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U. BRONOWICKA-MIELNICZUK, J. MIELNICZUK and
T. PRZYBYSZ (Lublin)

EFFICIENCY OF CROPPING SYSTEM DESIGNS VIA BASE CONTRAST

Abstract. The present article is a continuation of previous papers by the same authors devoted to the efficiency of crop rotation experiments. We focus on plans distinguished by the cyclical pattern of the incidence matrix. For practical reasons, we slightly modify the efficiency coefficient. The relation between the resulting efficiency coefficients is examined. In addition, we provide a background material on crop rotation experiments.

1. Preliminaries. In the preliminary section, we establish conventions of notation, state the problem and give an outline of the paper.

1.1. *Notation.* Let $\mathcal{D}_1$ be a cyclic block design with v_1 treatments labelled $0, \ldots, v_1 - 1$ (cf. John [2]). The blocks of such a design are obtained by successive addition of unity to the elements of the initial block and reduction modulo v_1 if necessary. It may be that the same block is obtained after a certain number of steps, but our convention is that the initial block generates $v_1 - 1$ further blocks. We denote by r_1 the number of treatments attached to the initial block and by $\mathbf{N}_1$ the $v_1 \times v_1$ incidence matrix of such a proper design.

Let $\mathcal{D}_2$ be a cyclic design with a similar pattern, but with v_2 blocks, each of size r_2, and with incidence matrix $\mathbf{N}_2$.

Given the designs $\mathcal{D}_1$ and $\mathcal{D}_2$, we will focus our attention on block designs $\mathcal{D}^*$ with incidence matrix

$$\mathbf{N} = \begin{pmatrix} \mathbf{1}'_{\delta_2} \otimes \mathbf{N}_1 \\ \cdots\cdots \\ \mathbf{1}'_{\delta_1} \otimes \mathbf{N}_2 \end{pmatrix}. \tag{1}$$

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Since there is no restriction $v_1 = v_2$, the dimensions δ_1, δ_2 of the respective vectors of ones should be such that $\delta_2 v_1 = \delta_1 v_2$. The design $\mathcal{D}^*$ has the following parameters: $v = v_1 + v_2$, $b = \delta_2 v_1 = \delta_1 v_2$, $\mathbf{r}' = (\delta_2 r_1 \mathbf{1}'_{v_1}, \delta_1 r_2 \mathbf{1}'_{v_2})$ and $k = r_1 + r_2$. Its information matrix $\mathbf{C}$ can be partitioned into circulant submatrices as follows:

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_1 & \vdots & \mathbf{C}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{C}'_{12} & \vdots & \mathbf{C}_2 \end{pmatrix},$$

where the diagonal subblocks are square matrices of order v_1 and v_2, respectively. It will be convenient to use the symbols s_1, s_2 to denote row sums of the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$. Moreover, we let $\mathbf{C}^+ = [c^{ij}]$ be the Moore–Penrose generalized inverse of the information matrix and we denote by var_{ij} the variance of the elementary contrast with non-zero entries at positions specified by subscripts. Let us recall that

$$\mathrm{var}_{ij} = (c^{ii} + c^{jj} - 2c^{ij})\sigma^2, \tag{2}$$

where σ^2 is the variance of the error component in the model. For the sake of convenience, we shall omit this factor in further considerations.

Finally, let us introduce the following *base contrast*:

$$\boldsymbol{\gamma}'\boldsymbol{\alpha} = [\delta \mathbf{1}'_{v_1} \vdots - \mathbf{1}'_{v_2}]\, \boldsymbol{\alpha}, \tag{3}$$

expressing the overall comparison of the effects of competing cyclical systems. Here δ stands for the fraction v_2/v_1 and $\boldsymbol{\alpha}$ denotes the vector of treatment parameters in the model of the design $\mathcal{D}^*$.

1.2. *Efficiency factors.* In a series of papers (cf. [4], [5]), we considered the efficiency of the block designs $\mathcal{D}^*$. To assess the performance of a design, we employed the usual measure of *efficiency relative to an orthogonal design* as the inverse ratio of mean variances with which elementary contrasts are estimated in both designs. In some cases of interest (cf. Section 5), there emerges some preference in the paired comparisons the experimenter judges. Namely, we are primarily interested in comparisons of treatments from different groups. These pairwise comparisons are referred to as *inter-contrasts*. Because of our practical bias, we propose an efficiency factor based on mean variances resulting from *inter-contrast* estimates:

$$E^* = \left(\frac{1}{\delta_2 r_1} + \frac{1}{\delta_1 r_2}\right) \Big/ \Big((v_1 v_2)^{-1} \sum_{i \le v_1 < j} \mathrm{var}_{ij}\Big). \tag{4}$$

Note that all these variances obtained from an orthogonal design are equal to the sum of the replication inverses (cf. John [2]).

1.3. *Motivation and outline of the paper.* Our chief interest is to re-examine the concepts introduced in recent work. Our theoretical investigations centre on performance properties of efficiency factors. The rest of the paper is organised as follows. In Section 2, attention is given to the variance of the base contrast. Section 3 provides a detailed study of the case $v_1 = v_2$. Section 4 contains some numerical examples. Some issues related to applications are discussed in Section 5. Finally, a few concluding remarks are given in Section 6.

2. Variance of base contrast. Since the rows of the information matrix add to zero, i.e., $\mathbf{C1} = \mathbf{0}$, short calculations yield $s_1 = \delta s_2$. As a consequence, we obtain

$$\mathbf{C}\boldsymbol{\gamma} = (\delta+1)\mathbf{C}\begin{pmatrix}\mathbf{1}_{v_1}\\ \mathbf{0}_{v_2}\end{pmatrix} - \mathbf{C}\mathbf{1}_v = (\delta+1)\begin{pmatrix}s_1\mathbf{1}_{v_1}\\ -s_2\mathbf{1}_{v_2}\end{pmatrix} = (\delta^{-1}+1)s_1\boldsymbol{\gamma}.$$

Thus $\boldsymbol{\gamma}$ is an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda = (\delta^{-1}+1)s_1$, i.e., the normalised vector $\boldsymbol{\gamma}$ is the *basic contrast* of the design. According to John [2],

$$\text{var}\ \boldsymbol{\gamma}'\widehat{\boldsymbol{\alpha}} = \lambda^{-1}\|\boldsymbol{\gamma}\|^2 = \frac{v_1}{s_1}\,\delta^2. \tag{5}$$

In the following, we express (5) in terms of var_{ij}. To this end, we utilise similarity between the structures of $\mathbf{C}^+$ and $\mathbf{C}$. Let $\mathbf{C}_1^+$, $\mathbf{C}_2^+$, $\mathbf{C}_{12}^+$ be the respective subblocks of the matrix $\mathbf{C}^+$. By applying the decomposition of $\mathbf{C}^+$, we find

$$\begin{aligned}\text{var}\,\boldsymbol{\gamma}'\widehat{\boldsymbol{\alpha}} = \boldsymbol{\gamma}'\mathbf{C}^+\boldsymbol{\gamma} &= \left((\delta+1)\begin{pmatrix}\mathbf{1}_{v_1}\\ \mathbf{0}_{v_2}\end{pmatrix} - \mathbf{1}_v\right)'\mathbf{C}^+\left((\delta+1)\begin{pmatrix}\mathbf{1}_{v_1}\\ \mathbf{0}_{v_2}\end{pmatrix} - \mathbf{1}_v\right)\\ &= (\delta+1)^2\mathbf{1}_{v_1}'\mathbf{C}_1^+\mathbf{1}_{v_1} = (\delta+1)^2 v_1 s_1^+,\end{aligned}$$

meaning that $\mathbf{C}^+\mathbf{1}_v = \mathbf{0}$. Here, s_1^+ denotes the row sum in the submatrix $\mathbf{C}_1^+$.

For brevity, we write Σ, Σ^* for $\sum_{i<j\le v}\text{var}_{ij}$ and $\sum_{i\le v_1<j}\text{var}_{ij}$, respectively. Then, by applying (2) for all $i \le v_1 < j$, one gets

$$\Sigma^* = v_2\,\text{tr}\,\mathbf{C}_1^+ + v_1\,\text{tr}\,\mathbf{C}_2^+ + 2v_1 s_1^+.$$

Finally, combining the above observations, we derive the decomposition

$$\text{var}\,\boldsymbol{\gamma}'\widehat{\boldsymbol{\alpha}} = \frac{(\delta+1)^2}{2}(\Sigma^* - v_2\,\text{tr}\,\mathbf{C}_1^+ - v_1\,\text{tr}\,\mathbf{C}_2^+).$$

In particular, for $v_1 = v_2$ we obtain

$$\text{var}\,\boldsymbol{\gamma}'\widehat{\boldsymbol{\alpha}} = 2\Sigma^* - v\,\text{tr}\,\mathbf{C}^+ = 2\Sigma^* - \Sigma. \tag{6}$$

A detailed explanation for the second equality can be found in [1]. The formula (6) gives an interesting interpretation of the eigenvalues of the matrix $\mathbf{C}$.

3. Main result. In this section we discuss the case $v_1 = v_2$. To gain a better insight into the structure of the information matrix, we make a few observations which follow immediately from the cyclical set-up of the design. First, observe that $c_{ii} = r_j\,(k-1)/k$, where $i = 1, \ldots, v$, while $j = 1$ if $i \le v/2$ and $j = 2$ if $i > v/2$. This gives

$$\sum_{i=1}^{v} \frac{1}{c_{ii}} = \frac{v}{2} \cdot \frac{k^2}{r_1 r_2 (k-1)}. \tag{7}$$

We now turn to the row sum s_1. Notice that

$$\sum_{j=2}^{v/2} \lambda_{1j} = r_1(r_1 - 1),$$

where $\lambda_{(\cdot)}$ denote the entries of the concurrence matrix $\mathbf{N}\mathbf{N}'$. In view of the above, the resulting formula is

$$s_1 = c_{11} - k^{-1} \sum_{j=2}^{v/2} \lambda_{1j} = \frac{r_1 r_2}{k}. \tag{8}$$

Having established some elementary properties of the information matrix, we can now investigate the efficiency properties of the design $\mathcal{D}^*$. We prove the following theorem.

THEOREM 3.1. *For any connected design with incidence matrix* $\mathbf{N}$, *given in* (1), *if* $v_1 = v_2$ *then*

$$E^* \ge E. \tag{9}$$

Proof. In the following, we denote by $\overline{\Sigma^*}$, $\overline{\Sigma}$ the respective mean variances. First, by combining (5) and (6), we derive

$$\begin{aligned} \overline{\Sigma^*} - \overline{\Sigma} &= \binom{v}{2}^{-2} \Sigma^* - \binom{v}{2}^{-1} \Sigma = \frac{2}{v^2}\left(\operatorname{var} \boldsymbol{\gamma}'\widehat{\boldsymbol{\alpha}} - \frac{1}{v-1}\Sigma \right) \\ &= \frac{2}{v^2}\left(\frac{v}{2s_1} - \frac{v}{v-1} \operatorname{tr} \mathbf{C}^+ \right). \end{aligned} \tag{10}$$

We need the following lemma due to Shah and Sinha [6].

LEMMA 3.2. *Let a function* f *be convex and nonincreasing on* $[0, \infty)$. *Then for any connected design*

$$\sum_{i=1}^{v-1} f(\lambda_i) \ge \frac{v-1}{v} \sum_{i=1}^{v} f\left(\frac{v}{v-1} c_{ii} \right),$$

where $\lambda_1, \ldots, \lambda_{v-1}$ *are the nonzero eigenvalues of the information matrix.* ■

Applying this result to $f(x) = 1/x$, we find

$$\operatorname{tr} \mathbf{C}^{+} \geq \frac{(v-1)^2}{v^2} \sum \frac{1}{c_{ii}}.$$

Inserting this lower bound in (10), we obtain

$$\overline{\Sigma^*} - \overline{\Sigma} \leq \frac{2}{v^2}\left(\frac{v}{2s_1} - \frac{v-1}{v} \sum \frac{1}{c_{ii}}\right).$$

The relations (7) and (8) yield

$$\overline{\Sigma^*} - \overline{\Sigma} \leq \frac{2}{v^2}\left(\frac{vk}{2r_1r_2} - \frac{(v-1)k^2}{2r_1r_2(k-1)}\right) = \frac{1}{v^2} \cdot \frac{k}{r_1r_2} \cdot \frac{k-v}{k-1} \leq 0 \tag{11}$$

since $k \leq v$. Moreover, it is straightforward to verify the equality of the respective mean variances in an orthogonal design. This completes the proof of Theorem 3.1. ∎

4. Examples. We conducted some numerical studies to assess the performance of the bound (11). Tables 1 and 2 list the results. Calculations were carried out for the specific case $r_1 = r_2$. The results reported in the two rightmost columns are of prime interest. The comparison arranged in Tables 1 and 2 reveals a significant discrepancy in the case of $v_1 = v_2 = 6$, $r = 3$. Both the lower bound and the differences calculated are very similar in the remaining cases.

Table 3 summarises similar calculations for the case $v_1 \neq v_2$. Numbers in bold exemplify important cases. We conclude that the resulting inequality (11) fails to hold for arbitrary v_1, v_2.

TABLE 1. Summary of comparison for $v_1 = v_2 = 6$ and $r_1 = r_2 = r$. Asterisk stands for the lower bound.

r	initial blocks		$\overline{\text{var}}$	$\overline{\text{var}}^*$	diff.	
$r = 3$	(012)	(013)	.7451	.7384	.0067	**.0056***
	(012)	(024)	.7771	.7679	.0092	
	(013)	(024)	.7738	.7649	.0089	
$r = 4$	(0123)	(0124)	.5255	.5234	.0021	**.0020***
	(0123)	(0134)	.5262	.5241	.0021	

TABLE 2. Comparison for $v_1 = v_2 = 7$

r	initial blocks		$\overline{\text{var}}$	$\overline{\text{var}}^*$	diff.	
$r = 3$	(012)	(015)	.7654	.7584	.0070	**.0068***
	(012)	(025)	.7632	.7564	.0068	
$r = 4$	(0123)	(0124)	.5346	.5322	.0024	**.0022***
	(0123)	(0235)	.5345	.5320	.0025	

TABLE 3. Comparison for $v_1 = 3$, $v_2 = 4$

initial blocks		$\overline{\text{var}}$	$\overline{\text{var}}^*$	E	E^*
$v_1 = 3$	$v_2 = 4$				
(0)	(0)	1.095	1.000	.543	.583
(0)	(02)	.599	.583	.676	.715
(0)	(01)	.539	**.542**	.751	.769
(0)	(012)	.407	**.424**	.838	.851

5. Applications—crop rotation experiments. In this section, we discuss briefly the issue of applications. Crop rotation is growing two or more crops sequentially on the same area in successive years. We start the experiment with all plants included in a sequence. The key assumption is that different sequences of plants create various levels of soil fertility. We treat such additional effects of soil fertility accumulated during the full rotation as treatment effects. The experiment aims at comparing the effects of competing rotations. An illustration is provided by Table 4.

Some difficulties arise in the statistical analysis of crop rotation experiments. It is a reasonable requirement that all the successive crops should be taken into account. For this purpose, one can apply some uniform measure to combine yields of all species. On the other hand, selection of such a measure depends on the general objective of cultivation and should be treated as rather subjective. In the approach undertaken, statistical inference is made on the basis of the responses of some distinguished species that is referred

TABLE 4. Comparison of 5-course cropping systems

Plot	Year				
	1	2	3	4	5
1	**wheat**	rape	**wheat**	oats	beans
2	beans	**wheat**	rape	**wheat**	oats
3	oats	beans	**wheat**	rape	**wheat**
4	**wheat**	oats	beans	**wheat**	rape
5	rape	**wheat**	oats	beans	**wheat**

TABLE 4 (cont.)

Plot	Year				
	1	2	3	4	5
6	**wheat**	potato	**wheat**	pulses	**wheat**
7	**wheat**	**wheat**	potato	**wheat**	pulses
8	pulses	**wheat**	**wheat**	potato	**wheat**
9	**wheat**	pulses	**wheat**	**wheat**	potato
10	potato	**wheat**	pulses	**wheat**	**wheat**

to as the test crop. In the illustrative example, wheat is treated as the test crop.

The usual linear response model for the crop rotation experiment (cf. Przybysz [3]) specifies the observation as follows:

$$y_{ijk} = \mu + \varrho_i + \alpha_j + e_{ij} + \beta_k + \varphi_{jk} + \varepsilon_{ijk},$$

where μ is the general level effect, ϱ_i, α_j, β_k represent effects due to the ith replication ($i = 1, \ldots, p$), jth plot ($j = 1, \ldots, v$) and kth year ($k = 1, \ldots, b$). In addition, e_{ij} and ε_{ijk} are random errors of experimental units and random technical errors with classical assumptions of independence and normal distribution with zero mean and constant variances σ_1^2, σ_2^2, respectively. Finally, φ_{jk} stands for the interaction component.

The model of the experiment can be rewritten in matrix notation as

$$\mathbf{y} = \mathbf{1}\mu + \mathbf{R}'\boldsymbol{\varrho} + \boldsymbol{\Delta}'\boldsymbol{\alpha} + \mathbf{e} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{G}'\boldsymbol{\varphi} + \boldsymbol{\varepsilon},$$

where $\mathbf{y}$ is an $N \times 1$ vector of observations, with $N = pvb$, $\mathbf{1}$ is an $N \times 1$ vector of units, while $\mathbf{R}'$, $\boldsymbol{\Delta}'$, $\mathbf{D}'$ and $\mathbf{G}'$ are known design matrices of $N \times p$, $N \times v$, $N \times b$ and $N \times n$, respectively. We point out that the parameters α_j are of primary interest in studies of this sort. To specify the model completely, we mention that

$$\boldsymbol{\Delta}' = \mathbf{1} \otimes \widetilde{\boldsymbol{\Delta}}', \qquad \mathbf{D}' = \mathbf{1} \otimes \widetilde{\mathbf{D}}',$$

where $\widetilde{\boldsymbol{\Delta}}'$ and $\widetilde{\mathbf{D}}'$ are design $n \times v$ and $n \times b$ matrices, respectively, where $n = bv$. The binary matrices $\widetilde{\boldsymbol{\Delta}}$ and $\widetilde{\mathbf{D}}$ take a prominent place in our efficiency investigations. By means of them, we can describe the model of one replication as follows:

$$\widetilde{\mathbf{y}} = \mathbf{1}\mu + \widetilde{\boldsymbol{\Delta}}'\boldsymbol{\alpha} + \widetilde{\mathbf{D}}'\boldsymbol{\beta} + \widetilde{\boldsymbol{\varphi}}, \tag{12}$$

where $\widetilde{\mathbf{y}}$ is an $n \times 1$ vector of observations, $\mathbf{1}$ is an $n \times 1$ vector of units, $\widetilde{\boldsymbol{\Delta}}', \widetilde{\mathbf{D}}'$ are known design matrices for treatments and years, respectively, while $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $v \times 1$, $b \times 1$ vectors of treatment parameters and year effects, $\widetilde{\boldsymbol{\varphi}}$ is a standard random vector of errors with zero expectation and a common variance σ^2. Note that the vector $\widetilde{\mathbf{y}}$ is modelled by the equation of incomplete block designs.

We remark that applications are related to the model (12). The incidence matrix $\widetilde{\mathbf{N}} = \widetilde{\mathbf{\Delta}}\ \widetilde{\mathbf{D}}'$ is of the form (1) and remains in a close relation with the arrangement of plants on the plots in the respective rotations (cf. Table 1). Namely, its rows are related to plots, the columns correspond to years, "one" indicates that wheat was on the plot, zero refers to other species. To give a complete picture of this relationship, let us mention that the incidence matrix corresponding to our example has the structure (1) with the initial blocks (03) and (013), respectively, where $\delta_1 = \delta_2 = 1$.

6. Concluding remarks. Through detailed exploration of the efficiency criteria, we derived some characterization of the class of designs with the cyclic structure (1). The case with $v_1 = v_2$ is covered by Theorem 3.1. A weak lower bound on the discrepancy between average variances is obtained. One of the features of this result is easy implementation. The inequality (9) is obtained by direct application of the bound (11). Notice also that for both E and E^*, the only calculations required concern the eigenvalues of the matrix $\mathbf{C}$.

We close this summary with the conjecture that the conclusion (9) can be extended to block designs having $v_1 \neq v_2$. This is supported by an extensive numerical study.

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U. Bronowicka-Mielniczuk, J. Mielniczuk and T. Przybysz
Department of Applied Mathematics
Agricultural University of Lublin
Akademicka 13
P.O. Box 158
20-934 Lublin, Poland
E-mail: jackmiel@ursus.ar.lublin.pl

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