Abstract. We show how to use the Gaussian HJM model to price Polish three-year bonds. A Polish Treasury bond is treated as a risk-free security.

1. Treasury bonds. Since 1994, a stable growth trend has been observed for the number of Treasury bond series issued in Poland. Bonds are perceived as a safe investment, in contrast to shares, and represent an attractive alternative to bank deposits.

By 31 March 1998, the National Depository for Securities (KDPW SA) had registered 50 Treasury bond series including 43 listed on Warsaw Stock Exchange. Capitalisation of Treasury bonds registered at the National Depository was PLN 9.095 million. The following types of bonds were registered: one-year bonds with a variable interest rate and par value of PLN 100 (IR) and PLN 1,000 (RP); two-year bonds with a fixed interest rate and par value PLN 1,000 (OS); three-year bonds with a variable interest rate and par value PLN 100 (TZ) and PLN 1,000 (TP); five-year bonds with a fixed interest rate and par value of PLN 1,000 (OS), and ten-year bonds with a variable interest rate and par value of PLN 1,000 (DZ). Prices of bonds vary depending on their type. Those with a variable interest rate have the most stable prices, while bonds with a fixed interest rate seem to fluctuate most strongly, sometimes even up to several percentage points. This is caused by differing opinions among investors as to the future inflation rate in Poland.

First, we give some information from the prospectus of the three-year bonds issue. This instrument pays coupons quarterly and after three years the face value. These bonds can be purchased not only by Polish investors but also by investors from abroad. According to Polish law the TZ bonds are allowed to be traded in the secondary market.

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The coupons are paid in arrears, i.e., several days after the end of the quarter (see Table 1.1). The nominal value is the base for computing the coupon rate. The coupon is computed as follows: The basic rate is calculated separately for each interest period as the arithmetic mean of weighted average rates of 13-week T-bills sold by the four consecutive Treasury bill auctions, the last of which taking place not later than two weeks before the beginning of the given interest period. The formula is

\[ J = \frac{r_1 + r_2 + r_3 + r_4}{4}, \]

where \( J \) is the basic rate, and \( r_k \) are the weighted average rates of 13-week T-bills.

Each \( r_k, k = 1, \ldots, 4 \), is calculated as

\[ r_k = \frac{100 - p_k}{p_k} \cdot \frac{360}{91}, \]

where \( p_k \) is the weighted average price of 13-week T-bills sold at the four consecutive sales, \( k = 1, \ldots, 4 \).

Finally, the coupon \( C_i \) corresponding to the interest period \( k \) with duration of \( n_i \) days is equal to

\[ C_i = \theta \cdot J_i \cdot \frac{n_i}{360}, \]

where \( \theta = 1.05 \) and \( J_i \) denotes the basic rate for the \( i \)th interest period.

Table 1.1 presents the dates of beginning and end for each interest period and interest payment dates for TZ bonds with maturity on 6 May, 2002.

<table>
<thead>
<tr>
<th>Interest period</th>
<th>Beginning</th>
<th>End</th>
<th>Length ( n_k ) [days]</th>
<th>Acquiring of interest right</th>
<th>Interest payment date</th>
</tr>
</thead>
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<tr>
<td>TZ0502/1</td>
<td>04.05.1999</td>
<td>05.08.1999</td>
<td>93</td>
<td>30.07.1999</td>
<td>19.08.1999</td>
</tr>
<tr>
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<td>05.11.1999</td>
<td>92</td>
<td>25.10.1999</td>
<td>08.11.1999</td>
</tr>
<tr>
<td>TZ0502/3</td>
<td>05.11.1999</td>
<td>05.02.2000</td>
<td>92</td>
<td>25.01.2000</td>
<td>07.02.2000</td>
</tr>
<tr>
<td>TZ0502/4</td>
<td>05.02.2000</td>
<td>05.05.2000</td>
<td>90</td>
<td>20.04.2000</td>
<td>08.05.2000</td>
</tr>
<tr>
<td>TZ0502/5</td>
<td>05.05.2000</td>
<td>05.08.2000</td>
<td>92</td>
<td>25.07.2000</td>
<td>07.08.2000</td>
</tr>
<tr>
<td>TZ0502/6</td>
<td>05.08.2000</td>
<td>05.11.2000</td>
<td>92</td>
<td>23.10.2000</td>
<td>06.11.2000</td>
</tr>
<tr>
<td>TZ0502/7</td>
<td>05.11.2000</td>
<td>05.02.2001</td>
<td>92</td>
<td>24.01.2001</td>
<td>06.02.2001</td>
</tr>
<tr>
<td>TZ0502/8</td>
<td>05.02.2001</td>
<td>05.05.2001</td>
<td>89</td>
<td>20.04.2001</td>
<td>07.05.2001</td>
</tr>
<tr>
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<td>05.05.2001</td>
<td>05.08.2001</td>
<td>92</td>
<td>24.07.2001</td>
<td>06.08.2001</td>
</tr>
<tr>
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<td>92</td>
<td>23.10.2001</td>
<td>06.11.2001</td>
</tr>
<tr>
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<td>05.11.2001</td>
<td>05.02.2002</td>
<td>92</td>
<td>24.01.2002</td>
<td>06.02.2002</td>
</tr>
<tr>
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<td>05.02.2002</td>
<td>05.05.2002</td>
<td>89</td>
<td>19.04.2002</td>
<td>06.05.2002</td>
</tr>
</tbody>
</table>

2. The Gaussian HJM model. There is a relatively extensive theoretical and empirical literature on the Heath–Jarrow–Morton model for
bond pricing and the term structure [1–6]. Below we concentrate on the so-called Gaussian HJM model. Let $[0, \tau]$, for $\tau > 0$, be a trading interval and $(\Omega, \{\mathcal{F}_t : t \in [0, \tau]\}, Q)$ be a probability space, where $\mathcal{F}_t$ is the $Q$-augmentation of the natural filtration generated by an $n$-dimensional Brownian motion $W(t) = \{W_1(t), \ldots, W_n(t)\}$. Assume that there are zero-coupon bonds in the market with all maturities and face value 1. For $t \leq T$, let $P(t, T)$ denote the $t$ time price of such a $T$ maturity bond. Of course $P(T, T) \equiv 1$ must hold for all $T$. We require that $\partial \log P(t, T)/\partial T$ exists.

The instantaneous forward rate $f(t, T)$ for $t < T$ is defined as

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T};$$

then

$$P(t, T) = \exp \left( -\int_t^T f(t, s) \, ds \right).$$

When $t = T$, the forward rate $f(t, t)$ is called the spot rate and is denoted by $r(t)$. Similarly one can interpret $r(t)$ as the rate of loan at moment $t$ which is returned an instant later. Assuming arbitrage-free market, we obtain the following dynamics of the forward rate $f(t, T)$:

$$df(t, T) = \sigma(t, T)^* \int_t^T \sigma(t, y) \, dy \, dt + \sigma(t, T)^* dW(t),$$

where $^*$ denotes transposition. In this paper we assume that the volatility $\sigma(t, T)$ of the forward rate is a deterministic function. In the literature such a case is referred to as the Gaussian HJM model (see [5]).

Let the price process of a savings account be given by

$$B(t) = \exp \left( \int_0^t r(s) \, ds \right).$$

If we invest 1 unit in the cash market, we will get the amount $B(t)$ after time $t$.

In this framework each financial instrument with pay-off function $X$, which is an $\mathcal{F}_T$-measurable random variable, can be priced at time $t$ using the formula

$$\mathbb{E}^Q \left( \frac{B(t)}{B(T)} X \bigg| \mathcal{F}_t \right).$$

3. Pricing three-year bonds

**Theorem 3.1.** A three-year bond is considered. There are $n+1$ available coupons for an investor who is about to buy the bond in the secondary market at time $t$. The coupons $C_i$ are paid at $T_i$, $i = 0, \ldots, n$. At time $T_n$ the Face
Value FV is also paid. Let \( t_i^1, t_i^2, t_i^3, t_i^4 \) denote the dates of Treasury bill auctions related to coupon \( C_i, i = 1, \ldots, n \). Let \( t_1^1 > t \). Then the fair price is given by the formula

\[
TZ_t = \sum_{i=0}^{n} P(t, T_i) \sum_{k=1}^{4} (c(i, k)P(t, t_k^i)/P(t, t_k^{i+1}) - 1) + FVP(t, T_n)
\]

where

\[
c(i, k) = \frac{\theta FV \delta_i}{4\delta} \exp(\text{Cov}(\log P(t_k^i, t_k^i + \delta), \log(P(t_k^i, t_k^i + \delta)/P(t_k^i, T_i)|F_t))
\]

with \( \delta_i = n_i/360, \delta = 91/360 \).

Proof. We need the following lemma:

**Lemma 3.1.** Define a new forward measure by setting

\[
Q_T(A) = \int_A (P(0, T)B(T))^{-1} dQ.
\]

Then for any \( F_T \)-measurable random variable \( X \) such that \( E|X|^p < \infty \) for some \( p > 1 \) we have

\[
E^Q\left(\frac{B(t)}{B(T)}X \bigg| F_t\right) = P(t, T)E^{Q_T}(X | F_t).
\]

For the proof, see [5], p. 317.

We price this bond using the arbitrage-free pricing formula

\[
(3.3) \quad TZ_t = E\left(\sum_{i=1}^{n} \frac{B(t)}{B(T_i)} C_i \cdot FV + \frac{B(t)}{B(T_n)} FV \bigg| F_t\right).
\]

According to the way of calculating coupons, described above, we have

\[
TZ_t = E\left(\sum_{i=1}^{n} \frac{B(t)}{B(T_i)} FV \frac{\theta \delta_i}{4} \sum_{k=1}^{4} r_k^i + \frac{B(t)}{B(T_n)} FV \bigg| F_t\right).
\]

In our notation (see (1.1)),

\[
r_k^i = \frac{1 - P(t_k^i, t_k^i + \delta)}{\delta P(t_k^i, t_k^i + \delta)}, \quad i = 1, \ldots, 12, \ k = 1, \ldots, 4.
\]

Therefore

\[
(3.4) \quad TZ_t = E\left(\sum_{i=1}^{n} \frac{B(t)}{B(T_i)} FV \frac{\theta \delta_i}{4\delta} \sum_{k=1}^{4} (P(t_k^i, t_k^i + \delta)^{-1} - 1) \bigg| F_t\right)
\]

\[+ FV \cdot E\left(\frac{B(t)}{B(T_n)} \bigg| F_t\right)\]
\[ c_{ik} = E \left( \frac{B(t)}{B(t_k)} \right) \left\{ \frac{P(t_k^i, t_k^i + \delta)}{P(t_k^i, T_k)} \right\} \exp \left( - \frac{1}{2} \int_t^{t_k^i} \int_s^{t_k^i + \delta} \sigma(s, u) \, du \, ds \right) \]

Because (see [2]) for any \( t < T \),

\[ P(t, T) = P(0, T) \exp \left( \int_0^t r(s) \, ds - \frac{1}{2} \int_0^T \sigma(s, u) \, du \, ds \right) \]

we can rewrite the expression (3.5) as

\[ c_{ik} = E \left( \frac{B(t)}{B(t_k)} \right) \left\{ \frac{P(t_k^i, t_k^i + \delta)}{P(t_k^i, T_k)} \right\} \exp \left[ - \frac{1}{2} \int_t^{t_k^i} \int_s^{t_k^i + \delta} \sigma(s, u) \, du \, ds \right] \].

Since \( P(t, T) \) is \( \mathcal{F}_t \)-measurable we have

\[ c_{ik} = \frac{P(t, T_k)}{P(t, t_k^i + \delta)} \exp \left[ - \int_t^{t_k^i} \int_s^{t_k^i + \delta} \sigma(s, u) \, du \, ds \right] \]

Using Lemma 3.1 we get

\[ c_{ik} = \frac{P(t, T_k)}{P(t, t_k^i + \delta)} \exp \left[ - \int_t^{t_k^i} \int_s^{t_k^i + \delta} \sigma(s, u) \, du \, ds \right] \]

\[ + \frac{1}{2} \int_t^{t_k^i} \int_s^{t_k^i + \delta} \sigma(s, u) \, du \, ds \]
Moreover, the Girsanov theorem shows that

\[ W(t,t_i^k) = W(t) + \int_0^t \sigma(s,u) \, ds \, du \]

is a Brownian motion with respect to \( Q_{t_i^k} \). Hence

\[
c_{ik} = \frac{P(t,T_i)}{P(t,t_i^k + \delta)} P(t,t_i^k) \mathbb{E}_{t_i^k} \left( \exp \left[ X + \int_t^{t_i^k} \int_{t_i^k + \delta}^s \sigma(s,u)* \, du \int_s^t \sigma(s,u) \, du \, ds \right] \right) \]

\[
+ \frac{1}{2} \left\{ \int_t^{t_i^k} \int_{t_i^k + \delta}^s \sigma(s,u)^* \, du \int_s^t \sigma(s,u) \, du \right\} ds \bigg) \bigg| \mathcal{F}_t \bigg),
\]

where

\[
X = - \int_t^{t_i^k} \int_{t_i^k + \delta}^s \sigma(s,u)^* \, du \, dW(s,t_i^k).
\]

The random variable \( X \) has normal \( N(0, \int_t^{t_i^k} \int_{t_i^k + \delta}^s \sigma(s,u) du^2 \, ds) \) distribution under the measure \( Q_{t_i^k} \) and is independent of \( \mathcal{F}_t \). Therefore we obtain

\begin{equation}
(3.7) \quad c_{ik} = \frac{P(t,T_i)}{P(t,t_i^k + \delta)} P(t,t_i^k) \exp \left( \frac{1}{2} \int_t^{t_i^k} \int_{t_i^k + \delta}^s \sigma(s,u) du \right)^2

+ \frac{1}{2} \left\{ \int_t^{t_i^k} \int_{t_i^k + \delta}^s \sigma(s,u)^* \, du \int_s^t \sigma(s,u) \, du \right\} ds \bigg) \bigg| \mathcal{F}_t \bigg),
\end{equation}

Let \( a = \int_s^{t_i^k} \sigma(s,u) du \), \( b = \int_{t_i^k + \delta}^s \sigma(s,u) du \) and \( c = \int_{t_i^k + \delta}^s \sigma(s,u) du \). Then

\begin{equation}
(3.8) \quad c_{ik} = \frac{P(t,T_i)P(t,t_i^k)}{P(t,t_i^k + \delta)} \times \exp \left( \frac{1}{2} \int_t^{t_i^k} \left[ c^2 + 2c^*a + (a+b)^2 - (a+b+c)^2 \right] ds \right)

= \frac{P(t,T_i)P(t,t_i^k)}{P(t,t_i^k + \delta)} \times \exp \left( \frac{1}{2} \int_t^{t_i^k} \left[ c^2 + 2c^*a - 2(a+b)^*c - c^2 \right] ds \right)
\end{equation}
Pricing bonds in the HJM framework

\[
= \frac{P(t, T_i)P(t, t_k)}{P(t, t_k + \delta)} \exp \left( \int_t^{t_k} [-b^e c] \, ds \right)
\]

\[
= \frac{P(t, T_i)P(t, t_k)}{P(t, t_k + \delta)} \exp \left( - \int_t^{t_k} \int_{t_k + \delta}^{T_i} \sigma(s, u) \, du \int_{t_k + \delta}^{T_i} \sigma(s, u) \, du \, ds \right).
\]

Because for any \( t < s < u < v \),

\[
\text{Cov}(\log P(s, u), \log P(s, v) \mid \mathcal{F}_t) = \int_t^{s} \int_s^{u} \int_t^{v} \sigma(x, y)^* \, dy \int_t^{t_k} \sigma(s, u) \, du \, dx
\]

(see [2]), combining (3.4) and (3.8) proves the theorem. ■

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References


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