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GOODNESS-OF-FIT TESTS BASED ON CHARACTERIZATIONS OF CONTINUOUS DISTRIBUTIONS

Abstract. We construct goodness-of-fit tests for continuous distributions using their characterizations in terms of moments of order statistics and moments of record values. Our approach is based on characterizations presented in [2]–[4], [5], [9].

1. Introduction. Let $(X_1, \ldots, X_n)$ be a random sample from a distribution $F(x) = P[X \le x]$, $x \in \mathbb{R}$, and let $X_{k:n}$ denote the kth smallest order statistic of the sample. In what follows we use the following characterizations of continuous distributions via moments of functions of order statistics.

THEOREM 1 (cf. [9]). *Let m be a positive integer and $EX_{k:n}^2 < \infty$ for some pair (k,n). Then*

$$\frac{(k-1)!}{n!}EX_{k:n}^2 - 2\frac{(k+m-1)!}{(n+m)!}EX_{k+m:n+m} + \frac{(k+2m-1)!}{(n+2m)!} = 0$$

iff $F(x) = x^{1/m}$ on $(0,1)$.

Taking $k = n = 1$, we get

COROLLARY 1. *$F(x) = x^{1/m}$ on $(0,1)$ iff*

$$\frac{2}{m+1}EX_{m+1:m+1} - EX^2 = \frac{1}{2m+1}.$$

In particular, $X \sim U(0,1)$ iff $EX_{2:2} - EX^2 = 1/3$.

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In the following theorems, X denotes a random variable with distribution F, and $I(F)$ denotes the minimal interval containing the support of F.

THEOREM 2 (cf. [5]). *Let n, k, l be given integers such that $n \geq k \geq l \geq 1$. Assume that G is a nondecreasing right-continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then the relations*

$$EG^l(X_{k+1:n+1}) = \frac{(k+1)\ldots(k+l)}{(n+2)\ldots(n+l+1)},$$
$$EG^{2l}(X_{k+1-l:n+1-l}) = \frac{(k-l+1)\ldots(k+l)}{(n-l+2)\ldots(n+l+1)}$$

hold iff $F(x) = G(x)$ on $I(F)$ and F is continuous on $\mathbb{R}$.

Taking $n = k = l = 1$, we get

COROLLARY 2. *$F(x) = G(x)$ on $I(F)$ and F is continuous on $\mathbb{R}$ iff $EG^2(X) = 1/3$ and $EG(X_{2:2}) = 2/3$.*

THEOREM 3 (cf. [2], [3]). *Under the assumptions of Theorem 2, $F(x) = G(x)$ on $I(F)$ and F is continuous on $\mathbb{R}$ iff*

$$\frac{(k-l)!}{(n-l+1)!}EG^{2l}(X_{k+1-l:n+1-l}) - \frac{2k!}{(n+1)!}EG^l(X_{k+1:n+1}) + \frac{(k+l)!}{(n+l+1)!} = 0.$$

Taking $n = k = l = 1$, we get

COROLLARY 3. *$F(x) = G(x)$ on $I(F)$ and F is continuous on $\mathbb{R}$ iff*

$$EG(X_{2:2}) - EG^2(X) = 1/3. \tag{1.1}$$

Before quoting characterization theorems in terms of moments of record values we give the definition of k-record values (cf. [1]).

Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables with cdf F and pdf f. For a fixed $k \geq 1$ we define the sequence $U_k(1), U_k(2), \ldots$ of k-(upper) record times of $X_1, X_2, \ldots$ as follows: $U_k(1) = 1$, and for $n = 2, 3, \ldots$,

$$U_k(n) = \min\{j > U_k(n-1) : X_{j:j+k-1} > X_{U_k(n-1):U_k(n-1)+k-1}\}.$$

Write

$$Y_n^{(k)} := X_{U_k(n):U_k(n)+k-1}, \qquad n \geq 1.$$

The sequence $\{Y_n^{(k)},\ n \geq 1\}$ is called the sequence of k-(upper) *record values* of the above sequence. For convenience we also take $Y_0^{(k)} = 0$ and note that $Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$.

We shall apply the following characterization results:

THEOREM 4 (cf. [3], [4]). *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf F. Assume that G is a nondecreasing right-continuous* *function from* $\mathbb{R}$ *to* $(-\infty, 1]$, *and let* n, k, l *be given integers such that* $k \geq 1$ *and* $n \geq l \geq 1$. *Then* $F(x) = G(x)$ *on* $I(F)$ *iff the following relations hold*:

$$E[-\log(1 - G(Y_{n+1}^{(k)}))]^l = \frac{(n+l)!}{n!k!},$$

$$E[-\log(1 - G(Y_{n-l}^{(k)}))]^{2l} = \frac{(n+l)!}{(n-l)!k^{2l}}.$$

THEOREM 4′ (cf. [3], [4]). *Under the assumptions of Theorem* 4, $F(x) = G(x)$ *on* $I(F)$ *iff*

$$k^{2l}(n-l)!EH_l^2(Y_{n-l+1}^{(k)}) - 2n!k^l EH_l(Y_{n+1}^{(k)}) + (n+l)! = 0,$$

where $H_l(x) = (-\log(1 - G(x)))^l$, $x \in \mathbb{R}$.

In particular X *has df* F *iff*

$$E[-\log(1 - F(Y_1^{(k)}))]^2 - \frac{2}{k}E[-\log(1 - F(Y_2^{(k)}))] + \frac{2}{k^2} = 0.$$

COROLLARY 4. (a) $F(x) = x^\alpha$ *on* $(0, 1)$, $\alpha > 0$, *iff*

$$E(-\log(1 - (Y_1^{(k)})^\alpha))^2 - \frac{2}{k}E(-\log(1 - (Y_2^{(k)})^\alpha)) + \frac{2}{k^2} = 0.$$

In particular, $X \sim U(0, 1)$ *iff*

$$E(-\log(1 - Y_1^{(k)}))^2 - \frac{2}{k}E(-\log(1 - Y_2^{(k)})) + \frac{2}{k^2} = 0.$$

(b) $F(x) = 1 - e^{-(1/\lambda)x^\alpha}$, $x > 0$, $\alpha > 0$, $\lambda > 0$, *iff*

$$E(Y_1^{(k)})^{2\alpha} - \frac{2\lambda}{k}E(Y_2^{(k)})^\alpha + \frac{2\lambda^2}{k^2} = 0.$$

In particular, $X \sim \mathrm{Exp}(1/\lambda)$, *i.e.* $F(x) = 1 - e^{-x/\lambda}$, *iff*

$$E(Y_1^{(k)})^2 - \frac{2\lambda}{k}EY_2^{(k)} + \frac{2\lambda^2}{k^2} = 0.$$

(c) $F(x) = 1 - (x_0/x)^a$, $x > x_0$, $a > 0$, *iff*

$$E\left[-\log\left(\frac{x_0}{Y_1^{(k)}}\right)\right]^2 - \frac{2}{ka}E\left[-\log\left(\frac{x_0}{Y_2^{(k)}}\right)\right] + \frac{2}{k^2a^2} = 0.$$

2. Goodness-of-fit tests based on characterizations via moments of order statistics.

First note that (1.1) can be written in the form

$$E(F(X_{2:2})) - \frac{1}{2}(E(F^2(X_1)) + E(F^2(X_2))) = \frac{1}{3}$$

as X_1 and X_2 are distributed as X.

Let $(X_1, \ldots, X_{2n})$ be a sample. Write

$$Y_j = F^2(X_{2j-1}) + F^2(X_{2j}),$$
$$Z_j = F(\max(X_{2j-1}, X_{2j})), \quad j = 1, \ldots, n.$$

Letting $Y := Y_1 = F^2(X_1) + F^2(X_2)$, $Z := Z_1 = F(\max(X_1, X_2))$, we quote the following result (cf. [6]).

LEMMA 1. *Under the above assumptions, the density of* (Y, Z) *is given by*

$$f(y,z) = \begin{cases} 1/\sqrt{y - z^2}, & 0 \le y \le 2,\ 0 \le z \le 1,\ z^2 \le y \le 2z^2, \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$EY = 2/3, \quad \mathrm{Var}(Y) = 8/45,$$
$$EZ = 2/3, \quad \mathrm{Var}(Z) = 1/18, \quad \mathrm{Cov}(Y, Z) = 4/45.$$

Put

$$D_j = Z_j - \frac{1}{2} Y_j, \quad j = 1, \ldots, n.$$

We see that

$$ED_j = EZ_j - \frac{1}{2} EY_j = \frac{1}{3},$$
$$\mathrm{Var}\, D_j = \mathrm{Var}\, Z_j + \frac{1}{4}\mathrm{Var}\, Y_j - \mathrm{Cov}(Z_j, Y_j) = \frac{1}{90}, \quad j = 1, \ldots, n.$$

Now define

$$V_n = 3\sqrt{10n}(\overline{D_n} - 1/3),$$

where $\overline{D_n} = (1/n)\sum_{j=1}^n D_j$.

Setting $X_j^* = \max(X_{2j-1}, X_{2j})$, $j = 1, \ldots, n$, we note that V_n can be written as

$$V_n = 3\sqrt{10n}\left(\frac{1}{n}\sum_{j=1}^{n} F(X_j^*) - \frac{1}{2n}\sum_{j=1}^{2n} F^2(X_j) - \frac{1}{3}\right).$$

Taking into account that

$$X_j^* = (X_{2j-1} + X_{2j})/2 + |X_{2j} - X_{2j-1}|/2$$

and writing

$$X_j^0 = (X_{2j-1} + X_{2j})/2, \quad X_j^+ = |X_{2j} - X_{2j-1}|/2$$

we obtain

$$V_n = 3\sqrt{10n}((\overline{F(X_n^0 + X_n^+)} - \overline{F^2(X_{2n})}) - 1/3),$$

where

$$\overline{F(X_n^0 + X_n^+)} = \frac{1}{n}\sum_{j=1}^{n} F(X_j^0 + X_j^+),$$

$$\overline{F^2(X_{2n})} = \frac{1}{2n}\sum_{j=1}^{2n} F^2(X_j).$$

Moreover, we conclude from the CLT that

$$V_n \xrightarrow{D} V \sim N(0,1), \tag{2.1}$$

and hence that

$$V_n^2 \xrightarrow{D} \chi^2(1),$$

which provides a simple asymptotic test of the hypothesis $X \sim F$ when the parameters of F are specified.

Special cases:

(a) If $F(x) = x^{1/m}$, $x \in (0,1)$, m is a positive integer, then

$$V_n = 3\sqrt{10n}\bigg(\frac{1}{n}\sum_{j=1}^{n}((X_{2j} + X_{2j-1} + |X_{2j} - X_{2j-1}|)/2)^{1/m} - \frac{1}{2n}\sum_{j=1}^{2n} X_j^{2/m} - \frac{1}{3}\bigg).$$

In particular, for $X \sim U(0,1)$,

$$V_n = 3\sqrt{10n}(\overline{X_{2n}} + \overline{X_n^+} - \overline{X_{2n}^2} - 1/3).$$

(b) If $F(x) = x/\beta$, $x \in (0,\beta)$, $\beta > 0$, then

$$V_n = \frac{3\sqrt{10n}}{\beta}\bigg(\overline{X_{2n}} + \overline{X_n^+} - \frac{1}{\beta}\overline{X_{2n}^2} - \frac{\beta}{3}\bigg),$$

(c) If $F(x) = \frac{x-\alpha}{\beta-\alpha}$, $x \in (\alpha,\beta)$, then

$$V_n = \frac{3\sqrt{10n}}{\beta-\alpha}\bigg(\frac{\beta+\alpha}{\beta-\alpha}\overline{X_{2n}} + \overline{X_n^+} - \frac{1}{\beta-\alpha}\overline{X_{2n}^2} - \frac{\alpha\beta}{\beta-\alpha} - \frac{\beta-\alpha}{3}\bigg).$$

(d) If $F(x) = 1 - e^{-(1/\lambda)x^\alpha}$, $x \geq 0$, $\alpha > 0$, $\lambda > 0$, then

$$V_n = 3\sqrt{10n}\bigg(\frac{1}{n}\sum_{j=1}^{n}(1 - \exp(-(X_{2j} + X_{2j-1} + |X_{2j} - X_{2j-1}|)^\alpha/(2^\alpha\lambda))) - \frac{1}{2n}\sum_{j=1}^{2n}(1 - \exp(-X_j^\alpha/\lambda))^2 - \frac{1}{3}\bigg).$$

In particular, for $X \sim \mathrm{Exp}(1/\lambda)$,

$$V_n = 3\sqrt{10n}\bigg(\frac{1}{n}\sum_{j=1}^{n}(1-\exp(-(X_{2j}+X_{2j-1}+|X_{2j}-X_{2j-1}|)/(2\lambda))) - \frac{1}{2n}\sum_{j=1}^{2n}(1-\exp(-X_j/\lambda))^2 - \frac{1}{3}\bigg),$$

(e) If $F(x) = 1-(x_0/x)^a$, $x \ge x_0$, $a > 0$, then

$$V_n = 3\sqrt{10n}\bigg(\frac{1}{n}\sum_{j=1}^{n}\bigg(1-\bigg(\frac{2x_0}{X_{2j-1}+X_{2j}+|X_{2j}-X_{2j-1}|}\bigg)^a\bigg) - \frac{1}{2n}\sum_{j=1}^{2n}\bigg(1-\bigg(\frac{x_0}{X_j}\bigg)^a\bigg)^2 - \frac{1}{3}\bigg).$$

From (2.1) we see that in each special case V_n converges weakly to the standard normal distribution, and so provides an asymptotic test of the hypothesis H that X has df F in the case when the parameter values are specified by H. When H does not specify the parameter values we consider the test statistic obtained from V_n by replacing the parameters by estimators. In this case we have the following results.

Proposition 1. *When* $F(x) = x/\beta$, $x \in (0,\beta)$, $\beta > 0$, *the resulting test statistic is*

$$V_n(\widehat{\beta}_n) := \frac{3\sqrt{10n}}{\widehat{\beta}_n}\bigg(\overline{X_{2n}} + \overline{X_n^+} - \frac{1}{\widehat{\beta}_n}\overline{X_{2n}^2} - \frac{\widehat{\beta}_n}{3}\bigg) \xrightarrow{D} V \sim N(0,1),$$

where $\widehat{\beta}_n = \max(X_1,\ldots,X_{2n})$.

Proof. We write

$$V_n(\widehat{\beta}_n) = \frac{\beta}{\widehat{\beta}_n}\bigg(\frac{3\sqrt{10n}}{\beta}\bigg(\overline{X_{2n}} + \overline{X_n^+} - \frac{1}{\beta}\overline{X_{2n}^2} - \frac{\beta}{3}\bigg)\bigg) - \frac{3\sqrt{10n}}{\widehat{\beta}_n}\bigg(\frac{1}{\widehat{\beta}_n} - \frac{1}{\beta}\bigg)\overline{X_{2n}^2} - \frac{\sqrt{10n}}{\widehat{\beta}_n}(\widehat{\beta}_n - \beta).$$

Note that

$$-\frac{\sqrt{10n}}{\widehat{\beta}_n}\bigg(\frac{1}{\widehat{\beta}_n} - \frac{1}{\beta}\bigg)\overline{X_{2n}^2} = \frac{\sqrt{10n}(\widehat{\beta}_n - \beta)}{\widehat{\beta}_n^2\beta}\overline{X_{2n}^2} \xrightarrow{P} 0,$$

as

$$\overline{X_{2n}^2} \xrightarrow{P} EX^2 \quad \text{and} \quad 2n(\beta - \widehat{\beta}_n) \xrightarrow{D} W \sim \mathrm{Exp}(1/\beta).$$

The assertion then follows from Slutsky's theorem.

PROPOSITION 2. *When* $F(x) = \frac{x-\alpha}{\beta-\alpha}$, $x \in (\alpha, \beta)$, *the resulting test statistic is*

$$V_n(\widehat{\alpha}_n, \widehat{\beta}_n) = \frac{3\sqrt{10n}}{\widehat{\beta}_n - \widehat{\alpha}_n}\bigg(\frac{\widehat{\beta}_n + \widehat{\alpha}_n}{\widehat{\beta}_n - \widehat{\alpha}_n}\overline{X_{2n}} + \overline{X_n^+} - \frac{1}{\widehat{\beta}_n - \widehat{\alpha}_n}\overline{X_{2n}^2} - \frac{\widehat{\alpha}_n\widehat{\beta}_n}{\widehat{\beta}_n - \widehat{\alpha}_n} - \frac{\widehat{\beta}_n - \widehat{\alpha}_n}{3}\bigg) \xrightarrow{D} V \sim N(0,1),$$

where $\widehat{\beta}_n = \max(X_1, \ldots, X_{2n})$ *and* $\widehat{\alpha}_n = \min(X_1, \ldots, X_{2n})$.

Proof. The proof is similar to the proof of Proposition 1, since

$$2n(\widehat{\alpha}_n - \alpha) \xrightarrow{D} W_1 \sim \mathrm{Exp}(1/\alpha), \quad 2n(\beta - \widehat{\beta}_n) \xrightarrow{D} W_2 \sim \mathrm{Exp}(1/\beta),$$
$$\overline{X_{2n}} \xrightarrow{P} EX, \quad \overline{X_{2n}^2} \xrightarrow{P} EX^2.$$

REMARK. From the above proof we see that one can use estimators $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ such that

$$\sqrt{n}(\widehat{\alpha}_n - \alpha) \xrightarrow{P} 0 \quad \text{and} \quad \sqrt{n}(\widehat{\beta}_n - \beta) \xrightarrow{P} 0.$$

NOTE. It appears that a similar result holds when $X \sim \mathrm{Exp}(1/\lambda)$, but the proof is too long for inclusion here.

3. Goodness-of-fit tests based on characterizations via moments of record values. From Corollary 4(b) we know that $X \sim \mathrm{Exp}(1/\lambda)$ iff

$$E(Y_1^{(k)})^2 - \frac{2\lambda}{k}EY_2^{(k)} + \frac{2\lambda^2}{k^2} = 0.$$

Consider the case $\lambda = 1$. Then we see that $X \sim \mathrm{Exp}(1)$ iff

$$E(\min(X_1, \ldots, X_k))^2 - \frac{2}{k}EY_2^{(k)} + \frac{2}{k^2} = 0. \tag{3.1}$$

The idea is to use the sample to obtain an estimate, θ_n say, of the expected value of $(Y_1^{(k)})^2 - (2/k)Y_2^{(k)} + 2/k^2$ and reject H if θ_n^2 is large. Since record values are defined in terms of an infinite sequence, it is not clear how one can get estimates of the associated expected values from a finite sample. But they can be estimated indirectly here because when H is true then for each k,

$$EY_2^{(k)} = EX_{1:k} + \frac{1}{k} \quad \text{(cf. [7], [8])},$$

and so (3.1) has the form

$$E\bigg(X_{1:k}^2 - \frac{2}{k}X_{1:k}\bigg) = 0. \tag{3.2}$$

Now suppose that $X \sim \mathrm{Exp}(1/\lambda)$. Since $X \sim \mathrm{Exp}(1/\lambda) \Leftrightarrow X/\lambda \sim \mathrm{Exp}(1)$, it follows from (3.2) that

$$E\left(X_{1:k}^2 - \frac{2\lambda}{k} X_{1:k}\right) = 0. \tag{3.2$'$}$$

Consider first the case $k = 1$. Then

$$E(X_1^2 - 2\lambda X_1) = 0.$$

The sample $(X_1, \ldots, X_n)$ provides an estimator of EW_1, where $W_1 = X_1^2 - 2\lambda X_1$, of the form

$$\overline{W_n} = \overline{X_n^2} - 2\lambda\overline{X_n},$$

where $\overline{X_n^2} = (1/n)\sum_{j=1}^n X_j^2$. It follows from the CLT that

$$\sqrt{n}\,\overline{W_n} \xrightarrow{D} W \sim N(0, \mathrm{Var}(W_1)),$$

and hence that

$$T_n^{(1)}(\lambda) := n(\overline{W_n}\,)^2/\mathrm{Var}(W_1) \xrightarrow{D} \chi^2(1),$$

which provides a simple asymptotic test of the hypothesis $X \sim \mathrm{Exp}(1/\lambda)$ when λ is specified. Here

$$\mathrm{Var}(W_1) = EX_1^4 - 4\lambda EX_1^3 + 4\lambda^2 EX_1^2 = 8\lambda^4$$

since $X_1 \sim \mathrm{Exp}(1/\lambda)$ gives $EX_1^m = m!\lambda^m$, $m = 1, 2, \ldots$, and so

$$T_n^{(1)}(\lambda) = \frac{n}{8}\left(\frac{1}{\lambda^2}\overline{X_n^2} - \frac{2}{\lambda}\overline{X_n}\right)^2. \tag{3.3}$$

Thus we have proved

Proposition 3. *If* $X_n \sim \mathrm{Exp}(1/\lambda)$, $n \ge 1$, *are independent then*

$$T_n^{(1)}(\lambda) = \frac{n}{8}\left(\frac{1}{\lambda^2}\overline{X_n^2} - \frac{2}{\lambda}\overline{X_n}\right)^2 \xrightarrow{D} \chi^2(1).$$

Now consider the case $k = 2$. Write $U_1 := X_{1:2} = \min(X_1, X_2)$. Here from (3.2$'$) we have to estimate EW_1', where $W_1' = U_1^2 - \lambda U_1$. The sample $X_1, \ldots, X_{2n}$ provides the sample $W_1', \ldots, W_n'$, where $W_j' = U_j^2 - \lambda U_j$ and $U_j = \min(X_{2j-1}, X_{2j})$, $j = 1, \ldots, n$. Then EW_1' is estimated by

$$\overline{W_n'} = \overline{U_n^2} - \lambda\overline{U_n},$$

and

$$T_n^{(2)}(\lambda) := n(\overline{W_n'})^2/\mathrm{Var}(W_1') \xrightarrow{D} \chi^2(1).$$

Taking into account that $U_1 \sim \mathrm{Exp}(2/\lambda)$ we see that $\mathrm{Var}(W_1') = \lambda^4/2$. Thus another simple asymptotic test is provided by

PROPOSITION 4. *If* $X_n \sim \mathrm{Exp}(1/\lambda)$, $n \geq 1$, *are independent then*

$$T_n^{(2)}(\lambda) = \frac{2n}{\lambda^4}(\overline{U_n^2} - \lambda\overline{U_n})^2 \xrightarrow{D} \chi^2(1). \tag{3.4}$$

The same argument leads to a similar test for the case $k = 3, \ldots, n-1$ based on a sample of size kn.

We now consider the case $k = n$. Write $U_n = \min(X_1, \ldots, X_n)$. Then by (3.2′) we have to estimate $E(U_n^2 - (2\lambda/n)U_n)$. The obvious estimate is $U_n^2 - (2\lambda/n)U_n$ itself, and then when λ is specified the test statistic is

$$T_n^{(n)}(\lambda) := \left(U_n^2 - \frac{2\lambda}{n}U_n\right)^2.$$

As above, under H, $U_n \sim \mathrm{Exp}(n/\lambda)$, whence

$$U := \frac{n}{\lambda}U_n \sim \mathrm{Exp}(1), \quad n \geq 1. \tag{3.5}$$

It follows that

$$T_n^{(n)}(\lambda) = \frac{\lambda^4}{n^4}(U^2 - 2U)^2$$

and so an equivalent test statistic is $T := (U^2 - 2U)^2$, which provides an exact test for $H:\ X \sim \mathrm{Exp}(1/\lambda)$.

PROPOSITION 5. *The significance probability of the test using* T *is*

$$P_t := P[T > t] = \begin{cases} e^{-1-\sqrt{1+\sqrt{t}}} & \text{if } t > 1, \\ e^{-1-\sqrt{1+\sqrt{t}}} + e^{-1+\sqrt{1-\sqrt{t}}} - e^{-1-\sqrt{1-\sqrt{t}}} & \text{if } 0 < t < 1. \end{cases}$$

Proof. The first statement is obtained from the positive root of the equation $u^2 - 2u - \sqrt{t} = 0$, and the second from the positive roots of the equation $(u^2 - 2u)^2 = t$.

In particular we consider the 5% test of H, i.e. $P_t = 0.05$. But since

$$P[T > 1] = e^{-(1+\sqrt{2})} > 0.05,$$

the 5% test rejects when $U > x_0$, where $e^{-x_0} = 0.05$, i.e. when $x_0 = 3.00$. Thus the exact 5% test rejects when $(n/\lambda)U_n > 3$.

We now consider corresponding tests when λ is not specified. The general idea is to consider the statistics obtained by replacing λ in (3.3) and (3.4) by an estimate $\widehat{\lambda}_n$ obtained from the sample.

In this case we have the following results based on $T_n^{(1)}(\lambda)$ and $T_n^{(2)}(\lambda)$.

PROPOSITION 6. *When* $F(x) = 1 - e^{-x/\lambda}$, $x > 0$, $\lambda > 0$, *the resulting test statistic is*

$$2\widehat{T}_n^{(1)} := 2T_n^{(1)}(\widehat{\lambda}_n) = \frac{n}{4}(\overline{X_n^2}/(\overline{X_n})^2 - 2)^2 \xrightarrow{D} \chi^2(1),$$

where $\widehat{\lambda}_n = \overline{X_n}$.

PROPOSITION 7. *When* $F(x) = 1 - e^{-x/\lambda}$, $x > 0$, $\lambda > 0$, *the resulting statistic is*

$$\frac{4}{3}\widehat{T}_n^{(2)} := \frac{4}{3}T_n^{(2)}(\widehat{\lambda}_n) = \frac{8n}{3\widehat{\lambda}_n^4}(\overline{U_n^2} - \widehat{\lambda}_n\overline{U_n}\,)^2$$
$$= \frac{8n}{3}\left(\frac{\overline{U_n^2}}{(\overline{X_{2n}})^2} - \frac{\overline{U_n}}{\overline{X_{2n}}}\right)^2 \xrightarrow{D} \chi^2(1),$$

where $\widehat{\lambda}_n = \overline{X_{2n}}$.

Proof of Proposition 6. Consider $\mathbf{V} = \binom{X^2}{X}$. Then $\overline{\mathbf{V}_n} = \binom{\overline{X_n^2}}{\overline{X_n}}$ and by the CLT,

$$\sqrt{n}(\overline{\mathbf{V}_n} - \mu) \xrightarrow{D} N(0, \Sigma),$$

where

$$\mu = E\mathbf{V} = \begin{pmatrix} 2\lambda^2 \\ \lambda \end{pmatrix} \quad \text{and} \quad \Sigma = \mathrm{Var}(\mathbf{V}) = \begin{pmatrix} 20\lambda^4 & 4\lambda^3 \\ 4\lambda^3 & \lambda^2 \end{pmatrix}.$$

We now use a theorem on asymptotic distributions of functions of statistics (cf. [10], p. 260), with $g(\mathbf{x}) = x_1/x_2^2$. Then

$$g(\mu) = 2, \quad \gamma := \left(\frac{\partial g}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mu} = \begin{pmatrix} 1/\lambda^2 \\ -4/\lambda \end{pmatrix}, \quad \gamma'\Sigma\gamma = 4,$$

and so

$$\sqrt{n}(g(\overline{\mathbf{V}_n}) - g(\mu)) = \sqrt{n}(\overline{X_n^2}/(\overline{X_n})^2 - 2) \xrightarrow{D} W \sim N(0, 4),$$

and

$$2\widehat{T}_n^{(1)} = \frac{n}{4}(\overline{X_n^2}/(\overline{X_n}\,)^2 - 2)^2 \xrightarrow{D} \chi^2(1).$$

In the proof of Proposition 7 we shall apply the following

LEMMA 2. *Let* $X_1 \sim \mathrm{Exp}(1/\lambda)$, $X_2 \sim \mathrm{Exp}(1/\lambda)$ *be independent and put* $U := U_1 = \min(X_1, X_2)$, $Y = (X_1 + X_2)/2$. *Then the pdf of* U *and* Y *is*

$$h(u, y) = \frac{4}{\lambda^2}e^{-2y/\lambda}, \quad 0 < u < y, \ y > 0,$$

and

$$\mathrm{Cov}(U, Y) = \lambda^2/4, \quad \mathrm{Cov}(U^2, Y) = \lambda^3/2.$$

Proof of Proposition 7. We now consider

$$\mathbf{V} = \begin{pmatrix} U_1^2 \\ U_1 \\ (X_1 + X_2)/2 \end{pmatrix}, \quad \overline{\mathbf{V}_n} = \begin{pmatrix} \overline{U_n^2} \\ \overline{U_n} \\ \overline{X_{2n}} \end{pmatrix}.$$

By the CLT,

$$\sqrt{n}(\overline{\mathbf{V}_n} - \mu) \xrightarrow{D} N(\mathbf{0}, \Sigma),$$

where $\mu = E\mathbf{V}$ and $\Sigma = \operatorname{Var}\mathbf{V}$. Now using Lemma 2 we get

$$\mu = \begin{pmatrix} \lambda^2/2 \\ \lambda/2 \\ \lambda \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 5\lambda^4/4 & \lambda^3/2 & \lambda^3/2 \\ \lambda^3/2 & \lambda^2/4 & \lambda^2/4 \\ \lambda^3/2 & \lambda^2/4 & \lambda^2/2 \end{pmatrix}.$$

Using the above theorem of Wilks [10] with $g(\mathbf{x}) = x_1/x_3^2 - x_2/x_3$ we have

$$g(\mu) = 0, \qquad \gamma := \left(\frac{\partial g}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mu} = \begin{pmatrix} 1/\lambda^2 \\ -1/\lambda \\ -1/2\lambda \end{pmatrix}, \qquad \gamma' \Sigma \gamma = 3/8.$$

Thus

$$\sqrt{n}(\overline{U_n^2}/(\overline{X_{2n}}\,)^2 - \overline{U_n}/\overline{X_{2n}}\,) \xrightarrow{D} N(0, 3/8),$$

and so

$$\frac{8n}{3}(\overline{U_n^2}/(\overline{X_{2n}}\,)^2 - \overline{U_n}/\overline{X_{2n}}\,)^2 = \frac{4}{3}\widehat{T}_n^{(2)} \xrightarrow{D} \chi^2(1).$$

REMARK. Instead of the MLE $\widehat{\lambda}_n$ one could consider the corresponding estimate $\lambda_n^* := 2\overline{U_n}$ obtained from $U_1, \ldots, U_n$. But since $\operatorname{Var}(\lambda_n^*) > \operatorname{Var}(\widehat{\lambda}_n)$, one would expect intuitively that the resulting test would be in some respect poorer. This leads to

$$T_n^{*(2)} := \frac{n}{8}(\overline{U_n^2}/(\overline{U_n}\,)^2 - 2)^2.$$

Then it follows as in the discussion of Proposition 6 that

$$2T_n^{*(2)} \xrightarrow{D} \chi^2(1).$$

Referring to (3.5), in the case when $k = n$ we use the statistic $\widehat{U}_n = nU_n/\widehat{\lambda}_n$ where $\widehat{\lambda}_n = \overline{X_n}$. Consider the test that rejects when $\widehat{U}_n > 3$. Now $\widehat{U}_n = (\lambda/\widehat{\lambda}_n)U$ where $U = nU_n/\lambda \sim \operatorname{Exp}(1)$, and $\widehat{U}_n \xrightarrow{D} U$ since $\widehat{\lambda}_n \xrightarrow{P} \lambda$. Thus

$$\lim_{n\to\infty} P(\widehat{U}_n > 3) = P(U > 3) = 0.05$$

and so this is an asymptotic 5% test.

Moreover, we have

PROPOSITION 8. *Let* $\widehat{T}_n := (\widehat{U}_n^2 - 2\widehat{U}_n)^2$ *and let* $\widehat{P}_t := P[\widehat{T}_n > t]$ *stand for the associated significance probability. Then* $\lim_{n\to\infty} \widehat{P}_t = P_t$, *where* P_t *is given by Proposition* 5.

Proof. Since $\widehat{U}_n = (\lambda/\widehat{\lambda}_n)U$, we have

$$\widehat{T}_n = [(\lambda/\widehat{\lambda}_n)^2 U^2 - 2(\lambda/\widehat{\lambda}_n)U]^2 \xrightarrow{D} T,$$

which ends the proof.

4. Simulations. Here we consider tests of $\mathrm{Exp}(1/\lambda)$.

First note that some goodness-of-fit tests based on a characterization were also proposed in [6] where the $\chi^2(2)$ approximation was used. Here we observe that a long and complicated argument shows that

$$D_n(\widehat{\lambda}_n) = 45n\bigg[\frac{110}{19}\bigg(\frac{1}{2n}\sum_{j=1}^{2n}(1-\exp(-X_j/\widehat{\lambda}_n))^2 - \frac{1}{3}\bigg)^2 + 2\bigg(\frac{1}{n}\sum_{j=1}^{n}\exp(-(X_{2j}+X_{2j-1}+|X_{2j}-X_{2j-1}|)/(2\widehat{\lambda}_n)) - \frac{1}{3}\bigg)^2 - 4\bigg(\frac{1}{2n}\sum_{j=1}^{2n}(1-\exp(-X_j/\widehat{\lambda}_n))^2 - \frac{1}{3}\bigg) \times \bigg(\frac{1}{n}\sum_{j=1}^{n}\exp(-(X_{2j}+X_{2j-1}+|X_{2j}-X_{2j-1}|)/(2\widehat{\lambda}_n)) - \frac{1}{3}\bigg)\bigg] \to \chi^2(2), \quad \text{where } \widehat{\lambda}_n = \overline{X}_{2n}.$$

Simulation strongly confirms that indeed $D_n(\widehat{\lambda}_n) \xrightarrow{D} \chi^2(2)$, and so $D_n(\widehat{\lambda}_n)$ provides a simple test for $X \sim \mathrm{Exp}(1/\lambda)$. We see that $D_n(\widehat{\lambda}_n)$ differs from D_n of [6] by having leading coefficient $\frac{110}{19}$ instead of $\frac{5}{2}$.

The test statistics investigated here are: $D_n(\widehat{\lambda}_n)$,

$$V_n^2(\widehat{\lambda}_n) = 90n\bigg[\frac{1}{n}\sum_{j=1}^{n}(1-\exp(-(X_{2j}+X_{2j-1}+|X_{2j}-X_{2j-1}|)/(2\widehat{\lambda}_n))) - \frac{1}{2n}\sum_{j=1}^{2n}(1-\exp(X_j/\widehat{\lambda}_n))^2 - \frac{1}{3}\bigg]^2,$$

$\widehat{T}_n^{(1)}$, $\widehat{T}_n^{(2)}$, $T_n^{*(2)}$ and $\widehat{T}_n$ from Propositions 8 and 5.

Firstly, 2000 samples of size 20 were obtained from an exponential distribution and the 6 statistics evaluated for each sample, and tested for significance at the 10%, 5% and 1% levels approximately. For $D_n(\widehat{\lambda}_n)$ the $\chi^2(2)$ approximation was used, so that for the approximate 10% test the observed value is significant if it exceeds 4.605 etc. Then for $V_n^2(\widehat{\lambda}_n)$, $\widehat{T}_n^{(1)}$, $\widehat{T}_n^{(2)}$ and $T_n^{*(2)}$ the $\chi^2(1)$ approximation was used, and for $\widehat{T}_n$ the approximation obtained from Propositions 5 and 8. In each case the percentage of significant samples is shown in the table below. Then this was repeated for samples of size 40, 100, 200.

	n	$D_n(\widehat{\lambda}_n)$	$V_n^2(\widehat{\lambda}_n)$	$\widehat{T}_n^{(1)}$	$\widehat{T}_n^{(2)}$	$T_n^{*(2)}$	$\widehat{T}_n$
10%	20	8.1	11.7	3.2	3.6	1.4	10.0
	40	8.6	9.6	5.8	4.5	3.8	10.8
	100	9.6	8.7	7.8	6.5	5.5	9.6
	200	9.3	9.4	8.9	7.9	7.9	10.5
5%	20	3.6	5.7	2.0	2.5	0.8	4.6
	40	4.3	4.8	3.5	3.1	2.5	5.5
	100	5.0	4.1	3.8	3.8	2.9	5.3
	200	4.2	4.3	3.9	4.5	4.1	5.5
1%	20	0.8	0.9	0.7	0.8	0.4	1.0
	40	0.7	1.1	1.4	1.8	1.0	0.7
	100	1.2	0.9	1.4	1.7	1.3	1.3
	200	0.7	0.9	0.8	2.1	1.8	1.2

It appears that $\widehat{T}_n$ performs the best, followed by $D_n(\widehat{\lambda}_n)$ and $V_n^2(\widehat{\lambda}_n)$, and the other tests are poorer when n is small.

REMARK. The above statistics $D_n(\widehat{\lambda}_n)$ and $V_n^2(\widehat{\lambda}_n)$ are derived from the statistics

$$D_n = 45n\left[\frac{110}{19}\left(\overline{Y}_n - \frac{1}{3}\right)^2 + 2\left(\overline{Z}_n - \frac{2}{3}\right)^2 - 4\left(\overline{Y}_n - \frac{1}{3}\right)\left(\overline{Z}_n - \frac{2}{3}\right)\right],$$

$$V_n^2 = 90n\left(\overline{Z}_n - \frac{1}{2}\overline{Y}_n - \frac{1}{3}\right)^2,$$

respectively.

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