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RANDOM PRIORITY TWO-PERSON FULL-INFORMATION BEST CHOICE PROBLEM WITH IMPERFECT OBSERVATION

Abstract. The following version of the two-player best choice problem is considered. Two players observe a sequence of i.i.d. random variables with a known continuous distribution. The random variables cannot be perfectly observed. Each time a random variable is sampled, the sampler is only informed whether it is greater than or less than some level specified by him. The aim of the players is to choose the best observation in the sequence (the maximal one). Each player can accept at most one realization of the process. If both want to accept the same observation then a random assignment mechanism is used. The zero-sum game approach is adopted. The normal form of the game is derived. It is shown that in the fixed horizon case the game has a solution in pure strategies whereas in the random horizon case with a geometric number of observations one player has a pure strategy and the other one has a mixed strategy from two pure strategies. The asymptotic behaviour of the solution is also studied.

1. Introduction. The paper deals with the following zero-sum game version of the full-information best choice problem. Two players observe sequentially N i.i.d. random variables from a known continuous distribution with the objective of choosing the largest. The random variables cannot be perfectly observed. Players specify their sensitivity (impressionability) levels and each time a random variable is sampled the sampler is informed only whether it is greater than or less than the level he specified. Each of the players can choose at most one observation. Neither recall nor uncertainty of selection is allowed. When some player accepts an observation at time n,

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then the other one will investigate the sequence of future realizations having an opportunity to accept one of them. The players cannot choose the same state of the process at the same moment. When both want to accept such an observation a random assignment mechanism is used. A zero-sum game model is adopted. A class of suitable strategies and a gain function for the problem is constructed. It is shown that the game has a solution in pure strategies. In the random horizon case with a geometric number of observations one player has a pure strategy and the other one has a mixed strategy from two pure strategies. For the fixed horizon case the asymptotic behaviour of the solution is also studied.

The results of the paper extend those obtained in the paper by Neumann, Porosiński and Szajowski [12]. The games with priority for one player, in other game versions of the best choice problem, have been considered for example by Enns and Ferenstein [5]–[7], Majumdar [10], [11], Sakaguchi [21], [23], [24]. A relation of the priority games in the best choice problem to Dynkin's game has been shown by Ravindran and Szajowski [18] and Szajowski [25]. Random priority has been considered by Radzik and Szajowski [17] and Szajowski [27]. Imperfect observation for one decision maker problem has been investigated by Enns [4], Porosiński [14], Sakaguchi [20], [22]. One decision maker problems with random horizon were treated by Presman and Sonin [15], Cowan and Zabczyk [2] for the no-information case of the best choice problem and by Porosiński [14] for the full-information best choice problem with imperfect observation. An extensive review of generalizations of the best choice problem can be found in Freeman [9], Rose [19] and Ferguson [8].

A rigorous formulation of the problem, with the definition of strategies, is the subject of Section 2. In Section 3, for each assignment mechanism and fixed horizon it is shown that the problem is equivalent to a zero-sum game on the unit square. The normal form of the game is derived and the optimal pure strategies are pointed out. The asymptotic behaviour of the finite horizon problem is investigated in Section 4. The random horizon case is formulated and solved in Section 5 for N having the geometric distribution.

2. The priority game with imperfect observation. Let $X_1,\ldots,X_N$, $N \in \mathbb{N}$, be a sequence of i.i.d. random variables with a common known continuous distribution defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The sequence is sequentially sampled one by one by two decision makers (players). However the observations are imperfect and the exact realized values are not known. Players specify only their levels of impressionability and they are able to know whether the observed random variable is greater than or less than the levels they chose. After X_n is observed the players are informed whether X_n exceeds or not their sensitivity levels. If only one player decides to accept the state at time n, he gets it at once, the other player is informed of this fact and continues the observation of the process. On the other hand, if both decision makers decide to accept the state at the same moment, a random mechanism chooses one of them to benefit and the other decision maker can continue sequential observation of the process to choose the best one. One can say that the players have random priority to accept a realization. Neither recall nor uncertainty of selection is allowed. The aim of the players is to choose the best observation (the maximal one).

In this paper we assume that the problem is modeled by a two-person zero-sum game. Similar models for the no-information case have been considered in [27]. The structure of strategy sets and the form of the gain functions are different in those problems.

Let $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$, $n = 1, \ldots, N$, and let $\mathcal{S}$ be the set of stopping times with respect to $\{\mathcal{F}_n\}_{n=1}^N$. Since the observations are imperfect we take

$$\mathcal{S}_0 = \{\tau \in \mathcal{S} : \tau = \inf\{1 \le n \le N : X_n \ge x\},\ x \in \mathbb{R}\}$$

as the class of strategies for the one-person decision problem. For the two-person problem this class is not suitable. The proper class is the set of pairs

$$\mathfrak{S} = \{(\tau, \{\sigma_n\}_{n=1}^{N-1}) : \tau \in \mathcal{S}_0,\ \sigma_n \in \mathcal{S}_0,\ \sigma_n > n\}$$

(cf. [26]). The stopping time τ_i is the strategy of the first attempt of acceptance for Player i. If the state is accepted by Player i at time n then Player j, $j \neq i$, is using the strategy σ_n^j. Further we restrict ourselves to a narrower class $\mathfrak{S}_0$ of strategies based on one fixed level only. This means that the stopping times τ and σ_n, $n = 1, \ldots, N-1$, are defined by the same constant x. Let $\mathfrak{S}_0^1$ and $\mathfrak{S}_0^2$ be the sets of strategies for Player 1 and Player 2, respectively. The random assignment mechanism is defined by the lottery described by a random variable ξ with uniform distribution on $[0,1]$ and a number $\pi \in [0,1]$. If both players wish to select the state of the process at the same moment then Player 1 benefits if $\xi \le \pi$; otherwise Player 2 gets the observation.

Based on the above formulation the gain function can be described as follows. Let Player 1 and Player 2 choose $x \in \mathbb{R}$ and $y \in \mathbb{R}$, respectively. This defines strategies $\mathsf{x} \in \mathfrak{S}_0^1$ and $\mathsf{y} \in \mathfrak{S}_0^2$.

Player 1 gets $+1$ when he accepts the first $X_s \ge x$ (if $X_s < y$ or $X_s \ge y$ and the lottery chooses Player 1) and all further observations are less than X_s or when Player 2 accepts the first $X_s \ge y$ (if $X_s < x$ or $X_s \ge x$ and the lottery chooses Player 2) and the first, after X_s, observation $X_t \ge x$ is also greater than X_s and there is no observation greater than X_t later. Since this is a zero-sum game, Player 1 gets -1 when Player 2 gets $+1$ (and in the description of the winning events for Player 1, x is interchanged with y and "Player 1" with "Player 2"). In other cases Player 1 gets 0.

Taking into account the above considerations we can assume without loss of generality that the observed random variables have the standard uniform distribution and the set $\mathfrak{S}_0$ of strategies is equivalent to the interval $[0,1]$. This reduces the problem to a zero-sum game on the unit square. The expected payoff function is the expected value of the described payoff function of Player 1. The aims of the players are the same, but different priorities for the players give them unequal opportunities. The value of the game is the difference between the probability of success for Player 1 and Player 2.

3. The finite horizon case. Let the horizon N of observation be fixed. The expected payoff described in Section 2, which is the payoff function of the auxiliary game, has the form

$$
\begin{aligned}
\text{(1a)}\quad & h_N(x,y,\pi) \\
&= \sum_{s=1}^{N} \mathbf{P}\{X_1<x,\ldots,X_{s-1}<x,\ x\le X_s<y,\ X_{s+1}<X_s,\ldots,X_N<X_s\} \\
&\quad + \sum_{s=1}^{N} \mathbf{P}\{X_1<x,\ldots,X_{s-1}<x,\ X_s\ge y,\ \xi\le\pi, \\
&\qquad\qquad X_{s+1}<X_s,\ldots,X_N<X_s\} \\
&\quad + \sum_{s=1}^{N-1}\sum_{t=s+1}^{N} \mathbf{P}\{X_1<x,\ldots,X_{s-1}<x,\ X_s\ge y,\ \xi>\pi, \\
&\qquad\qquad X_{s+1}<x,\ldots,X_{t-1}<x,\ X_t>X_s,\ X_{t+1}<X_t,\ldots,X_N<X_t\} \\
&\quad - \sum_{s=1}^{N-1}\sum_{t=s+1}^{N} \mathbf{P}\{X_1<x,\ldots,X_{s-1}<x,\ x\le X_s<y, \\
&\qquad\qquad X_{s+1}<y,\ldots,X_{t-1}<y,\ X_t>y,\ X_{t+1}<X_t,\ldots,X_N<X_t\} \\
&\quad - \sum_{s=1}^{N-1}\sum_{t=s+1}^{N} \mathbf{P}\{X_1<x,\ldots,X_{s-1}<x,\ X_s\ge y,\ \xi\le\pi, \\
&\qquad\qquad X_{s+1}<y,\ldots,X_{t-1}<y,\ X_t>X_s,\ X_{t+1}<X_t,\ldots,X_N<X_t\} \\
&\quad - \sum_{s=1}^{N} \mathbf{P}\{X_1<x,\ldots,X_{s-1}<x,\ X_s\ge y,\ \xi>\pi, \\
&\qquad\qquad X_{s+1}<X_s,\ldots,X_N<X_s\}
\end{aligned}
$$

for $0\le x<y\le 1$ and

(1b) $h_N(x,y,\pi)$

$$
\begin{aligned}
&= \sum_{s=1}^{N} \mathbf{P}\{X_1 < y, \ldots, X_{s-1} < y, X_s \geq x, \xi \leq \pi, X_{s+1} < X_s, \ldots, X_N < X_s\} \\
&\quad + \sum_{s=1}^{N-1} \sum_{t=s+1}^{N} \mathbf{P}\{X_1 < y, \ldots, X_{s-1} < y,\ y \leq X_s < x, \\
&\qquad\qquad X_{s+1} < x, \ldots, X_{t-1} < x,\ X_t \geq x,\ X_{t+1} < X_t, \ldots, X_N < X_t\} \\
&\quad + \sum_{s=1}^{N-1} \sum_{t=s+1}^{N} \mathbf{P}\{X_1 < y, \ldots, X_{s-1} < y,\ X_s \geq x,\ \xi > \pi, \\
&\qquad\qquad X_{s+1} < x, \ldots, X_{t-1} < x,\ X_t \geq X_s,\ X_{t+1} < X_t, \ldots, X_N < X_t\} \\
&\quad - \sum_{s=1}^{N-1} \sum_{t=s+1}^{N} \mathbf{P}\{X_1 < y, \ldots, X_{s-1} < y,\ X_s > x,\ \xi \leq \pi, \\
&\qquad\qquad X_{s+1} < y, \ldots, X_{t-1} < y,\ X_t > X_s,\ X_{t+1} < X_t, \ldots, X_N < X_t\} \\
&\quad - \sum_{s=1}^{N} \mathbf{P}\{X_1 < y, \ldots, X_{s-1} < y,\ X_s \geq x,\ \xi > \pi, \\
&\qquad\qquad X_{s+1} < X_s, \ldots, X_N < X_s\} \\
&\quad - \sum_{s=1}^{N} \mathbf{P}\{X_1 < y, \ldots, X_{s-1} < y,\ y \leq X_s < x,\ X_{s+1} < X_s, \ldots, X_N < X_s\}
\end{aligned}
$$

for $x \geq y$. By (1) the payoff of Player 1 for $0 \leq x < y \leq 1$ has the form

(2a) $h_N(x,y,\pi)$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \frac{x^{N-i} - x^N}{i} - (1-\pi)\Bigg(2\sum_{i=1}^{N} x^{N-i}\frac{1-y^i}{i} \\
&\quad + \sum_{j=2}^{N} \sum_{i=j-1}^{N} (x^{N-j}y^{i-j-1} + x^{N-i-1})\left(\frac{1-y}{j} - \frac{1-y^{j+1}}{j(j+1)}\right)\Bigg) \\
&\quad - \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \frac{x^{N-i}y^{i-j-1}}{j}\left(1 - x - (y-x)y^j - \frac{1-y^{j+1}}{1+j}\right)
\end{aligned}
$$

and, for $x \geq y$,

(2b) $$\begin{aligned} h_N(x,y,\pi) &= \sum_{i=1}^{N} \frac{y^{N-i}}{i}(1-2x^i+y^i) - (1-\pi)\Big(2\sum_{j=1}^{N} y^{N-j}\frac{1-x^j}{j} \\ &+ \sum_{i=1}^{N-1}\sum_{j=1}^{i} y^{N-i}(y^{i-j}+x^{i-j})\Big(\frac{1-x}{j} - \frac{1-x^j}{j(j+1)}\Big)\Big) \\ &+ \sum_{j=1}^{N-1}\sum_{i=j+1}^{N}\Big(y^{N-i}x^{i-j-1}(x-y)\frac{1-x^j}{j} \\ &- \frac{y^{N-j-1}}{j}\Big(1-x-\frac{1-x^{j+1}}{j+1}\Big)\Big). \end{aligned}$$

In this way the game is transformed to a zero-sum game on the unit square. Since the relation $h_N(x,y,\pi)+h_N(y,x,1-\pi)=0$ follows, we restrict our considerations to $\pi \in [0.5,1]$.

The existence and form of equilibrium for such a game can be found in Parthasarathy and Raghavan [13], where a generalization of the well known theorem of Bohnenblust, Karlin and Shapley [1] (see also Radzik [16], Dresher [3]) is formulated and proved. Denote by $\mathbb{I}_a$ the probability distribution function concentrated at a.

Theorem 1. *Let $F(x,y)$ be a continuous function on the unit square $I\times I$ and let $F(x,y)$ be concave in x for each y. Then the zero-sum game $\Gamma=(I,I,F(x,y))$ has an equilibrium of the form*

$$(\mathbb{I}_a, \beta\mathbb{I}_c + (1-\beta)\mathbb{I}_d)$$

for some $0\le a,c,d,\beta\le 1$.

Proposition 1. *The payoff function of Player 1 in the full-information best choice problem for the two-person zero-sum game with imperfect information, fixed horizon N and priority $\pi\in[0,1]$ for Player 1 is given by (2). The game has a solution $(x^*(\pi),y^*(\pi))$ in pure strategies.*

Proof. Fix π. The gain function $h_N(x,y)$ given by (2) is concave in x for each y and continuous in both variables. It has two minima for each x at points $0<y_1(x)\le x\le y_2(x)\le 1$ and $h_N(x,y_1(x))<h_N(x,y_2(x))$ for $\pi\ne 0.5$. For $\pi=0.5$ and each x there is one minimum in y. From the proof of Theorem 1 (see Dresher [3], pp. 119–122) and by the above facts we get the assertion. ∎

The probability of success for Player 1 when both players are using the equilibrium strategy is

$$
\begin{aligned}
(3)\quad & P_N(\pi) \\
= & \sum_{s=1}^{N} \mathbf{P}\{X_1 < y^*, \dots, X_{s-1} < y^*,\ X_s \ge x^*,\ \xi \le \pi, \\
& \qquad\qquad\qquad\qquad X_{s+1} < X_s, \dots, X_N < X_s\} \\
& + \sum_{s=1}^{N-1} \sum_{t=s+1}^{N} \mathbf{P}\{X_1 < y^*, \dots, X_{s-1} < y^*,\ y^* \le X_s < x^*, \\
& \qquad X_{s+1} < x^*, \dots, X_{t-1} < x^*, X_t \ge x^*, X_{t+1} < X_t, \dots, X_N < X_t\} \\
& + \sum_{s=1}^{N-1} \sum_{t=s+1}^{N} \mathbf{P}\{X_1 < y^*, \dots, X_{s-1} < y^*,\ X_s \ge x^*,\ \xi > \pi, \\
& \qquad X_{s+1} < X_s, \dots, X_{t-1} < X_s,\ X_t \ge X_s,\ X_{t+1} < X_t, \dots, X_N < X_t\} \\
= & \pi \sum_{s=1}^{N} y^{*N-s} \frac{1 - x^{*s}}{s} + \sum_{s=1}^{N-1} \sum_{t=s+1}^{N} y^{*N-t} x^{*t-s-1} (x^* - y^*) \frac{1 - x^{*s}}{s} \\
& + (1 - \pi) \sum_{i=1}^{N-1} \sum_{j=1}^{i} y^{*N-j-1} x^{*i-j} \left(\frac{1 - x^*}{j} - \frac{1 - x^{*j+1}}{j(j+1)} \right).
\end{aligned}
$$

An interesting problem is the asymptotic behaviour of the equilibria and of the value of the game as $N \to \infty$. This is investigated in the next section.

4. Asymptotic solution. As $N \to \infty$, the optimal decision levels $x_N^*(\pi)$ and $y_N^*(\pi)$ tend to 1 and they are approximately linear in $1/N$: $x_N^*(\pi) = 1 - a(\pi)/N + o(1/N)$, $y_N^*(\pi) = 1 - b(\pi)/N + o(1/N)$ (in other words: $N(1 - x_N^*(\pi)) \to a(\pi)$ or $x_N^{*N}(\pi) \to e^{-a(\pi)}$) for some constants $0 < a(\pi) \le b(\pi)$ (and $a(\pi) = b(\pi)$ for $\pi = 0.5$ only). The analysis of the solution for $\pi \in [0, 0.5]$ is analogous. It is enough to change the roles of the players. The asymptotic behaviour of the payoff function can be investigated based on approximation of $x_N^*(\pi)$ and $y_N^*(\pi)$.

For example the third component in the payoff function in (26) for $x = x_N^*(\pi)$, $y = y_N^*(\pi)$ tends to some integral:

$$
\begin{aligned}
& \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} y^{N-i} x^{i-j-1} (x - y) \frac{1 - x^j}{j} \\
& \qquad \to \int_0^1 \int_t^1 \left(e^{-b+bs} e^{-as+at} (b - a) \frac{1 - e^{-at}}{t} \right) ds\, dt \\
& \qquad = e^{-a} I(a) - e^{-b} I(b) + e^{-b} I(b - a),
\end{aligned}
$$

where $a = a(\pi)$, $b = b(\pi)$ and $I(c) = \int_0^c ((e^t - 1)/t)\, dt$. In this calculation we use the identity $\int_0^c ((e^t - t - 1)/t^2)\, dt - \int_0^c ((e^t - 1)/t)\, dt = 1 + (1 - e^{-c})/c$.

Applying these in (2) we find $\lim_{N\to\infty} h_N(x_N^*, y_N^*, \pi) = h(a, b, \pi)$, where

$$h(a,b,\pi) = \begin{cases} e^{-a}I(a) - e^{-b}I(b) + (1-\pi)\frac{b}{b-a}(e^{-b} - e^{-a}) & \\ \quad +\pi(b-a)e^{-b}(I(b) - I(b-a)) & \\ \quad +\pi\big(e^{-b}\big(a - \frac{a}{b}\big) + e^{-a} - 1 + \frac{a}{b}\big) & \text{for } 0 \le a < b, \\ (2\pi - 1)ae^{-a} & \text{for } 0 \le a = b, \\ e^{-a}I(a) - e^{-b}I(b) - \pi\frac{b}{a-b}(e^{-a} - e^{-b}) & \\ \quad -(1-\pi)(a-b)e^{-a}(I(a) - I(a-b)) & \\ \quad -(1-\pi)\big(e^{-a}\big(b - \frac{b}{a}\big) + e^{-b} - 1 + \frac{b}{a}\big) & \text{for } 0 \le b < a. \end{cases}$$

The function $h(a, b, \pi)$ has a unique saddle point $(a^*, b^*) = (a^*(\pi), b^*(\pi))$. Examples of saddle points and values, for some $\pi \in [0.5, 1]$, are shown in Table 1. The equilibria for $\pi \in [0, 0.5]$ can be obtained from the relation $h(a, b, \pi) + h(b, a, 1 - \pi) = 0$.

The asymptotic behaviour of the solution for finite N and given π can be described as follows.

PROPOSITION 2. *For fixed N the decision levels $x_N = 1 - a^*/N$, $y_N = 1 - b^*/N$ are asymptotically optimal. The asymptotic value of the game is $h(a^*, b^*, \pi)$.*

The limit probability of success for Player 1 (remember $\pi \in [0.5, 1]$) is

$$\begin{aligned} P(\pi) &= \lim_{N\to\infty} P_N(\pi) \\ &= \pi e^{-b^*}(I(b^*) - I(b^* - a^*)) + e^{-a^*}I(a^*) - e^{-b^*}(I(b^*) - I(b^* - a^*)) \\ &\quad + (1-\pi)\left(e^{-a^*} - e^{-b^*} + e^{-b^*}(I(b^*) - I(b^* - a^*)) + b^*\frac{e^{-b^*} - e^{-a^*}}{b^* - a^*}\right). \end{aligned}$$

Examples of limit equilibria, the value of the game and the probability of success for Player 1 are given in Table 1.

TABLE 1. Asymptotic behaviour of the solution of the game with random priority

π	a^*	b^*	Value of the game	$P(\pi)$
0.5	1.3834	1.3834	0.0000	0.3424
0.6	1.4303	1.5183	0.0677	0.3858
0.7	1.4667	1.6649	0.1319	0.4251
0.8	1.4965	1.8226	0.1924	0.4602
0.9	1.5201	1.9941	0.2493	0.4908
1	1.5404	2.1812	0.3026	0.5172

5. Geometric N. Let N be geometric with parameter $p \in [0,1]$, i.e. $p_k = P(N = k) = pq^k$, $p > 0$, $q = 1-p$, $k \geq 0$. The payoff function in this case is $h(x,y,\pi) = \sum_{k=0}^{\infty} p_k h_k(x,y,\pi)$, where $h_k(x,y,\pi)$ stands for the gain function for $N = k$ fixed given by (2). The function $h(x,y,\pi)$ can be written, after simplifications, as a function $g(s,t,\pi)$ of new coordinate variables $s = p/(1-qx)$, $t = p/(1-qy)$ on $[p,1]\times[p,1]$ (this transformation preserves monotonicity)

$$g(s,t,\pi) = \begin{cases} (1-\pi)\big(2t\ln s + \big(\frac{t^2}{s} + t\big)(s - 1 - \ln s)\big) & \\ \quad + t\ln t - s\ln s - t\ln s + \frac{t^2}{s} + \frac{t^2}{s}\ln s - t^2 & \text{if } p \leq t \leq s \leq 1, \\ (1-\pi)\big(2s\ln t + \big(\frac{s^2}{t} + s\big)(t - 1 - \ln t)\big) & \\ \quad + t\ln t - s\ln s + s - st & \text{if } p \leq s < t \leq 1. \end{cases}$$

It is very interesting and important that the gain function $g(s,t,\pi)$ depends on p only via its domain. The game with the gain function $g(s,t,\pi)$ considered on the unit square has the following solution. Player 1 has an optimal pure strategy $s^*(\pi)$ and Player 2 has an optimal mixed strategy $Q^*(t) = \alpha^*(\pi)\mathbb{I}_{t_1^*(\pi)}(t) + (1-\alpha^*(\pi))\mathbb{I}_{t_2^*(\pi)}(t)$. Examples of solutions for different π are given in Table 2. Let

$$v(p,\pi) = \max_{p\leq s\leq 1}\min_{p\leq t\leq 1} g(s,t,\pi) \tag{4}$$

denote the value of the game. The optimal strategy for Player 1 for given π is obtained as the unique s^* such that $\min_t g(s^*,t,\pi) = v(p,\pi)$. The parameters $\alpha^* = \alpha^*(\pi)$, $t_1^* = t_1^*(\pi) \leq s^*$, $t_2^* = t_2^*(\pi) \geq s^*$ of the best strategy for Player 2 are obtained from the conditions

$$\begin{aligned} & g(s^*,t_1,\pi) = g(s^*,t_2,\pi) = v(p,\pi), \\ & \frac{\partial g(s^*,t_1,\pi)}{\partial s} \leq 0 \leq \frac{\partial g(s^*,t_2,\pi)}{\partial s}, \\ & \alpha\frac{\partial g(s^*,t_1,\pi)}{\partial s} + (1-\alpha)\frac{\partial g(s^*,t_2,\pi)}{\partial s} = 0. \end{aligned} \tag{5}$$

The solution obtained on $[0,1]\times[0,1]$ is also valid on $[p,1]\times[p,1]$ for $p \leq t_1^*(\pi)$. For $p > t_1^*(\pi)$ the players have to modify their strategies according to the above conditions applied on $[p,1]\times[p,1]$. Player 1 changes his strategy to some $s^*(p,\pi) \geq s^*(\pi)$. Player 2 uses a mixed strategy consisting of $t_1^*(p,\pi) = p$ and $t_2^*(p,\pi) \geq t_2^*(\pi)$ for p up to some $p_1(\pi)$ when $s^*(p,\pi) = p$ and Player 2 uses the pure strategy $t_2^*(p,\pi) \geq p$. For some $p_2(\pi) > p_1(\pi)$, if $p > p_2(\pi)$ then both players use the pure strategy $s^*(p,\pi) = t_2^*(p,\pi) = p$. As a result of such considerations we obtain the following solution.

Proposition 3. *Let $\pi \in [0.5,1]$. For geometric N there exists a solution of the game and it is of the following form depending on p and π:*

1. *If* $p \le t_1^*(\pi)$ *then the optimal level for Player* 1 *is* $x^* = x^*(\pi) = (s^*(\pi)-p)/(qs^*(\pi))$ *and Player* 2 *uses the mixed strategy* $Q^*(y) = \alpha^*\mathbb{I}_{y_1^*} + (1-\alpha^*)\mathbb{I}_{y_2^*}$ *with* $\alpha^* = \alpha^*(\pi)$, $y_1^* = y_1^*(\pi) = (t_1^*(\pi)-p)/(qt_1^*(\pi))$ *and* $y_2^* = y_2^*(\pi) = (t_2^*(\pi)-p)/(qt_2^*(\pi))$, *where* $s^*(\pi)$, $t_1^*(\pi)$, $t_2^*(\pi)$, $\alpha^*(\pi)$ *are parameters of the solution of the game on* $[0,1]\times[0,1]$. *The value of the game* (4) *and the strategies are independent of* p,

$$v(p,\pi) = \alpha^*(\pi)g(s^*(\pi),t_1^*(\pi),\pi) + (1-\alpha^*(\pi))g(s^*(\pi),t_2^*(\pi),\pi)$$

and the probability of success for Player 1 *when both players use the equilibrium strategy is*

$$P_p = -s^*(\pi)\ln s^*(\pi) - (1-\pi)(\alpha^*(\pi)t_1^*(\pi) + (1-\alpha_2^*(\pi))t_2^*(\pi))(1-s^*(\pi)).$$

2. *If* $t_1^*(\pi) < p \le p_1(\pi)$ *then the optimal strategy for Player* 1 *is* $x^*(p,\pi) = (s^*(p,\pi)-p)/(qs^*(p,\pi))$, *where* $s^*(p,\pi)$ *is a solution of the equation* $v(p,\pi) = \min_{t\in[p,1]} g(s^*,t,\pi)$. *The optimal strategy* $Q^*(y)$ *for Player* 2 *is of the form* $Q^*(y) = \alpha^*\mathbb{I}_0 + (1-\alpha^*)\mathbb{I}_{y_2^*}$, *where* $y_2^* = (t_2^*(p,\pi)-p)/(qt_2^*(p,\pi))$, $\alpha^* = \alpha^*(p,\pi)$ *and the parameters* $\alpha^*(p,\pi)$, $t_1^*(p,\pi) = p$ *and* $t_2^*(p,\pi)$ *fulfil* (5) *on* $[p,1]\times[p,1]$,

$$v(p,\pi) = \alpha^*(p,\pi)g(s^*(p,\pi),p,\pi) + (1-\alpha^*(p,\pi))g(s^*(p,\pi),t_2^*(p,\pi),\pi),$$

and

$$P_p = -s^*\ln s^* - (1-\pi)(\alpha^*(p,\pi)p + (1-\alpha^*(p,\pi))t_2^*(p,\pi)(1-s^*(p,\pi)).$$

3. *If* $p_1(\pi) < p \le p_2(\pi)$ *then the equilibrium is pure and has the form* $(0,y^*(p,\pi))$. *The optimal strategy is* $y^*(p,\pi) = (t_2^*(p,\pi)-p)/(qt_2^*(p,\pi))$, *where* $t_2^*(p,\pi)$ *satisfies* $g(p,t_2^*(p,\pi)) = v(p,\pi)$ *and* $\partial g(p,t_2^*(p,\pi))/\partial s \le 0$. *The value of the game is* $g(p,t_2^*(p,\pi),\pi)$, *and*

$$P_p = -p\ln p - (1-\pi)p(1-p).$$

4. *If* $p > p_2(\pi)$ *then* $(0,0)$ *is a pure equilibrium*, $v(p,\pi) = g(p,p,\pi)$, *and*

$$P_p = -p\ln p - (1-\pi)p(1-p).$$

Numerical examples are given in Table 2 (the result for $\pi = 1$ was obtained in [12]). Let us mention that $g(s,t,\pi) + g(t,s,1-\pi) = 0$ and the solution for $\pi\in[0,0.5]$ can be constructed using this remark.

It is interesting and quite unexpected that in all *natural* situations (i.e. when p is small, $p \le t_1^*(\pi)$, see Table 2 for examples of $t_1^*(\pi)$) both the value of the game and P_p are constant (independent of p). It is also surprising that the probability of success P_p for the two-person model with $\pi = 1$ (Player 1 has priority) (≈ 0.3646) is only a little less than that for the one-person model ($= e^{-1} \approx 0.3679$, see Porosiński [14]).

T A B L E 2. Solution of the game with horizon N having geometric distribution distribution with parameter p and random priority π

π	p	Strategies Player 1 $s^*(p,\pi)$	$\alpha^*(p,\pi)$	Player 2 $t_1^*(p,\pi)$	$t_2^*(p,\pi)$	$v(p,\pi)$	P_p
0.5	$p \le 0.4237$	0.4237			0.4237	0	0.2418
	0.5	0.5			0.5	0	0.2216
	0.75	0.75			0.75	0	0.1220
	1	1			1	0	0
0.6	$p \le 0.3820$	0.4104	0.5376	0.3820	0.4497	0.0477	0.2680
	0.4	0.4130	0.5939	0.4	0.4499	0.0479	0.2666
	0.5	0.5			0.5	0.0500	0.2466
	0.75	0.75			0.75	0.0375	0.1408
0.75	$p \le 0.3135$	0.3951	0.4939	0.3135	0.4837	0.1138	0.3065
	0.4	0.4180	0.4265	0.4	0.4859	0.1183	0.2992
	0.5	0.5			0.5	0.1250	0.2841
	0.75	0.75			0.75	0.0936	0.1689
0.9	$p \le 0.2453$	0.3805	0.4907	0.2453	0.5142	0.1717	0.3440
	0.3	0.3863	0.5085	0.3	0.5163	0.1739	0.3425
	0.4	0.4279	0.3149	0.4	0.5314	0.1863	0.3352
	0.5	0.5048			0.5611	0.2349	0.3185
	0.6	0.6			0.6011	0.1928	0.2825
	0.7	0.7			0.7	0.1680	0.2287
	0.8	0.8			0.8	0.1280	0.1625
	0.9	0.9			0.9	0.0720	0.0858
1	$p \le 0.2030$	0.3702	0.7555	0.2030	0.5327	0.2030	0.3679
	0.3	0.3872	0.7168	0.3	0.5418	0.2127	0.3674
	0.4	0.4372	0.5197	0.4	0.5696	0.2293	0.3617
	0.5	0.5139	0.1966	0.5	0.6150	0.2410	0.3421
	0.5671	0.5671	0.0002	0.5671	0.6486	0.2401	0.3217
	0.6	0.6			0.6704	0.2362	0.3065
	0.7	0.7			0.7408	0.2088	0.2497
	0.8	0.8			0.8187	0.1598	0.1785
	0.9	0.9			0.9048	0.0900	0.0948
	1	1			1	0	0

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