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SOME REMARKS ON EQUILIBRIA IN SEMI-MARKOV GAMES

Abstract. This paper is a first study of correlated equilibria in nonzero-sum semi-Markov stochastic games. We consider the expected average payoff criterion under a strong ergodicity assumption on the transition structure of the games. The main result is an extension of the correlated equilibrium theorem proven for discounted (discrete-time) Markov games in our joint paper with Raghavan. We also provide an existence result for stationary Nash equilibria in the limiting average payoff semi-Markov games with state independent and nonatomic transition probabilities. A similar result was proven for discounted Markov games by Parthasarathy and Sinha.

1. Introduction and the model. This paper is a first study of correlated equilibria in nonzero-sum semi-Markov stochastic games. We consider the expected average payoff criterion and assume some strong ergodicity condition on the transition structure of the games. Nash equilibria were studied in several classes of (discrete-time) general state space Markov stochastic games but they are known to exist only when some specific conditions (especially concerning the transition probability functions) are satisfied. Also some results on ε-equilibria in Markov games are available (see [2, 17, 18] and their references). Nash equilibria are known to exist in some classes of ergodic semi-Markov games with countable state spaces [14, 22]. A broad discussion of nonzero-sum Markov games can be found in [20].

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The main purpose of this paper is to extend the correlated equilibrium theorem established for discounted Markov games in our joint paper with Raghavan [19] and for a class of ergodic Markov games in our article [16]. Combining our result with an observation due to Parthasarathy and Sinha [21], we also prove the existence of stationary Nash equilibria in a class of semi-Markov games with state independent and nonatomic transition probabilities. This is a counterpart of the result obtained for discounted Markov games in [21].

Let X be a metric space, and (S, Σ) a measurable space. A multivalued mapping Φ from S into a family of subsets of X is said to be *lower measurable* if for any open subset U of X the set $\{s \in S : \Phi(s) \cap U \neq \emptyset\}$ belongs to Σ. For a thorough discussion of lower measurable multivalued mappings with some applications to control and optimization theory we refer to [4].

An *N-person nonzero-sum semi-Markov game* is defined by the following objects:

• (S, Σ) is a measurable space, where S is the *set of states* for the game, and Σ is a countably generated σ-algebra of subsets of S.

• X_k is a nonempty compact metric *space of actions* for player k. We put $X = X_1 \times \ldots \times X_N$.

• A_k is a lower measurable multivalued mapping from S into nonempty compact subsets of X_k. For each $s \in S$, $A_k(s)$ represents the *set of actions available* to player k in state s. We put

$$A(s) = A_1(s) \times \ldots \times A_N(s), \quad s \in S.$$

• $r_k : S \times X \to \mathbb{R}$ is a bounded nonnegative product measurable *payoff function* for player k.

• q is a product measurable transition probability from $S \times X$ to S, called the *law of motion* among states.

• $F(t \mid s, x, y)$ is a product measurable *distribution function of the transition time.*

If s is a state at some stage of the game and the players select an $x \in A(s)$, then every player k receives the immediate payoff $r_k(s, x)$ and a new state y for the game is selected according to the probability distribution $q(\cdot \mid s, x)$. Conditional on the next state y the time until the transition from s to y actually occurs is a random variable having the distribution $F(t \mid s, x, y)$. For any $s \in S$ and $x \in A(s)$, the distribution of the *holding time* in the state s is

$$G(t \mid s, x) = \int_S F(t \mid s, x, y)\, q(dy \mid s, x)$$

and the *mean holding time* in the state s is

$$T(s,x) = \int_0^\infty t\,dG(t \mid s,x).$$

We now formulate our basic regularity assumptions.

A1: For every player k and $s \in S$, the function $r_k(s,\cdot)$ is continuous on X.

Usually, it is assumed that

$$r_k(s,x) = r_k^1(s,x) + T(s,x)r_k^2(s,x)$$

where $r_k^1(s,x)$ is the *immediate reward* at the transition time and $r_k^2(s,x)$ is the *reward rate* in the time interval between the successive transitions.

A2: The transition probability q has a density function, say z, with respect to a fixed probability measure μ on (S,Σ), satisfying the following L_1 continuity condition: If $x^n \to x^0$ in $A(s)$, then

$$\int_S |z(s,y,x^n) - z(s,y,x^0)|\,\mu(dy) \to 0 \quad \text{as } n \to \infty.$$

The L_1 continuity above is satisfied via Scheffe's theorem (see Theorem 16.11 of [3]) when $z(s,y,\cdot)$ is continuous on X. It implies the norm continuity of the transition probability $q(\cdot \mid s,x)$ with respect to $x \in X$.

A3: For any $s \in S$, the function $T(s,\cdot)$ is continuous on X. Moreover, there exist a and b such that $0 < a \le T(s,x) \le b$ for each $s \in S$ and $x \in X$.

The game is played over the infinite future with past history as common knowledge for all the players. A *strategy* for player k is a measurable mapping (transition probability) which associates with each given finite history $h^n = (s_1, x_1, \ldots, s_{n-1}, x_{n-1}, s_n)$ of the game (where $s_i \in S$, $x_i \in A(s_i)$) a probability distribution on the set $A_k(s_n)$ of actions available to him. A *stationary strategy* for player k is a mapping which associates with each state $s \in S$ a probability distribution on the set $A_k(s)$, independent of the history that led to the state s. A stationary strategy for player k can thus be identified with a measurable transition probability f from S to X_k such that $f(A_k(s) \mid s) = 1$, for every $s \in S$.

Let $H = S \times X \times S \times \ldots$ be the space of all infinite histories of the game, endowed with the product σ-algebra. For any multi-strategy $\pi = (\pi_1, \ldots, \pi_N)$ for the players and every initial state $s_1 = s \in S$, a probability measure P_s^π and a stochastic process $\{\sigma_n, \alpha_n\}$ are defined on H in a canonical way, where the random variables σ_n and α_n describe the state and the actions chosen by the players, respectively, on the nth stage of the game (see Proposition V.1.1 of [15]). Thus, for each multi-strategy $\pi = (\pi_1, \ldots, \pi_N)$

and every initial state $s \in S$, the *expected average payoff to player k* is

$$J_k(s,\pi) = \liminf_{m\to\infty} \frac{E_s^\pi(\sum_{n=1}^m r_k(\sigma_n,\alpha_n))}{E_s^\pi(\sum_{n=1}^m T(\sigma_n,\alpha_n))}.$$

Here E_s^π means the expectation operator with respect to the probability measure P_s^π.

Let $\pi^* = (\pi_1^*,\ldots,\pi_N^*)$ be a fixed multi-strategy for the players. For any strategy π_k of player k, we write (π_{-k}^*,π_k) for the multi-strategy obtained from π^* by replacing π_k^* with π_k.

A multi-strategy $\pi^* = (\pi_1^*,\ldots,\pi_N^*)$ is called a *Nash equilibrium* for the average payoff semi-Markov game if no unilateral deviations from it are profitable, that is, for each $s \in S$,

$$J_k(s,\pi^*) \ge J_k(s,(\pi_{-k}^*,\pi_k)),$$

for every player k and any strategy π_k.

It is still an open problem whether Markov games with uncountable state space have stationary equilibrium points. A positive answer to this problem is known only for some special classes of games, where the transition probabilities satisfy certain specific conditions. For a good survey of the existing literature see [20, 16, 18]. An equilibrium is easier to obtain if we allow the players to communicate in some sense and correlate their choices. We now extend the approach taken in our joint paper with Raghavan [19] where discounted (discrete-time) Markov games were studied.

We extend the sets of strategies available to the players in the sense that we allow them to correlate their choices in a natural way described below. The resulting solution is a kind of extensive-form correlated equilibrium [7].

Suppose that $\{\xi_n : n \ge 1\}$ is a sequence of so-called *signals*, drawn independently from $[0,1]$ according to the uniform distribution. Suppose that at the beginning of each (random) period n of the game the players are informed not only of the outcome of the preceding period and the current state s_n, but also of ξ_n. Then the information available to them is a vector $h^n = (s_1,\xi_1,x_1,\ldots,s_{n-1},\xi_{n-1},x_{n-1},s_n,\xi_n)$ where $s_i \in S$, $x_i \in A(s_i))$, and $\xi_i \in [0,1]$. We denote the set of such vectors by H^n.

An *extended strategy* for player k is a sequence $\pi_k = (\pi_k^1,\pi_k^2,\ldots)$, where every π_k^n is a (product) measurable transition probability from H^n to X_k such that $\pi_k^n(A_k(s_n)\,|\,h^n) = 1$ for any history $h^n \in H^n$. (Here s_n is the last coordinate of h^n.) An *extended stationary strategy* for player k is a strategy $\pi_k = (\pi_k^1,\pi_k^2,\ldots)$ such that each π_k^n depends on the current state s_n and the last signal ξ_n only. In other words, a strategy π_k of player k is called stationary if there exists a transition probability f from $S\times[0,1]$ to X_k such that for every n and each history $h^n \in H^n$, we have $\pi_k^n(\cdot\,|\,h^n) = f(\cdot\,|\,s_n,\xi_n)$. Assuming that the players use extended strategies we actually assume that they play the semi-Markov game with the extended state space $S \times [0,1]$. The law of motion, say $\overline{q}$, in the extended state space model is obviously the product of the original law of motion q and the uniform distribution η on $[0,1]$. More precisely, for any $s \in S$, $\xi \in [0,1]$, $a \in A(s)$, any set $C \in \Sigma$ and any Borel measurable set $D \subseteq [0,1]$, $\overline{q}(C \times D \mid s, \xi, a) = q(C \mid s, a)\eta(D)$.

For any multi-strategy $\pi = (\pi_1, \ldots, \pi_N)$ of the players, the limiting expected average payoff to player k is a function of the initial state s_1 and the first signal ξ_1 and is denoted by $J_k(s_1, \xi_1, \pi)$.

We say that $f^* = (f_1^*, \ldots, f_N^*)$ is a *Nash equilibrium for the average payoff semi-Markov game in the class of extended strategies* if for each initial state $s_1 \in S$,

$$\int_0^1 J_k(s_1, \xi_1, f^*)\, \eta(d\xi_1) \geq \int_0^1 J_k(s_1, \xi_1, (f^*_{-k}, \pi_k))\, \eta(d\xi_1)$$

for every player k and any extended strategy π_k.

A Nash equilibrium in extended strategies is also called a *correlated equilibrium with public signals*. The reason is that after the outcome of any period of the game, the players can coordinate their next choices by exploiting the next (known to all of them, i.e. public) signal and using some coordination mechanism telling which (pure or mixed) action is to be played by each of them. In many applications, we are particularly interested in stationary equilibria. In such a case the coordination mechanism can be represented by a family of $N+1$ measurable functions $\lambda^1, \ldots, \lambda^{N+1} : S \to [0,1]$ such that $\sum_{i=1}^{N+1} \lambda^i(s) = 1$ for every $s \in S$. (We remind the reader that N is the number of players. The number $N+1$ appears in our definition because Carathéodory's theorem [4] is applied in the proof of our main result.) A stationary Nash equilibrium in the class of extended strategies can then be constructed by using a family of $N+1$ stationary strategies $f_k^1, \ldots, f_k^{N+1}$, given for each player k, and the following coordination rule. If the game is at a state s on the nth stage and a random number ξ_n is selected, then each player k is suggested to use $f_k^m(\cdot \mid s)$, where m is the least index for which $\sum_{i=1}^m \lambda^i(s) \geq \xi_n$. An extended stationary strategy f_k^* for each player k can be defined as follows:

$$(1) \qquad f_k^*(\cdot \mid s, \xi) = \begin{cases} f_k^1(\cdot \mid s) & \text{if } \xi \leq \lambda^1(s),\ s \in S, \\ f_k^m(\cdot \mid s) & \text{if } \sum_{i=1}^{m-1} \lambda^i(s) < \xi \leq \sum_{i=1}^m \lambda^i(s), \end{cases}$$

for $s \in S$, $2 \leq m \leq N+1$. Because the signals are independent and uniformly distributed in $[0,1]$, it follows that at any period of the game and for any current state s, the number $\lambda^i(s)$ can be interpreted as the probability that player k is suggested to use $f_k^i(\cdot \mid s)$ as his mixed action. It turns out (under our assumptions) that a multi-strategy $(f_1^*, \ldots, f_N^*)$ obtained by the above construction is a stationary Nash equilibrium in the class of extended strategies of the players in a game iff no player k can unilaterally improve upon his expected average payoff by changing any of his strategies f_k^i, $i = 1, \ldots, N+1$. A formal proof of this result is given in Section 2.

In the proofs we shall use the following notation. If $w : S \to \mathbb{R}$ is a bounded measurable function, $f = (f_1, \ldots, f_N)$ is a stationary (extended) multi-strategy and $s \in S$, $\xi \in [0,1]$, then

$$w(s,\xi,f) = \int \ldots \int w(s, x^1, \ldots, x^N)\, f_1(dx^1 \mid s, \xi) \ldots f_N(dx^N \mid s, \xi). \tag{2}$$

Clearly, $w(s, f)$ is defined in a similar way for a stationary multi-strategy f (which is independent of ξ).

2. Main results. We start with an assumption which implies the strong ergodicity property of the transition structure of the game.

A4: There exist a constant $\delta > 0$ and a probability measure ν on S such that

$$q(D \mid s, x) \geq \delta \nu(D),$$

for every $s \in S$, $x \in X$ and for each measurable subset D of S.

Condition A4 was often used in stochastic dynamic programming [6, 10, 23] and is satisfied in replacement models [10]. Bielecki [2] and Küenle [9] used condition A4 to study ε-equilibria in stochastic Markov games.

THEOREM 1. *Let* A1 *through* A4 *be satisfied. Then the semi-Markov game has a stationary Nash equilibrium in the class of extended strategies.*

Proof. Fix any $B > 0$ such that $0 \leq r_k(s,x) \leq B$ for every k and $s \in S$, $x \in X$. Define V to be the space of all $(\mu+\nu)$-equivalence classes of nonnegative measurable functions $w : S \to \mathbb{R}$ such that $w(s) \leq B$, $(\mu+\nu)$-a.e. Let $L_i^\infty = L_i^\infty(S, \Sigma, \mu_i)$ be the space of all μ_i-essentially bounded measurable functions $\phi : S \to \mathbb{R}$ ($\mu_1 = \mu$ and $\mu_2 = \nu$). Similarly, we put $L_i^1 = L_i^1(S, \Sigma, \mu_i)$. Assume that V is endowed with the topology $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2$ where $\mathcal{T}_i$ is the weak-star topology $\sigma(L_i^\infty, L_i^1)$. It is well known that V is a compact space and since Σ is countably generated the topology on V is metrizable [5]. Let $U = V \times \ldots \times V$ (N times). Assume that U is given the product topology.

Choose $c > 0$ such that $cb \leq \delta$ (recall A3). Then

$$q(D \mid s, x) \geq cT(s,x)\nu(D) \tag{3}$$

for every $s \in S$, $x \in X$ and $D \in \Sigma$. With any $s \in S$ and $u = (u_1, \ldots, u_N) \in U$, we associate the nonzero-sum (static) game $\Gamma_u(s)$ in which the payoff function to player k (defined on $A(s)$) is

$$car_k(s,\cdot) + \int_S u_k(y)\, q(dy \mid s, \cdot) - cT(s,\cdot) \int_S u_k(y)\, \nu(dy).$$

Let $coP_u(s)$ be the set of all convex combinations of Nash equilibrium payoffs in the game $\Gamma_u(s)$. It turns out that $s \mapsto coP_u(s)$ is a weakly measurable compact valued mapping (see Section 4 of [19]). By M_u we denote the set of all $(\mu+\nu)$-equivalence classes of measurable selectors of coP_u. By the Kuratowski and Ryll-Nardzewski measurable selection theorem [11], M_u is nonempty (see also Lemma 7 of [19]). From our definition of the constant c and (3), it follows that $M_u \subset U$. Moreover, the correspondence $u \mapsto M_u$ is upper semicontinuous (has a closed graph). We have to show that if $u_n \to u_0$ in U, $w_n \in M_{u_n}$ for each n and $w_n \to w_0$ in U, then $w_0 \in M_{u_0}$. The proof of this fact is similar to that of Lemma 7 of [19]. The crucial part is that (by Mazur's theorem [5]) there exists a sequence of convex combinations of the functions w_n which converges $(\mu+\nu)$-a.e. By the Kakutani–Glicksberg fixed point theorem [8], there exists some $v = (v_1, \ldots, v_N) \in M_v$. Using a "random version" of Carathéodory's theorem [4] and the Kuratowski and Ryll-Nardzewski measurable selection theorem [11], it is possible to find a family of $N+1$ measurable functions $\lambda^1, \ldots, \lambda^{N+1} : S \to [0,1]$ such that $\sum_{i=1}^{N+1} \lambda^i(s) = 1$ for every $s \in S$ and a family of $N+1$ stationary strategies $f_k^1, \ldots, f_k^{N+1}$ for every player k such that the extended stationary strategies $f^*(\cdot \mid s, \xi)$ defined by (1) form a Nash equilibrium in the extended one-stage game $\Gamma_v(s)$ allowing for "public communication" using signals $\xi \in [0,1]$. The formal proof of this fact is similar to that of the Equilibrium Theorem in [19].

For every player k, $s \in S$ and $\xi \in [0,1]$, we have (recall (2))

$$v_k(s) = car_k(s,\xi,f^*) + \int_S v_k(y)\,\overline{q}(dy \mid s,\xi,f^*) - T(s,\xi,f^*)j_k \tag{4}$$

where $f^* = (f_1^*, \ldots, f_N^*)$ and

$$j_k = c\int_S v_k(y)\,\nu(dy).$$

Fix player k. Equation (4) and the fact that $f_1^*(\cdot \mid s,\xi), \ldots, f_N^*(\cdot \mid s,\xi)$ form a Nash equilibrium in the (extended) game $\Gamma_v(s)$ mean that the so-called optimality equation is satisfied for the corresponding semi-Markov control process with player k as the controller. Standard iteration arguments and this optimality equation (see Theorem 7.6 of [23] or [10]) give

$$\int_0^1 J_k'(s_1,\xi_1,f^*)\,\eta(d\xi_1) = \sup_{\pi_k} \int_0^1 J_k'(s_1,\xi_1,(f_{-k}^*,\pi_k))\,\eta(d\xi_1), \tag{5}$$

where J_k' is the expected average payoff function corresponding to the immediate payoff function car_k. Since $ca > 0$, (5) implies that f^* is also a Nash equilibrium (in the class of all extended strategies) for the original semi-Markov game. ■

Our second result is a counterpart of a theorem by Parthasarathy and Sinha [21] proved for discounted Markov games. We return to the standard model without assuming any communication device.

THEOREM 2. *Assume that* $A_k(s) = X_k$ *for each* $s \in S$ *and* X_k *is a finite set for every player* k. *Assume that* q *is independent of* $s \in S$, *that is*, $q(\cdot \mid s, x) = q(\cdot \mid x)$ *for all* $s \in S$ *and* $x \in X$. *If moreover* A3 *and* A4 *are satisfied and the probability measures* $q(\cdot \mid x)$ *and* ν *are nonatomic*, *then the semi-Markov game has a stationary Nash equilibrium.*

Proof. Let $P_u(s)$ be the set of all Nash equilibrium payoffs in the game $\Gamma_u(s)$ defined above for $u \in U$. From the proof of Theorem 1, we infer that there exists some vector-valued measurable function $v = (v_1, \ldots, v_N)$ such that $v(s) \in coP_v(s)$ for all $s \in S$. Using Lyapunov's theorem, one can prove that there exists some measurable function $u = (u_1, \ldots, u_N)$ such that $u(s) \in P_v(s)$ for all $s \in S$ and

$$\int_S u_k(y)\, q(dy|x) = \int_S v_k(y)\, q(dy \mid x) \tag{6}$$

for each $x \in X$ and

$$\int_S u_k(y)\, \nu(dy) = \int_S v_k(y)\, \nu(dy). \tag{7}$$

For the details consult for example [1]. (Here we use our assumption that X is finite and $q(\cdot \mid x)$ and ν are nonatomic.) From (6) and (7), it follows that the games $\Gamma_v(s)$ and $\Gamma_u(s)$ have identical payoff functions and consequently $P_u(s) = P_v(s)$ for each $s \in S$. This and the fact that $u(s) \in P_v(s)$ for all $s \in S$ imply that

$$u_k(s) = car_k(s, g^*) + \int_S u_k(y)\, q(dy \mid s, g^*) - T(s, g^*) j_k \tag{8}$$

for some stationary Nash equilibrium strategies $g_i^*(\cdot \mid s)$, $g^* = (g_1^*, \ldots, g_N^*)$ and

$$j_k := c \int_S u_k(y)\, \nu(dy).$$

Of course, to get g^* a measurable selection theorem must be applied (similar issues are considered in [19]). Combining (8) with standard results on the optimality equation for semi-Markov control processes [23, 10], we infer that g^* is a stationary Nash equilibrium in the game with the immediate payoffs car_k and thus in the original semi-Markov game. ■

REMARK. Assume that

$$\beta(s, x, y) = \int_0^\infty e^{-\alpha t}\, dF(t \mid s, x, y) \le 1 - \varepsilon \tag{9}$$

for some $\alpha > 0$, $\varepsilon > 0$ and for all $(s, x, y) \in S \times X \times S$. Assume also that $\beta(s, \cdot, y)$ is continuous on X. Then one can consider the semi-Markov games in which the payoffs are discounted by the discount function β given by (9) (see [14] or [22]). We remark that results closely related to Theorems 1 and 2 can be stated for discounted semi-Markov games satisfying the above assumptions. Condition A4 can be dropped in that case. The proofs can be based on the methods developed in [19, 21] and standard results on discounted semi-Markov control processes [10, 23].

3. Concluding remarks. The existence of stationary Nash equilibria in general classes of stochastic Markov (or more general semi-Markov) games with uncountable state spaces is a challenging open problem. In this paper, we provide a first result on correlated equilibria in the semi-Markov setting. Our ergodicity assumption A4 is rather strong. However, the equilibrium strategies obtained in such an approach are "pathwise optimal" (see [10] for a definition). This interesting property follows from our theorems and Corollary 2.1 of [10]. Some results on the existence of Nash equilibria in semi-Markov games with a metric state space are stated in [12, 13]. However, the proofs given there are erroneous. They are based on a sequential compactness argument in a space of measurable functions which is completely incorrect.

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