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PRICING POLISH THREE-YEAR BONDS IN THE HJM FRAMEWORK

Abstract. We show how to use the Gaussian HJM model to price Polish three-year bonds. A Polish Treasury bond is treated as a risk-free security.

1. Treasury bonds. Since 1994, a stable growth trend has been observed for the number of Treasury bond series issued in Poland. Bonds are perceived as a safe investment, in contrast to shares, and represent an attractive alternative to bank deposits.

By 31 March 1998, the National Depository for Securities (KDPW SA) had registered 50 Treasury bond series including 43 listed on Warsaw Stock Exchange. Capitalisation of Treasury bonds registered at the National Depository was PLN 9.095 million. The following types of bonds were registered: one-year bonds with a variable interest rate and par value of PLN 100 (IR) and PLN 1,000 (RP); two-year bonds with a fixed interest rate and par value PLN 1,000 (OS); three-year bonds with a variable interest rate and par value PLN 100 (TZ) and PLN 1,000 (TP); five-year bonds with a fixed interest rate and par value of PLN 1,000 (OS), and ten-year bonds with a variable interest rate and par value of PLN 1,000 (DZ). Prices of bonds vary depending on their type. Those with a variable interest rate have the most stable prices, while bonds with a fixed interest rate seem to fluctuate most strongly, sometimes even up to several percentage points. This is caused by differing opinions among investors as to the future inflation rate in Poland.

First, we give some information from the prospectus of the three-year bonds issue. This instrument pays coupons quarterly and after three years the face value. These bonds can be purchased not only by Polish investors but also by investors from abroad. According to Polish law the TZ bonds are allowed to be traded in the secondary market.

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The coupons are paid in arrears, i.e. several days after the end of the quarter (see Table 1.1). The nominal value is the base for computing the coupon rate. The coupon is computed as follows: The basic rate is calculated separately for each interest period as the arithmetic mean of weighted average rates of 13-week T-bills sold by the four consecutive Treasury bill auctions, the last of which taking place not later than two weeks before the beginning of the given interest period. The formula is

$$J = \frac{r_1 + r_2 + r_3 + r_4}{4},$$

where J is the basic rate, and r_k are the weighted average rates of 13-week T-bills.

Each r_k, $k = 1, \ldots, 4$, is calculated as

$$r_k = \frac{100 - p_k}{p_k} \cdot \frac{360}{91}, \tag{1.1}$$

where p_k is the weighted average price of 13-week T-bills sold at the four consecutive sales, $k = 1, \ldots, 4$.

Finally, the coupon C_i corresponding to the interest period k with duration of n_i days is equal to

$$C_i = \theta \cdot J_i \cdot \frac{n_i}{360},$$

where $\theta = 1.05$ and J_i denotes the basic rate for the ith interest period.

Table 1.1 presents the dates of beginning and end for each interest period and interest payment dates for TZ bonds with maturity on 6 May, 2002.

TABLE 1.1. The term structure of TZ bonds under study

Interest period	Beginning	End	Length n_k [days]	Acquiring of interest right	Interest payment date
TZ0502/1	04.05.1999	05.08.1999	93	30.07.1999	19.08.1999
TZ0502/2	05.08.1999	05.11.1999	92	25.10.1999	08.11.1999
TZ0502/3	05.11.1999	05.02.2000	92	25.01.2000	07.02.2000
TZ0502/4	05.02.2000	05.05.2000	90	20.04.2000	08.05.2000
TZ0502/5	05.05.2000	05.08.2000	92	25.07.2000	07.08.2000
TZ0502/6	05.08.2000	05.11.2000	92	23.10.2000	06.11.2000
TZ0502/7	05.11.2000	05.02.2001	92	24.01.2001	06.02.2001
TZ0502/8	05.02.2001	05.05.2001	89	20.04.2001	07.05.2001
TZ0502/9	05.05.2001	05.08.2001	92	24.07.2001	06.08.2001
TZ0502/10	05.08.2001	05.11.2001	92	23.10.2001	06.11.2001
TZ0502/11	05.11.2001	05.02.2002	92	24.01.2002	06.02.2002
TZ0502/12	05.02.2002	05.05.2002	89	19.04.2002	06.05.2002

2. The Gaussian HJM model. There is a relatively extensive theoretical and empirical literature on the Heath–Jarrow–Morton model for bond pricing and the term structure [1–6]. Below we concentrate on the so-called Gaussian HJM model. Let $[0,\tau]$, for $\tau > 0$, be a trading interval and $(\Omega, \{\mathcal{F}_t : t \in [0,\tau]\}, \boldsymbol{Q})$ be a probability space, where $\mathcal{F}_t$ is the $\boldsymbol{Q}$-augmentation of the natural filtration generated by an n-dimensional Brownian motion $W(t) = \{W_1(t), \dots, W_n(t)\}$. Assume that there are zero-coupon bonds in the market with all maturities and face value 1. For $t \le T$, let $P(t,T)$ denote the t time price of such a T maturity bond. Of course $P(T,T) \equiv 1$ must hold for all T. We require that $\partial \log P(t,T)/\partial T$ exists. The instantaneous *forward rate* $f(t,T)$ for $t < T$ is defined as

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T};$$

then

$$P(t,T) = \exp\Big(-\int_t^T f(t,s)\,ds\Big).$$

When $t = T$, the forward rate $f(t,t)$ is called the *spot rate* and is denoted by $r(t)$. Similarly one can interpret $r(t)$ as the rate of loan at moment t which is returned an instant later. Assuming arbitrage-free market, we obtain the following dynamics of the forward rate $f(t,T)$:

$$df(t,T) = \sigma(t,T)^* \int_t^T \sigma(t,y)\,dy\,dt + \sigma(t,T)^* dW(t),$$

where * denotes transposition. In this paper we assume that the volatility $\sigma(t,T)$ of the forward rate is a deterministic function. In the literature such a case is referred to as the *Gaussian HJM model* (see [5]).

Let the price process of a savings account be given by

$$B(t) = \exp\Big(\int_0^t r(s)\,ds\Big).$$

If we invest 1 unit in the cash market, we will get the amount $B(t)$ after time t.

In this framework each financial instrument with pay-off function X, which is an $\mathcal{F}_T$-measurable random variable, can be priced at time t using the formula

$$\mathrm{E}^Q\Big(\frac{B(t)}{B(T)}X \,\Big|\, \mathcal{F}_t\Big). \tag{2.2}$$

3. Pricing three-year bonds

THEOREM 3.1. *A three-year bond is considered. There are $n+1$ available coupons for an investor who is about to buy the bond in the secondary market at time t. The coupons C_i are paid at T_i, $i = 0, \dots, n$. At time T_n the Face*

Value FV *is also paid. Let* t_1^i t_2^i, t_3^i, t_4^i *denote the dates of Treasury bill auctions related to coupon* C_i, $i = 1, \ldots, n$. *Let* $t_1^1 > t$. *Then the fair price is given by the formula*

$$\mathbf{TZ}_t = \sum_{i=0}^{n} P(t, T_i) \sum_{k=1}^{4} (c(i,k) P(t, t_k^i)/P(t, t_k^{i+1}) - 1) + \mathrm{FV} P(t, T_n)$$

where

$$c(i,k) = \frac{\theta \mathrm{FV} \delta_i}{4\delta} \exp(\mathrm{Cov}(\log P(t_k^i, t_k^i + \delta), \log(P(t_k^i, t_k^i + \delta)/P(t_k^i, T_i)) \mid \mathcal{F}_t)$$

with $\delta_i = n_i/360$, $\delta = 91/360$.

Proof. We need the following lemma:

LEMMA 3.1. *Define a new forward measure by setting*, *for all* $A \in \mathcal{F}_T$,

$$Q_T(A) = \int_A (P(0,T)B(T))^{-1}\, dQ.$$

Then for any $\mathcal{F}_T$-*measurable random variable* X *such that* $\mathrm{E}|X|^p < \infty$ *for some* $p > 1$ *we have*

$$\mathrm{E}^Q\bigg(\frac{B(t)}{B(T)} X \,\bigg|\, \mathcal{F}_t\bigg) = P(t,T)\mathrm{E}^{Q_T}(X \mid \mathcal{F}_t).$$

For the proof, see [5], p. 317.

We price this bond using the arbitrage-free pricing formula

$$\mathbf{TZ}_t = \mathrm{E}\bigg(\sum_{i=1}^{n} \frac{B(t)}{B(T_i)} C_i \cdot \mathrm{FV} + \frac{B(t)}{B(T_n)} \mathrm{FV} \,\bigg|\, \mathcal{F}_t\bigg). \tag{3.3}$$

According to the way of calculating coupons, described above, we have

$$\mathbf{TZ}_t = \mathrm{E}\bigg(\sum_{i=1}^{n} \frac{B(t)}{B(T_i)} \mathrm{FV} \frac{\theta\delta_i}{4} \sum_{k=1}^{4} r_k^i + \frac{B(t)}{B(T_n)} \mathrm{FV} \,\bigg|\, \mathcal{F}_t\bigg).$$

In our notation (see (1.1)),

$$r_k^i = \frac{1 - P(t_k^i, t_k^i + \delta)}{\delta P(t_k^i, t_k^i + \delta)}, \quad i = 1, \ldots, 12,\ k = 1, \ldots, 4.$$

Therefore

$$\begin{aligned} \mathbf{TZ}_t = {} & \mathrm{E}\bigg(\sum_{i=1}^{n} \frac{B(t)}{B(T_i)} \mathrm{FV} \frac{\theta\delta_i}{4\delta} \sum_{k=1}^{4} (P(t_k^i, t_k^i + \delta)^{-1} - 1) \,\bigg|\, \mathcal{F}_t\bigg) \\ & + \mathrm{FV} \cdot \mathrm{E}\bigg(\frac{B(t)}{B(T_n)} \,\bigg|\, \mathcal{F}_t\bigg) \end{aligned} \tag{3.4}$$

$$= \sum_{i=1}^{n} \mathrm{FV} \frac{\theta \delta_i}{4\delta} \sum_{k=1}^{4} \left(c_{ik} - \mathrm{E}\left(\frac{B(t)}{B(T_i)} \,\middle|\, \mathcal{F}_t \right) \right) + \mathrm{FV} \cdot \mathrm{E}\left(\frac{B(t)}{B(T_n)} \,\middle|\, \mathcal{F}_t \right),$$

where

$$\begin{aligned} c_{ik} &= \mathrm{E}\left(\frac{B(t)}{B(T_i)} P(t_k^i, t_k^i + \delta)^{-1} \,\middle|\, \mathcal{F}_t \right) \\ &= \mathrm{E}\left(\mathrm{E}\left[\frac{B(t)}{B(T_i)} P(t_k^i, t_k^i + \delta)^{-1} \,\middle|\, \mathcal{F}_{t_k^i} \right] \,\middle|\, \mathcal{F}_t \right) \\ &= \mathrm{E}\left(\frac{B(t)}{B(t_k^i)} P(t_k^i, t_k^i + \delta)^{-1} \mathrm{E}\left[\frac{B(t_k^i)}{B(T_i)} \,\middle|\, \mathcal{F}_{t_k^i} \right] \,\middle|\, \mathcal{F}_t \right) \\ &= \mathrm{E}\left(\frac{B(t)}{B(t_k^i)} \cdot \frac{P(t_k^i, T_i)}{P(t_k^i, t_k^i + \delta)} \,\middle|\, \mathcal{F}_t \right). \end{aligned} \tag{3.5}$$

Because (see [2]) for any $t < T$,

$$P(t,T) = P(0,T) \exp\left(\int_0^t r(s)\, ds - \frac{1}{2} \int_0^t \left| \int_s^T \sigma(s,u)\, du \right|^2 ds - \int_0^t \int_s^T \sigma(s,u)^*\, du\, dW(s) \right),$$

we can rewrite the expression (3.5) as

$$c_{ik} = \mathrm{E}\left(\frac{B(t)}{B(t_k^i)} \cdot \frac{P(t,T_i)}{P(t, t_k^i + \delta)} \times \frac{\exp\left[-\frac{1}{2} \int_t^{t_k^i} \left| \int_s^{T_i} \sigma(s,u)\, du \right|^2 ds - \int_t^{t_k^i} \int_s^{T_i} \sigma(s,u)^*\, du\, dW(s) \right]}{\exp\left[-\frac{1}{2} \int_t^{t_k^i} \left| \int_s^{t_k^i+\delta} \sigma(s,u)\, du \right|^2 ds - \int_t^{t_k^i} \int_s^{t_k^i+\delta} \sigma(s,u)^*\, du\, dW(s) \right]} \,\middle|\, \mathcal{F}_t \right). \tag{3.6}$$

Since $P(t,T)$ is $\mathcal{F}_t$-measurable we have

$$c_{ik} = \frac{P(t,T_i)}{P(t, t_k^i + \delta)} \mathrm{E}\left(\frac{B(t)}{B(t_k^i)} \exp\left[- \int_t^{t_k^i} \int_{t_k^i+\delta}^{T_i} \sigma(s,u)^*\, du\, dW(s) + \frac{1}{2} \int_t^{t_k^i} \left\{ \left| \int_s^{t_k^i+\delta} \sigma(s,u)\, du \right|^2 - \left| \int_s^{T_i} \sigma(s,u)\, du \right|^2 \right\} ds \right] \,\middle|\, \mathcal{F}_t \right).$$

Using Lemma 3.1 we get

$$c_{ik} = \frac{P(t,T_i)}{P(t, t_k^i + \delta)} P(t, t_k^i) \mathrm{E}_{t_k^i} \left(\exp\left[- \int_t^{t_k^i} \int_{t_k^i+\delta}^{T_i} \sigma(s,u)^*\, du\, dW(s) + \frac{1}{2} \int_t^{t_k^i} \left\{ \left| \int_s^{t_k^i+\delta} \sigma(s,u)\, du \right|^2 - \left| \int_s^{T_i} \sigma(s,u)\, du \right|^2 \right\} ds \right] \,\middle|\, \mathcal{F}_t \right).$$

Moreover, the Girsanov theorem shows that

$$W(t,t_k^i) = W(t) + \int_0^t \int_s^{t_k^i} \sigma(s,u)\,ds\,du$$

is a Brownian motion with respect to $Q_{t_k^i}$. Hence

$$\begin{aligned}c_{ik} = \frac{P(t,T_i)}{P(t,t_k^i+\delta)} P(t,t_k^i) \mathrm{E}_{t_k^i}\bigg(\exp\bigg[X + \int_t^{t_k^i} \int_{t_k^i+\delta}^{T_i} \sigma(s,u)^*\,du \int_s^{t_k^i} \sigma(s,u)\,du\,ds \\ + \frac{1}{2}\int_t^{t_k^i} \bigg\{\Big|\int_s^{t_k^i+\delta} \sigma(s,u)\,du\Big|^2 - \Big|\int_s^{T_i} \sigma(s,u)\,du\Big|^2\bigg\}\,ds\bigg] \,\bigg|\, \mathcal{F}_t\bigg),\end{aligned}$$

where

$$X = -\int_t^{t_k^i} \int_{t_k^i+\delta}^{T_i} \sigma(s,u)^*\,du\,dW(s,t_k^i).$$

The random variable X has normal $N(0, \int_t^{t_k^i} |\int_{t_k^i+\delta}^{T_i} \sigma(s,u)\,du|^2\,ds)$ distribution under the measure $Q_{t_k^i}$ and is independent of $\mathcal{F}_t$. Therefore we obtain

$$\begin{aligned}c_{ik} = \frac{P(t,T_i)}{P(t,t_k^i+\delta)} P(t,t_k^i) \exp\bigg(\frac{1}{2}\int_t^{t_k^i} \bigg[\Big|\int_{t_k^i+\delta}^{T_i} \sigma(s,u)\,du\Big|^2 \\ + 2\int_{t_k^i+\delta}^{T_i} \sigma(s,u)^*\,du \int_s^{t_k^i} \sigma(s,u)\,du \\ + \Big|\int_s^{t_k^i+\delta} \sigma(s,u)\,du\Big|^2 - \Big|\int_s^{T_i} \sigma(s,u)\,du\Big|^2\bigg]\,ds\bigg).\end{aligned} \tag{3.7}$$

Let $a = \int_s^{t_k^i} \sigma(s,u)\,du$, $b = \int_{t_k^i}^{t_k^i+\delta} \sigma(s,u)\,du$ and $c = \int_{t_k^i+\delta}^{T_i} \sigma(s,u)\,du$. Then

$$\begin{aligned}c_{ik} &= \frac{P(t,T_i)P(t,t_k^i)}{P(t,t_k^i+\delta)} \\ &\quad\times \exp\bigg(\frac{1}{2}\int_t^{t_k^i} [c^2 + 2c^*a + (a+b)^2 - (a+b+c)^2]\,ds\bigg) \\ &= \frac{P(t,T_i)P(t,t_k^i)}{P(t,t_k^i+\delta)} \exp\bigg(\frac{1}{2}\int_t^{t_k^i} [c^2 + 2c^*a - 2(a+b)^*c - c^2]\,ds\bigg)\end{aligned} \tag{3.8}$$

$$= \frac{P(t,T_i)P(t,t_k^i)}{P(t,t_k^i+\delta)} \exp\Big(\int\limits_t^{t_k^i}[-b^*c]\,ds\Big)$$

$$= \frac{P(t,T_i)P(t,t_k^i)}{P(t,t_k^i+\delta)} \exp\Big(-\int\limits_t^{t_k^i}\int\limits_{t_k^i}^{t_k^i+\delta}\sigma(s,u)^*\,du\int\limits_{t_k^i+\delta}^{T_i}\sigma(s,u)\,du\,ds\Big).$$

Because for any $t < s < u < v$,

$$\mathrm{Cov}(\log P(s,u),\log P(s,v)\,|\,\mathcal{F}_t) = \int\limits_t^s\int\limits_s^u \sigma(x,y)^*\,dy\int\limits_s^v \sigma(x,y)\,dy\,dx$$

(see [2]), combining (3.4) and (3.8) proves the theorem. ∎

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