observing that the function

\[ \varphi(t) = \frac{1}{1 - \psi(t) / \psi(y)} \psi(x) \]

where

\[ \varphi(t) = \log q \frac{\psi(t) / \psi(y)}{1 - \psi(t) / \psi(y)} \]

for \(\psi(t) / \psi(y) > 1\)
is regular in the whole plane, except for simple poles at \( s = 1 + 2k\pi i / \log q \) (\( k = 0, 1, 2, \ldots \)) with residue 1, we have, by (115)

\[
(116) \quad \log q \sum_{n=g}^{m} \frac{q^n - q^{n(\frac{1}{2} - \delta)}}{q^n} \exp \left( - \frac{1}{4g} \log^2 q^s / \sigma \right)
\]

\[
= -i \sqrt{\frac{2}{\pi}} \int_{1 - \Delta}^{1 + \Delta} \left( \psi(s) - \psi(s + \delta') \right) \frac{d^n \theta}{(2\pi)^n} d \delta'
\]

\[
= 2v \sqrt{\sigma} \psi (U_1 - U_2) - i \sqrt{\frac{2}{\pi}} \int_{1 - \Delta}^{1 + \Delta} \left( \psi(s) - \psi(s + \delta') \right) \frac{d^n \theta}{(2\pi)^n} d \delta'.
\]

Since by (17) on the line \( \sigma = c_1 \) we have \( \psi(s) - \psi(s + \delta') \ll \delta' \), the last term is \( \ll \). For \( \sigma = q^\nu \) (with \( \nu > 0 \)), the remainder after \( q^n \) terms of the series in (116) satisfies

\[
(117) \quad \sum_{n=q^n}^{m} (1 - \log q) \left( \psi(q^n) - \psi(q^n + \delta') \right) \ll \frac{1}{\log q}.
\]

The general term \( \psi(q^n) - \psi(q^n + \delta') \) in brackets attains its maxima at \( q = q_1 = \left[ \frac{2\nu}{\log q} \right] \) or \( \left[ \frac{2\nu}{\log q} \right] + 1 \) (or both). Therefore

\[
\sum_{n=q^n}^{m} \psi(q^n) \leq \frac{\nu}{\log q} \sum_{n=q^n}^{m} \psi(q^n) = \frac{1}{\log q} \sum_{n=q^n}^{m} \psi(q^n).
\]

for \( q < 1 \), and a sufficiently large \( \eta_q = \eta_q(q) < 1 \) the sum of that series is

\[
\geq 2 \frac{1}{\log q} \psi(q).
\]

Hence, by (116), (115)

\[
S > 2 \frac{1}{\log q} \left( 1 - \log q \right) > \frac{1}{2} \left( 1 - \log q \right)
\]

whence (since \( \sigma > \eta_q \) for \( q > 1 \)) we can deduce (114).

Note. On (117) may be based another proof of (99) in the case of \( 1 < q < D' \), \( \chi'(H) = 1 \). By the arguments of (116)

\[
(1 + \psi(q^{1/2 \log q}) \log q + \psi(q^{1/2 \log q}) \log q^2 + \ldots) \log q
\]

\[
= -i \sqrt{\frac{2}{\pi}} \int_{1 - \Delta}^{1 + \Delta} \left( \psi(s) - \psi(s + \delta') \right) \frac{d^n \theta}{(2\pi)^n} d \delta'
\]

\[
= 2v \sqrt{\sigma} \psi (U_1 - U_2) - i \sqrt{\frac{2}{\pi}} \int_{1 - \Delta}^{1 + \Delta} \left( \psi(s) - \psi(s + \delta') \right) \frac{d^n \theta}{(2\pi)^n} d \delta'.
\]

Hence, by (116) and (117)

\[
S > \left( 1 - \log q \right) \left\{ \sum_{k=0}^{n} e^{-\left( \frac{2k+1}{4\nu} \right)^2 q^{\nu H}} - \left( \frac{\log q}{\nu} \right) \right\}
\]

whence

\[
I_B S > \left( 1 - \log q \right) \left\{ \sum_{k=0}^{n} e^{-\left( \frac{2k+1}{4\nu} \right)^2 q^{\nu H}} - \left( \frac{\log q}{\nu} \right) \right\}
\]

and we may proceed as in § 23: For sufficiently large \( \eta_q, B < 1 \) the following theorem holds. Having proved the theorem for all \( \xi > 4 \eta_q \), we get it for all \( \xi > 0 \) after increasing \( \sigma \) by \( 4 \eta_q \).

**Appendix.** Let \( \epsilon < 1 \) and \( \sigma \) be the characters of the group \( K \), the principal character being \( \chi_1 \). We begin by proving that matrix

\[
M = \begin{pmatrix} \chi_1(H_1) & \cdots & \chi_1(H_n) \\ \vdots & \ddots & \vdots \\ \chi_n(H_1) & \cdots & \chi_n(H_n) \end{pmatrix}
\]

has the rank \( r(M) = \lambda - 1 \).

Suppose, that \( r(M) < \lambda - 1 \). Then, on the one hand, the determinants of the order \( \lambda - 1 \) belonging to \( M \) are zero, whence using the expansion corresponding to the first row in

\[
D = \begin{pmatrix} \chi_1(H_1) & \cdots & \chi_1(H_n) \\ \chi_2(H_1) & \cdots & \chi_2(H_n) \\ \vdots & \ddots & \vdots \\ \chi_n(H_1) & \cdots & \chi_n(H_n) \end{pmatrix}
\]
we deduce that \( D = 0 \). On the other hand, squaring \( D \) by the 'column by column' rule and observing that by (64)

\[
\sum_{x} \chi(H) \chi(H) = \begin{cases} h & \text{if } H_1, H_2 \text{ are inverse classes,} \\
0 & \text{otherwise} \end{cases}
\]

we get a determinant with the number \( h \) occurring once in each column and once in each row, all the other places being filled by zeros. This determinant being \( h \), we get a contradiction. Hence \( D \neq 0 \) and thus \( r(M) = h - 1 \).

If all the positive numbers \( a_1, \ldots, a_k \) are equal, then in the system of equations

\[
\begin{align*}
a_1 \chi_1(H_1) + a_2 \chi_2(H_2) + \cdots + a_k \chi_k(H_k) &= c_1, \\
\vdots & \\
a_1 \chi_1(H_1) + a_2 \chi_2(H_2) + \cdots + a_k \chi_k(H_k) &= c_k.
\end{align*}
\]

we have, by (14)

\[
c_1 > 0, \quad c_2 = \cdots = c_k = 0.
\]

Suppose, that there is a set of positive and unequal numbers \( a_i \) such that in (118) \( c_1 = \cdots = c_k = 0 \). Dropping the first line we get a homogeneous system of linear equations with a set of unequal solutions, whence \( r(M) < h - 1 \) — a contradiction. Hence, if in (118) \( a_1, \ldots, a_k \) are positive unequal numbers, then next to \( c_1 \neq 0 \) at least one of the numbers \( a_1, \ldots, a_k \) is \( \neq 0 \). Transposing (if necessary) the indices we may suppose that

\[
c_1, c_2, \ldots, c_k \quad (k > 2)
\]

are all the coefficients \( c \) in (118) which do not vanish. The corresponding characters \( \chi_1, \ldots, \chi_k \) form a group \( H \), by [11], Lemma A (5). The eta functions

\[
\zeta(s, \chi)(1 \leq j < k)
\]

with characters \( \chi_1, \ldots, \chi_k \) and (these alone) have a simple pole at \( s = 1 \) (cf. § 3). Hence the functions \( \zeta(\zeta \mid s, \chi) \) have a simple pole at \( s = 1 \) with residue \(-1\); the other \( \zeta(\zeta \mid s, \chi) \) are regular at \( s = 1 \). The main term in the asymptotic estimate of the sum \( \sum \log b \) over the generators \( b \leq x \) of any class \( H \) being

\[
-h^{-1} \sum_{x} \zeta(H) \frac{\log b}{x} = \frac{1}{h} \sum_{x \in \mathbb{C} \setminus \mathbb{C}^*} \zeta(H)
\]

(3) Although the starting point in [11] is different from that of the present paper (cf. [9]), it does not affect the proof concerning \( F \).

\[\text{(cf. [17], p. 134), for the principal class } H_1 \text{ is equal to } xk/h.\]

Hence the lemma would follow by proving that there is a class \( H \) such that

\[
\sum_{x \in \mathbb{C} \setminus \mathbb{C}^*} \zeta(H) = 0.
\]

By the isomorphism of the group \( K' \) of the characters on the group \( K \) (cf. [12], p. 36) there is a sub-group \( K_4 \) of \( K \) corresponding to \( \Gamma \). Hence the order of \( K_4 \) is \( k > 1 \) and thus there is a class \( H_4 \) such that for \( n \) running over all the characters of the group \( K_4 \) we have, by (64),

\[
\sum_{x \in \mathbb{C} \setminus \mathbb{C}^*} \zeta(H) = 0.
\]

whence

\[
\sum_{x \in \mathbb{C} \setminus \mathbb{C}^*} \zeta(H) = 0.
\]

Let \( x' \) run through all the characters of the factor-group \( K/K_4 \). Then

\[\zeta(x')(j = 1, \ldots, k)\]

apparently represent all the characters \( \zeta \) of \( K \). Since

\[\zeta(x') = \zeta(H) \zeta(H)
\]

(with an obvious interpretation of \( x'(H) \)), (120) follows from (121).

References

The discrepancy of random sequences \( \{ka\} \)

by

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1. Introduction. It was R. Bellman [2] who first suggested the investigation of the limit distribution of

\[
\sum_{k=1}^{N} f(y + kx; a, b) - N(b-a)
\]

if the pair \( a, b \) is a random variable, uniformly distributed in the unit square and if, for \( 0 \leq a \leq b \leq 1 \),

\[
f(\xi; a, b) = \begin{cases} 
1 & \text{if } a \leq \xi \leq b, \\
0 & \text{if } 0 \leq \xi < a \text{ or } b < \xi \leq 1,
\end{cases}
\]

\[
f(\xi+1; a, b) = f(\xi; a, b).
\]

If \( \{\xi\} = \xi - [\xi] \) denotes the fractional part of \( \xi \), then \( \sum_{k=1}^{N} f(y + kx; a, b) \) is simply the number of \( k, 1 \leq k \leq N \), with \( (y + kx) \in [a, b] \) and (1.1) measures the deviation of this number from its average.

In [4] and [5] the author found the limiting distribution of (1.1). In this note those results are extended by studying the discrepancy

\[
D_N(x) = \frac{1}{N} \sup_{0 \leq \chi \leq 1} \left| \sum_{k=1}^{N} f(kx; a, b) - N(b-a) \right|.
\]

(If \( 1 < b \leq 1 + a \) we define \( f(\xi; a, b) \) in an obvious way, namely as \( f(\xi; a+1, b) + f(\xi; 0, b-1) \).) Our main result is Theorem 2 below for which we consider \( x \) as a point from the measure space \([0, 1]\) with Lebesgue measure.

Theorem 2.

\[
\frac{N^{2}D_N(a)}{\log N \log \log N} \rightarrow \frac{2}{\pi} \text{ in measure on } [0, 1] \text{ as } N \rightarrow \infty.
\]

The first part of the proof (section 2) gives an asymptotic expression for \( D_N(a) \) in terms of the continued fraction denominators of \( a \), which