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MINIMUM DISTANCE ESTIMATOR FOR A HYPERBOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATION

Abstract. We study a minimum distance estimator in L_2-norm for a class of nonlinear hyperbolic stochastic partial differential equations, driven by a two-parameter white noise. The consistency and asymptotic normality of this estimator are established under some regularity conditions on the coefficients. Our results are applied to the two-parameter Ornstein–Uhlenbeck process.

1. Introduction. In recent years, there has been a growing interest in parameter estimation based on the minimum distance technique. For instance, Dietz and Kutoyants (1992, 1997) studied the problem of estimation of a parameter by the observations of an ergodic diffusion process. The Ornstein–Uhlenbeck process with small diffusion coefficients was treated by Kutoyants (1994), Kutoyants *et al.* (1994), Kutoyants and Pilibossian (1994) and Hénaff (1995). Models for random field diffusions were considered by Kutoyants and Lessi (1995) for the distance defined by Hilbert-type metrics.

The purpose of this paper is to extend their results to a more general class of random fields. More precisely, we deal with the following nonlinear hyperbolic stochastic partial differential equation: for any $(t^1, t^2) \in \mathbb{R}_+^2$,

$$(1) \qquad \frac{\partial^2 X_{t^1,t^2}}{\partial t^1 \partial t^2} = S^1(\theta_0, t^1, t^2)\frac{\partial X_{t^1,t^2}}{\partial t^2} + S^2(\theta_0, t^1, t^2)\frac{\partial X_{t^1,t^2}}{\partial t^1}$$
$$+ S^3(\theta_0, t^1, t^2, X) + \varepsilon \dot{W}_{t^1,t^2},$$

with the initial condition $X_{t^1,t^2} = x$ on the axes, $x \in \mathbb{R}$.

The coefficients are mesurable functions

$$S^i : \Theta \times [0, T^1] \times [0, T^2] \to \mathbb{R}, \quad i = 1, 2,$$
$$S^3 : \Theta \times [0, T^1] \times [0, T^2] \times \mathcal{C} \to \mathbb{R},$$

where $\Theta \subset \mathbb{R}^k$ and $\mathcal{C}$ stands for the set of all continuous real-valued functions defined on $[0, T^1] \times [0, T^2]$. $\{\dot{W}_{t^1, t^2} : (t^1, t^2) \in [0, T^1] \times [0, T^2]\}$ is a one-dimensional two-parameter white noise. Equations of this kind appear, for example, in the problem of constructing a Wiener sheet on manifolds (see Norris (1995)) and in nonlinear filtering theory for two-parameter processes (see Korezlioglu *et al.* (1983)). Their solutions are called two-parameter diffusion processes and there are two different approaches to solving them. The first one was introduced by Farré and Nualart (1993). By a solution they mean a random field $\{X_{t^1, t^2} : (t^1, t^2) \in [0, T^1] \times [0, T^2]\}$ adapted to the natural filtration associated with the Wiener sheet W and satisfying

$$
\begin{aligned}
X_{t^1, t^2} = x &+ \int_0^{t^1} \int_0^{t^2} S^1(\theta_0, s^1, s^2) \, X(s^1, ds^2) \, ds^1 \\
&+ \int_0^{t^2} \int_0^{t^1} S^2(\theta_0, s^1, s^2) \, X(ds^1, s^2) \, ds^2 \\
&+ \int_0^{t^1} \int_0^{t^2} S^3(\theta_0, s^1, s^2, X) \, ds^1 \, ds^2 + \varepsilon W_{t^1, t^2}.
\end{aligned}
$$

The other one is due to Rovira and Sanz-Solé (1995, 1996) who used a method based on the Green function $\gamma_{t^1, t^2}(\theta_0, s^1, s^2)$ associated with the second order differential operator

$$\mathcal{L}f(t^1, t^2) = \frac{\partial^2 f}{\partial t^1 \partial t^2}(t^1, t^2) - S^1(\theta_0, t^1, t^2)\frac{\partial f}{\partial t^2}(t^1, t^2) - S^2(\theta_0, t^1, t^2)\frac{\partial f}{\partial t^1}(t^1, t^2).$$

Note that $\gamma_{t^1, t^2}(\theta_0, s^1, s^2)$ is the solution to the partial differential equation

$$
\begin{cases}
\dfrac{\partial^2 \gamma_{t^1, t^2}}{\partial s^1 \partial s^2}(\theta_0, s^1, s^2) + \dfrac{\partial(S^1(\theta_0, s^1, s^2)\gamma_{t^1, t^2}(\theta_0, s^1, s^2))}{\partial s^2} \\
\qquad\qquad + \dfrac{\partial(S^2(\theta_0, s^1, s^2)\gamma_{t^1, t^2}(\theta_0, s^1, s^2))}{\partial s^1} = 0, \\
\dfrac{\partial \gamma_{t^1, t^2}}{\partial s^1}(\theta_0, s^1, s^2) + S^1(\theta_0, s^1, s^2)\gamma_{t^1, t^2}(\theta_0, s^1, s^2) = 0 \quad \text{if } s^2 = t^2, \\
\dfrac{\partial \gamma_{t^1, t^2}}{\partial s^2}(\theta_0, s^1, s^2) + S^2(\theta_0, s^1, s^2)\gamma_{t^1, t^2}(\theta_0, s^1, s^2) = 0 \quad \text{if } s^1 = t^1, \\
\gamma_{t^1, t^2}(\theta_0, s^1, s^2) = 1 \quad \text{if } s^1 = t^1, \ s^2 = t^2.
\end{cases}
$$

The solution to (1) is defined by

$$X_{t^1,t^2} = x + \int_0^{t^1}\int_0^{t^2} \gamma_{t^1,t^2}(\theta_0,s^1,s^2)S^3(\theta_0,s^1,s^2,X)\,ds^1\,ds^2$$

$$+ \varepsilon \int_0^{t^1}\int_0^{t^2} \gamma_{t^1,t^2}(\theta_0,s^1,s^2)\,W(ds^1,\,ds^2).$$

These two apparently different ways of solving equation (1) can be shown to be equivalent (see Rovira and Sanz-Solé 1996, Proposition 2.4).

Now, we state the problem. The coefficients S^i are supposed to be known but the value of the parameter θ_0 is unknown. Our aim is to estimate θ_0 by an L_2-minimum distance estimator (MDE) θ_ε^*. The case $S^2 = 0$ was treated by Kutoyants and Lessi (1995).

We define θ_ε^* by

$$\theta_\varepsilon^* = \arg\inf_{\theta\in\Theta} \|X - x(\theta)\|_{L^2(\mu)},$$

which means that θ_ε^* is a solution to the equation

$$\|X - x(\theta_\varepsilon^*)\|_{L^2(\mu)} = \inf_{\theta\in\Theta} \|X - x(\theta)\|_{L^2(\mu)}$$

where $\|\cdot\|_{L^2(\mu)}$ denotes the $L^2(\mu)$-norm associated with a finite measure μ and $\{x_{t^1,t^2}(\theta) : (t^1,t^2) \in [0,T^1]\times[0,T^2]\}$ is the solution of equation (1) when θ_0 is replaced by θ and $\varepsilon = 0$. Let us remark that $x_{t^1,t^2}(\theta)$ is a deterministic function.

The rest of the paper is organized as follows. In Section 2, we introduce notations and state some conditions on the coefficients which will be used throughout. We also recall some properties of the Green function $\gamma_{t^1,t^2}(\theta_0,s^1,s^2)$ and give some preliminary lemmas. Section 3 is devoted to the study of the asymptotic behavior of θ_ε^* as $\varepsilon \to 0$ through its consistency and asymptotic normality.

As usual, all constants appearing in the proofs are called C, although they may vary from one occurrence to another.

2. Notations and preliminaries. Let $\mathbb{R}_+ i = [0,\infty)$ and $T = (T^1,T^2) \in \mathbb{R}$ R_t stands for the rectangle $[0,t^1] \times [0,t^2]$. The set Θ of the parameters is a bounded open subset of $\mathbb{R}^k$, and $\varepsilon \in (0,1]$.

Now, we state the conditions on the coefficients.

• (H1) For any $\theta \in \Theta$, $S^i(\theta,\cdot)$, $i = 1,2$, is uniformly bounded and has uniformly bounded derivatives.

• (H2) There exists a constant $C > 0$ such for that for any $(x, y) \in \mathcal{C} \times \mathcal{C}$, $t \in R_T$ and $\theta \in \Theta$,

$$|S^3(\theta, t, x) - S^3(\theta, t, y)| \leq C|x_t - y_t|,$$
$$|S^3(\theta, t, x)| \leq C(1 + |x_t|).$$

• (H3) For any $t \in R_T$, $S^i(\cdot, t)$, $i = 1, 2$, has uniformly bounded first order and mixed second order partial dervatives.

Let $\dot{S}^3_\theta$ denote the vector function of the derivatives of $S^i, i = 1, 2$, with respect to θ:

$$\dot{S}^i_\theta(\theta, t) = \left(\frac{\partial S^i}{\partial \theta^1}(\theta, t), \ldots, \frac{\partial S^i}{\partial \theta^k}(\theta, t) \right)'$$

where A' stands for the transpose of the matrix A.

If $S^3(\theta, t, x) = S^3(\theta, t, x_t)$, we denote by $\dot{S}^3_\theta(\theta, t, x_t)$ the derivative of S^3 in x, i.e. $\dot{S}^3_\theta(\theta, t, x_t) = \frac{\partial S^3}{\partial x}(\theta, t, x)|_{x=x_t}$. We let $\dot{S}^3_\theta(\cdot, t, x_t)$ be the vector function of the derivatives of $S^3(\cdot, t, x)$ in θ.

• (H4) For any $x \in \mathcal{C}$, $\theta \in \Theta$ and $t \in R_T$, we have $S^3(\theta, t, x) = S^3(\theta, t, x_t)$ and:

(i) $S^3(\theta, t, \cdot)$ is differentiable with uniformly bounded derivatives, $\dot{S}^3_x(\theta, t, \cdot)$ is continuous and there exist constants $C > 0$, $a \in (0, 1]$ and $b \in (0, 1]$ such that for any $(x, y) \in \mathcal{C} \times \mathcal{C}$, $(\theta_1, \theta_2) \in \Theta^2$ and $t \in R_T$,

$$|\dot{S}^3_x(\theta_1, t, x_t) - \dot{S}^3_x(\theta_2, t, y_t)| \leq C(|x_t - y_t|^a + |\theta_1 - \theta_2|^b);$$

(ii) $S^3(\cdot, t, x_t)$ is differentiable, $\dot{S}^3_\theta(\cdot, t, x_t)$ is continuous and for any compact subset $\mathcal{K}$ of $\mathbb{R}$ there exist constants $C_\mathcal{K} > 0$, $c \in (0, 1]$ and $d \in (0, 1]$ such that for any $(x, y) \in \mathcal{K}^2$, $(\theta_1, \theta_2) \in \Theta^2$ and $t \in R_T$,

$$|\dot{S}^3_\theta(\theta_1, t, x) - \dot{S}^3_\theta(\theta_2, t, y)| \leq C_\mathcal{K}(|x - y|^c + |\theta_1 - \theta_2|^d).$$

Now, we recall the existence and uniqueness result for solutions to equation (1) and some properties of the Green function which will be needed.

THEOREM 1. *Under* (H1) *and* (H2), *there exists a unique continuous random field* $X = \{X_t : t \in R_T\}$ *which is a solution of* (1).

P r o o f. See Rovira and Sanz-Solé (1995), Proposition 2.1, or Farré and Nualart (1993), Theorem 2.1. ∎

LEMMA 2. *Under* (H1) *and* (H2), *we have*:

(i) *For any* $\theta \in \Theta$ *and* $t \in R_T$, *the function* $s \mapsto \gamma_t(\theta, s)$ *has uniformly bounded derivatives of first order and mixed partial derivatives of second order on* $\{s \in R_T : 0 \leq s^1 \leq t^1, \ 0 \leq s^2 \leq t^2\}$.

(ii) *There exists $C > 0$ such that*

$$\sup_{\theta \in \Theta} \sup_{t \in R_T} \sup_{s \in R_t} |\gamma_t(\theta, s)| \leq C.$$

P r o o f. This follows immediately from the boundedness of $S^i, i = 1, 2$, and techniques developed in the proofs of Propositions 3.1 and 3.2 of Rovira and Sanz-Solé (1995). ∎

LEMMA 3. *There exists $C > 0$ such that for any $t \in R_T$,*

$$\sup_{\theta \in \Theta} \left| \int_{R_t} \gamma_t(\theta, s) \, W(ds) \right| \leq C \sup_{s \in R_t} |W_s|.$$

P r o o f. In view of Lemma 2(i), for any $\theta \in \Theta$ and $t \in R_T$, the function $s \mapsto \gamma_t(\theta, s)$ has uniformly bounded first order derivatives and mixed second order derivatives. Therefore, we have

$$\int_{R_t} \gamma_t(\theta, s) \, W(ds) = W_t - \int_0^{t^1} \frac{\partial \gamma_t}{\partial s^1}(\theta, s^1, t^2) W_{s^1, t^2} \, ds^1$$

$$- \int_0^{t^2} \frac{\partial \gamma_t}{\partial s^2}(\theta, t^1, s^2) W_{t^1, s^2} \, ds^2 - \int_{R_t} \frac{\partial^2 \gamma_t}{\partial s^1 \partial s^2}(\theta, s) W_s \, ds.$$

The desired result follows immediately. ∎

LEMMA 4. *Under* (H1) *and* (H2), *there exists $C > 0$ such that*

$$\sup_{\theta \in \Theta} \sup_{t \in R_T} |X_t(\theta) - x_t(\theta)| \leq C\varepsilon \sup_{t \in R_T} |W_t|.$$

P r o o f. We have

$$|X_s(\theta) - x_s(\theta)| \leq \int_{R_s} |\gamma_s(\theta, u)[S^3(\theta, u, X(\theta)) - S^3(\theta, u, x(\theta))]| \, du$$

$$+ \varepsilon \left| \int_{R_s} \gamma_t(\theta, u) \, W(du) \right|.$$

By Lemmas 2(ii) and 3, we have

$$|X_s(\theta) - x_s(\theta)| \leq C \int_{R_s} |S^3(\theta, u, X(\theta)) - S^3(\theta, u, x(\theta))| \, du + \varepsilon C \sup_{u \in R_s} |W_u|.$$

Let

$$g_s = \sup_{\theta \in \Theta} \sup_{u \in R_s} |X_u(\theta) - x_u(\theta)|.$$

In view of (H2), we have

$$g_t \leq C \int_{R_t} g_u \, du + \varepsilon C \sup_{u \in R_t} |W_u|.$$

Now, by using the Gronwall Lemma (see Dozzi (1989), p. 91), we deduce that

$$g_t \leq \varepsilon C \sup_{u \in R_t} |W_u|. \quad \blacksquare$$

Let $Y = \{Y_t : t \in R_T\}$ be the solution of the following stochastic partial differential equation:

$$\frac{\partial^2 Y_t}{\partial t^1 \partial t^2} = S^1(\theta_0, t)\frac{\partial Y_t}{\partial t^2} + S^2(\theta_0, t)\frac{\partial Y_t}{\partial t^1} + \dot{S}_x^3(\theta_0, t, x_t(\theta_0))Y_t + \dot{W}_t,$$

with $Y_t = 0$ on the axes. We denote by $\widetilde{x}_t(\theta)$ the vector function of the derivatives of $x_t(\theta)$ with respect to θ and put

$$J(\theta) = \int_{R_T} \widetilde{x}_t(\theta)\widetilde{x}_t'(\theta) \, d\mu(t).$$

Let

$$\xi = J(\theta_0)^{-1} \int_{R_T} Y_t \, \widetilde{x}_t(\theta_0) \, d\mu(t) \quad \text{and} \quad Z_t = \varepsilon^{-1}(X_t - x_t(\theta_0)).$$

REMARK 5. Y is a centered Gaussian random field. Therefore, ξ is a centered Gaussian random variable with covariance matrix

$$\Gamma = J(\theta_0)^{-1}\left[\int_{R_T} \int_{R_T} E(Y_s Y_t) \, \widetilde{x}_t(\theta_0)\widetilde{x}_s'(\theta_0) \, d\mu(t) \, d\mu(s)\right] J(\theta_0)^{-1}.$$

LEMMA 6. *Under* (H1)–(H4), *there exists* $C > 0$ *such that*

(i) $\sup\limits_{t \in R_T} |Y_t| \leq C \sup\limits_{t \in R_T} |W_t|,$

(ii) $\sup\limits_{t \in R_T} |Z_t - Y_t| \leq C\varepsilon^a \sup\limits_{t \in R_T} |W_t|^{1+a},$

(iii) $\sup\limits_{t \in R_T} |\widetilde{x}_t(\theta) - \widetilde{x}_t(\theta_0)|$

$$\leq C(|\theta - \theta_0| + |\theta - \theta_0|^a + |\theta - \theta_0|^b + |\theta - \theta_0|^c + |\theta - \theta_0|^d).$$

P r o o f. (i) In view of Lemma 2(ii), Lemma 3 and (H4), we have

$$\sup_{s \in R_t} |Y_s| \leq C \int_{R_t} (\sup_{u \in R_s} |Y_u|) \, du + C \sup_{u \in R_t} |W_u|.$$

Now, the Gronwall Lemma leads to (i).

(ii) Let $g_t = |Z_t - Y_t|$. We have

$$g_t \leq \int_{R_t} |\gamma_t(\theta_0, s)|$$

$$\times |\varepsilon^{-1}[S^3(\theta_0, s, X_s) - S^3(\theta_0, s, x_s(\theta_0))] - \dot{S}_x^3(\theta_0, s, x_s(\theta_0))Y_s| \, ds.$$

By Lemma 2(ii), we have

$$g_t \le C \int\limits_{R_t} |\varepsilon^{-1}[S^3(\theta_0, s, X_s) - S^3(\theta_0, s, x_s(\theta_0))] - \dot{S}_x^3(\theta_0, s, x_s(\theta_0))Y_s|\,ds.$$

Therefore, there exists $\widetilde{X}_s = x_s(\theta_0) + \beta(X_s - x_s(\theta_0))$, $\beta \in (0,1)$, such that

$$g_t \le C \int\limits_{R_t} |\varepsilon^{-1}(X_s - x_s(\theta_0))\dot{S}_x^3(\theta_0, s, \widetilde{X}_s) - \dot{S}_x^3(\theta_0, s, x_s(\theta_0))Y_s|\,ds$$

$$\le C \int\limits_{R_t} |[\varepsilon^{-1}(X_s - x_s(\theta_0)) - Y_s]\dot{S}_x^3(\theta_0, s, \widetilde{X}_s)|\,ds$$

$$+ C \int\limits_{R_t} |[\dot{S}_x^3(\theta_0, s, \widetilde{X}_s) - \dot{S}_x^3(\theta_0, s, x_s(\theta_0))]Y_s|\,ds.$$

By using (H4), we obtain

$$g_t \le C \int\limits_{R_t} (\sup_{u \in R_s} g_u)\,ds + C \int\limits_{R_t} |\widetilde{X}_s - x_s(\theta_0)|^a |Y_s|\,ds.$$

Now, by Lemma 4, we have

$$\sup_{s \in R_t} g_s \le C \int\limits_{R_t} (\sup_{u \in R_v} g_u)\,dv + C\varepsilon^a \sup_{u \in R_t} |W_u|^a \sup_{u \in R_t} |Y_u|.$$

In view of the Gronwall Lemma and (i), we deduce that

$$\sup_{t \in R_T} g_t \le C\varepsilon^a \sup_{t \in R_T} |W_t|^{1+a}.$$

(iii) First of all, let us prove that for any $t \in R_T$ and $s \in R_t$, the function $\theta \mapsto \gamma_t(\theta, s)$ has uniformly bounded derivatives. To this end, recall that

$$\gamma_t(\theta, s) = \sum_{n=0}^{\infty} H_n(\theta, t, s),$$

where H_n is defined by

$$H_0(\theta, s, t) = 1,$$

$$H_{n+1}(\theta, s, t) = \int\limits_{s^1}^{t^1} S^1(\theta, u^1, s^2)H_n(\theta, t, (u^1, s^2))\,du^1$$

$$+ \int\limits_{s^2}^{t^2} S^2(\theta, s^1, u^2)H_n(\theta, t, (s^1, u^2))\,du^2, \quad n \ge 0.$$

By using (H1) and an induction argument, one can prove that for any $n \ge 0$

and $s \in R_t$,

$$(2) \qquad |H_n(\theta, t, s)| \leq C^n \sum_{j=0}^{n} \binom{n}{j} \frac{(t^1 - s^1)^j (t^2 - s^2)^{n-j}}{j!(n-j)!}.$$

Next, let us prove by induction that

$$(3) \qquad \left| \frac{\partial H_n}{\partial \theta}(\theta, t, s) \right| \leq nC^n \sum_{j=0}^{n} \binom{n}{j} \frac{(t^1 - s^1)^j (t^2 - s^2)^{n-j}}{j!(n-j)!}.$$

For $n = 0$, this is obvious. Now,

$$\frac{\partial H_{n+1}}{\partial \theta}(\theta, t, s) = \int_{s^1}^{t^1} H_n(\theta, t, (u^1, s^2)) \dot{S}_\theta^1(\theta, u^1, s^2)\, du^1$$

$$+ \int_{s^1}^{t^1} S^1(\theta, u^1, s^2) \frac{\partial H_n}{\partial \theta}(\theta, t, (u^1, s^2))\, du^1$$

$$+ \int_{s^2}^{t^2} H_n(\theta, t, (s^1, u^2)) \dot{S}_\theta^2(\theta, s^1, u^2)\, du^2$$

$$+ \int_{s^2}^{t^2} S^2(\theta, s^1, u^2) \frac{\partial H_n}{\partial \theta}(\theta, t, (s^1, u^2))\, du^2.$$

By using (H1), (H2), (2) and the induction hypothesis, we deduce that

$$\left| \frac{\partial H_{n+1}}{\partial \theta}(\theta, t, s) \right| \leq (n+1)C^{n+1} \left[\int_{s^1}^{t^1} \sum_{j=0}^{n} \binom{n}{j} \frac{(u^1 - s^1)^j (t^2 - s^2)^{n-j}}{j!(n-j)!}\, du^1 \right.$$

$$\left. + \int_{s^2}^{t^2} \sum_{j=0}^{n} \binom{n}{j} \frac{(t^1 - s^1)^j (u^2 - s^2)^{n-j}}{j!(n-j)!}\, du^2 \right]$$

$$= (n+1)C^{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{(t^1 - s^1)^j (t^2 - s^2)^{n+1-j}}{j!(n+1-j)!},$$

and (3) is proved.

It follows that

$$\left| \frac{\partial H_n}{\partial \theta}(\theta, t, s) \right| \leq n(2(T^1 + T^2)C)^n \max_{j \in \{0, \dots, n\}} \left(\frac{1}{j!}, \frac{1}{(n-j)!} \right).$$

Since $\max_{j \in \{0, \dots, n\}}(1/j!, 1/(n-j)!)$ is equal to $((n/2)!)^{-2}$ if n is even, and to $(((n+1)/2)!((n-1)/2)!)^{-1}$ if n is odd, we deduce that

$$\sum_{n=0}^{\infty} \left| \frac{\partial H_n}{\partial \theta}(\theta, t, s) \right| \leq C < \infty,$$

which implies that $\theta \mapsto \gamma_t(\theta, s)$ is differentiable and

$$\frac{\partial \gamma_t}{\partial \theta}(\theta, s) = \sum_{n=0}^{\infty} \frac{\partial H_n}{\partial \theta}(\theta, t, s).$$

Therefore

$$\sup_{\theta \in \Theta} \sup_{t \in R_T} \sup_{s \in R_t} \left| \frac{\partial \gamma_t}{\partial \theta}(\theta, s) \right| < \infty.$$

By (H3), (3) and noting that for any $(i, j) \in \{1, \ldots, k\}^2$,

$$\frac{\partial^2 H_{n+1}}{\partial \theta_i \partial \theta_j}(\theta, t, s) = \int_{s^1}^{t^1} \frac{\partial^2 S^1}{\partial \theta_i \partial \theta_j}(\theta, u^1, s^2) H_n(\theta, t, (u^1, s^2))\, du^1$$

$$+ \int_{s^1}^{t^1} \frac{\partial S^1}{\partial \theta_i}(\theta, u^1, s^2) \frac{\partial H_n}{\partial \theta_j}(\theta, t, (u^1, s^2))\, du^1$$

$$+ \int_{s^1}^{t^1} \frac{\partial S^1}{\partial \theta_j}(\theta, u^1, s^2) \frac{\partial H_n}{\partial \theta_i}(\theta, t, (u^1, s^2))\, du^1$$

$$+ \int_{s^1}^{t^1} S^1(\theta, u^1, s^2) \frac{\partial^2 H_n}{\partial \theta_i \partial \theta_j}(\theta, t, (u^1, s^2))\, du^1$$

$$+ \int_{s^2}^{t^2} \frac{\partial^2 S^2}{\partial \theta_i \partial \theta_j}(\theta, s^1, u^2) H_n(\theta, t, (s^1, u^2))\, du^2$$

$$+ \int_{s^2}^{t^2} \frac{\partial S^2}{\partial \theta_i}(\theta, s^1, u^2) \frac{\partial H_n}{\partial \theta_j}(\theta, t, (s^1, u^2))\, du^2$$

$$+ \int_{s^2}^{t^2} \frac{\partial S^2}{\partial \theta_j}(\theta, s^1, u^2) \frac{\partial H_n}{\partial \theta_i}(\theta, t, (s^1, u^2))\, du^2$$

$$+ \int_{s^2}^{t^2} S^2(\theta, s^1, u^2) \frac{\partial^2 H_n}{\partial \theta_i \partial \theta_j}(\theta, t, (s^1, u^2))\, du^2,$$

one can prove that for any $n \geq 0$, $\theta \in \Theta$, $t \in R_T$ and $s \in R_T$,

$$\left| \frac{\partial^2 H_n}{\partial \theta_i \partial \theta_j}(\theta, t, s) \right| \leq n(n+1) C^n \sum_{l=0}^{n} \binom{n}{l} \frac{(t^1 - s^1)^l (t^2 - s^2)^{n-l}}{l!(n-l)!}.$$

It follows that

$$\sup_{1 \leq i,j \leq k} \sup_{\theta \in \Theta} \sup_{t \in R_T} \sup_{s \in R_t} \left| \frac{\partial^2 \gamma_t}{\partial \theta_i \partial \theta_j}(\theta, s) \right| < \infty.$$

Now, let $\dot{\gamma}_{t,\theta}$ (resp. $\ddot{\gamma}_{t,\theta}$) stand for the vector (resp. matrix) function of the first (resp. second) order derivatives of γ_t in θ. We have

$$\widetilde{x}_t(\theta) = \int\limits_{R_t} S^3(\theta, s, x_s(\theta))\dot{\gamma}_{t,\theta}(\theta, s)\, ds$$
$$+ \int\limits_{R_t} \gamma_t(\theta, s)[\dot{S}_\theta^3(\theta, s, x_s(\theta)) + \dot{S}_x^3(\theta, s, x_s(\theta))\widetilde{x}_s(\theta)]\, ds.$$

Therefore

$$|\widetilde{x}_t(\theta) - \widetilde{x}_t(\theta_0)| \leq \int\limits_{R_t} |[S^3(\theta, s, x_s(\theta)) - S^3(\theta_0, s, x_s(\theta_0))]\dot{\gamma}_{t,\theta}(\theta, s)|\, ds$$
$$+ \int\limits_{R_t} |S^3(\theta_0, s, x_s(\theta_0))[\dot{\gamma}_{t,\theta}(\theta, s) - \dot{\gamma}_{t,\theta}(\theta_0, s)]|\, ds$$
$$+ \int\limits_{R_t} |\gamma_t(\theta, s)[\dot{S}_\theta^3(\theta, s, x_s(\theta)) - \dot{S}_\theta^3(\theta_0, s, x_s(\theta_0))]|\, ds$$
$$+ \int\limits_{R_t} |[\gamma_t(\theta, s) - \gamma_t(\theta_0, s)]\dot{S}_\theta^3(\theta_0, s, x_s(\theta_0))|\, ds$$
$$+ \int\limits_{R_t} |\gamma_t(\theta, s)\dot{S}_x^3(\theta, s, x_s(\theta))(\widetilde{x}_s(\theta) - \widetilde{x}_s(\theta_0))|\, ds$$
$$+ \int\limits_{R_t} |\gamma_t(\theta, s)\widetilde{x}_s(\theta_0)[\dot{S}_x^3(\theta, s, x_s(\theta)) - \dot{S}_x^3(\theta_0, s, x_s(\theta_0))]|\, ds$$
$$+ \int\limits_{R_t} |\dot{S}_x^3(\theta_0, s, x_s(\theta_0))\widetilde{x}_s(\theta_0)[\gamma_t(\theta, s) - \gamma_t(\theta_0, s)]|\, ds.$$

By using (H1)–(H4), the boundedness of $\dot{\gamma}_{t,\theta}$, $\ddot{\gamma}_{t,\theta}$ and noting that the function $(\theta, t) \mapsto x_t(\theta)$ is bounded, we deduce that

$$(4) \quad |\widetilde{x}_t(\theta) - \widetilde{x}_t(\theta_0)|$$
$$\leq C\Big(|\theta - \theta_0| + |\theta - \theta_0|^b + |\theta - \theta_0|^d + \int\limits_{R_t} |x_s(\theta) - x_s(\theta_0)|\, ds$$
$$+ \int\limits_{R_t} |x_s(\theta) - x_s(\theta_0)|^a\, ds + \int\limits_{R_t} |x_s(\theta) - x_s(\theta_0)|^c\, ds$$
$$+ \int\limits_{R_t} |\widetilde{x}_s(\theta) - \widetilde{x}_s(\theta_0)|\, ds\Big).$$

Now, it is not difficult to see that the function $(\theta, t) \mapsto \widetilde{x}_t(\theta)$ is bounded. So,

$$\sup_{t \in R_T} |x_t(\theta) - x_t(\theta_0)| \leq |\theta - \theta_0|.$$

Hence, from (4) and the Gronwall Lemma, we deduce that

$$\sup_{t \in R_T} |\widetilde{x}_t(\theta) - \widetilde{x}_t(\theta_0)| \le C(|\theta - \theta_0| + |\theta - \theta_0|^a + |\theta - \theta_0|^b + |\theta - \theta_0|^c + |\theta - \theta_0|^d). \quad \blacksquare$$

Now, since the ingredients for the proofs of the main results are assembled, we can deal with the asymptotic behavior of the estimator θ_ε^*.

3. Results. Let $g_{\theta_0}(\delta) = \inf_{|\theta - \theta_0| > \delta} \|x(\theta) - x(\theta_0)\|_{L^2(\mu)}$ and $g(\delta) = \inf_{\theta_0 \in K} g_{\theta_0}(\delta)$ where K is an arbitrary compact subset of Θ. The following theorem ensures the consistency of θ_ε^*.

THEOREM 7. *Under* (H1) *and* (H2), *there exists a constant* $C > 0$ (*independent of* K) *such that for any* $\delta > 0$ *and* $\varepsilon \in (0, 1]$,

$$\sup_{\theta_0 \in K} P_{\theta_0}^{(\varepsilon)}(|\theta_\varepsilon^* - \theta_0| \ge \delta) \le C \exp\left(-\frac{g^2(\delta)}{\varepsilon^2 C}\right).$$

Proof. The proof is similar to that of Kutoyants and Lessi (1995), Theorem 3.1, and uses Theorem 1 and an exponential inequality for the Wiener random field. ∎

The next result concerns the asymptotic law of θ_ε^* as $\varepsilon \to 0$.

THEOREM 8. *Assume that* (H1)–(H4) *are satisfied and for any* $\delta > 0$,

$$g(\delta) > 0, \qquad \inf_{|u|=1} \langle J(\theta_0)u, u \rangle > 0.$$

Then

$$P_{\theta_0}\text{-}\lim_{\varepsilon \to 0} \varepsilon^{-1}(\theta_\varepsilon^* - \theta_0) = \xi,$$

where P_{θ_0}*-lim denotes the convergence with respect to the probability* P_{θ_0}.

Proof. The proof is essentially based on Lemma 6 and follows the same lines as that of Kutoyants and Lessi (1995), Theorem 4.3. So, we only point out the minor adaptations needed. We make the change of variable $u = \varepsilon^{-1}(\theta - \theta_0)$ and put

$$U_{\theta_0, \varepsilon} = \{u \in \mathbb{R}^k : \theta_0 + \varepsilon u \in \Theta\},$$

$$u_\varepsilon^* = \arg \inf_{u \in U_{\theta_0, \varepsilon}} \left\| Z - \frac{x(\theta_0 + \varepsilon u) - x(\theta_0)}{\varepsilon} \right\|_{L^2(\mu)}.$$

We have $u_\varepsilon^* = \varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$.

Now, set

$$A_1 = \{\omega \in \Omega : \|X - x(\theta_0)\|_{L^2(\mu)} < \inf_{\substack{u \in U_{\theta_0, \varepsilon} \\ |u| > \lambda_\varepsilon}} \|X - x(\theta_0 + \varepsilon u)\|_{L^2(\mu)}\}$$

where $\lambda_\varepsilon = \varepsilon^{-\delta}$, $\delta \in (0,1]$. Following Kutoyants and Lessi (1995), we have

$$u_\varepsilon^* = J(\theta_0)^{-1} \int_{R_T} Z_t \widetilde{x}_s(\theta_\varepsilon^*)\, d\mu(t) - J(\theta_0)^{-1}[J(\theta_\varepsilon, \theta_\varepsilon^*)' - J(\theta_0)]u_\varepsilon^*.$$

Therefore

$$|u_\varepsilon^* - \xi| \le \left| J(\theta_0)^{-1}\left[\int_{R_T} (Z_t - Y_t)\widetilde{x}_t(\theta_\varepsilon^*)\, d\mu(t) + \int_{R_T} (\widetilde{x}_t(\theta_\varepsilon^*) - \widetilde{x}_t(\theta_0))Y_t\, d\mu(t)\right]\right|$$
$$+ |J(\theta_0)^{-1}[J(\theta_\varepsilon, \theta_\varepsilon^*)' - J(\theta_0)]u_\varepsilon^*|,$$

where $J(\theta_\varepsilon, \theta_\varepsilon^*)'$ is equal to the matrix $\int_{R_t} \widetilde{x}_t(\theta_\varepsilon^*)\widetilde{x}_t'(\theta_\varepsilon)\, d\mu(t)$ and $\theta_\varepsilon = \theta_0 + \gamma_t \varepsilon u$, $\gamma_t \in [0,1)$. By Lemma 6, on A_1 we have

$$|u_\varepsilon^* - \xi| \le C(\varepsilon^a \sup_{t \in R_T} |W_t|^{1+a} + \varepsilon^{1-\delta} + \varepsilon^{a-\delta} + \varepsilon^{b-\delta} + \varepsilon^{c-\delta} + \varepsilon^{d-\delta})$$
$$+ C(\varepsilon^{1-\delta} + \varepsilon^{a(1-\delta)} + \varepsilon^{b(1-\delta)} + \varepsilon^{c(1-\delta)} + \varepsilon^{d(1-\delta)}) \sup_{t \in R_T} |W_t|.$$

Now, put $e = a \wedge b \wedge c \wedge d$. Since $\varepsilon \in [0,1)$, we deduce that on A_1 we have

$$|u_\varepsilon^* - \xi| \le C(\varepsilon^a \sup_{t \in R_T} |W_t|^{1+a} + \varepsilon^{e-\delta} + \varepsilon^{e(1-\delta)} \sup_{t \in R_T} |W_t|).$$

Let $r = a \wedge (e - \delta)$ and choose $\delta = e/(1+a)$. Then on A_1 we have

$$|u_\varepsilon^* - \xi| \le C\varepsilon^{r/2}(\varepsilon^{a-r/2} \sup_{t \in R_T} |W_t|^{1+a} + \varepsilon^{e-\delta-r/2} + \varepsilon^{e(1-\delta)-r/2} \sup_{t \in R_T} |W_t|).$$

Now, set

$$A_2 = \{\omega \in \Omega : \sup_{t \in R_T} |W_t(\omega)| < \varepsilon^{-\varrho}\} \quad \text{and} \quad A = A_1 \cap A_2$$

where $\varrho = \min\{(a - r/2)(1+a)^{-1}, e - \delta - r/2\}$. Then on A we have

$$|u_\varepsilon^* - \xi| \le C\varepsilon^{r/2}(\varepsilon^{\alpha_1} + \varepsilon^{\alpha_2} + \varepsilon^{\alpha_3})$$

where

$$\alpha_1 = a - \varrho(1+a) - r/2 \ge 0, \ \alpha_2 = e - \delta - r/2 \ge 0, \ \alpha_3 = e - \delta - \varrho - r/2 \ge 0.$$

Following Kutoyants and Lessi (1995), one can prove that $P_{\theta_0}^{(\varepsilon)}(\overline{A}) \to 0$ as $\varepsilon \to 0$ where $\overline{A} = \Omega \setminus A$, which completes the proof. ∎

Now, let us apply the above results to the two-parameter Ornstein–Ulhenbeck process with parameter $(\theta^1, \theta^2, \varepsilon)$ which is a random field H defined in Dozzi (1989), p. 155, by

$$H_t = e^{\theta^1 t^1}x + e^{\theta^2 t^2}x - e^{\theta' t}x + \varepsilon e^{\theta' t} \int_{R_T} e^{-\theta' u}x\, dW_u.$$

We remark that $H_t = x$ on the axes. Itô's formula in Guyon and Prum (1981), p. 634, implies that H satisfies the nonlinear hyperbolic stochastic

partial differential equation

$$\frac{\partial^2 H_t}{\partial t^1 \partial t^2} = \theta^1 \frac{\partial H_t}{\partial t^2} + \theta^2 \frac{\partial H_t}{\partial t^1} - \theta^1 \theta^2 H_t + \varepsilon \dot{W}_t, \quad t \in \mathbb{R}_+^2.$$

Assume that $\theta = \binom{\theta^1}{\theta^2}$ is unknown, $\theta \in \Theta$, where Θ is a bounded open subset of $\mathbb{R}^2$, and put

$$\theta_\varepsilon^{**} = \arg \inf_{\theta \in \Theta} \|H - x(\theta)\|_{L^2(\mu)}$$

where

$$x_t(\theta) = e^{\theta^1 t^1} x + e^{\theta^2 t^2} x - e^{\theta' t} x.$$

We have

COROLLARY 9. (i) *There exists $C > 0$ (independent of the compact set K) such that*

$$\sup_{\theta_0 \in K} P_{\theta_0}^{(\varepsilon)} (|\theta_\varepsilon^{**} - \theta_0| \geq \delta) \leq \exp\left(-\frac{g^2(\delta)}{\varepsilon^2 C}\right).$$

(ii) $P_{\theta_0}\text{-}\lim_{\varepsilon \to 0} \varepsilon^{-1}(\theta_\varepsilon^{**} - \theta_0) = \xi$ *where ξ is a centered Gaussian random variable with covariance matrix*

$$\Gamma = J(\theta_0)^{-1}\left[\int_{R_T}\int_{R_T} \frac{1}{\theta_0^1 \theta_0^2} (e^{\theta_0^1 |t^1 - s^1|} - e^{\theta_0^1 (t^1 + s^1)})\right.$$

$$\left. \times (e^{\theta_0^2 |t^2 - s^2|} - e^{\theta_0^2 (t^2 + s^2)}) \widetilde{x}_t(\theta_0) \widetilde{x}_s'(\theta_0) \, d\mu(t) \, d\mu(s)\right] J(\theta_0)^{-1}.$$

Proof. It suffices to verify that in this case (H1)–(H4) are satisfied and

$$Y_t = e^{\theta_0' t} \int_{R_t} e^{-\theta_0' u} \, dW_u,$$

$$E(Y_t Y_s) = \frac{1}{\theta_0^1 \theta_0^2} (e^{\theta_0^1 |t^1 - s^1|} - e^{\theta_0^1 (t^1 + s^1)})(e^{\theta_0^2 |t^2 - s^2|} - e^{\theta_0^2 (t^2 + s^2)}).$$

Then use Remark 5 and apply Theorems 7 and 8. ∎

References

H. D i e t z and Y. K u t o y a n t s (1992), *A minimum-distance estimator for diffusion processes with ergodic properties*, Tech. Report 11, Inst. Appl. Analysis and Stochastics, Berlin.

H. D i e t z and Y. K u t o y a n t s (1997), *A class of minimum-distance estimators for diffusion processes with ergodic properties*, Statistics and Decisions 15, 211–217.

M. D o z z i (1989), *Stochastic Processes with a Multidimensional Parameter*, Longman Sci. Tech.

M. F a r r é and D. N u a l a r t (1993), *Nonlinear stochastic integral equations in the plane*, Stochastic Process. Appl. 46, 219–239.

X. G u y o n and B. P r u m (1981), *Semimartingales à deux indices*, Ph.D. Thesis, Univ. de Paris VI.

S. H é n a f f (1995), *On minimum distance estimate of the parameter of the Ornstein–Uhlenbeck process*, preprint, Univ. of Angers.

H. K o r e z l i o g l u, G. M a z z i o t t o and J. S z p i r g l a s (1983), *Nonlinear filtering equations for two parameter semimartingales*, Stochastic Process. Appl. 15, 239–269.

Y. K u t o y a n t s (1994), *Identification of Dynamical Systems with Small Noise*, Kluwer, Dordrecht.

Y. K u t o y a n t s and O. L e s s i (1995), *Minimum distance estimation for diffusion random fields*, Publ. Inst. Statist. Univ. Paris 29, fasc. 3, 3–20.

Y. K u t o y a n t s, A. N e r c e s s i a n and P. P i l i b o s s i a n (1994), *On limit distribution of the minimum sup norm estimate of the parameter of the Ornstein–Uhlenbeck process*, Romanian J. Pure Appl. Math. 39, 119–139.

Y. K u t o y a n t s and P. P i l i b o s s i a n (1994), *On minimum L_1 estimate of the parameter of the Ornstein–Uhlenbeck process*, Statist. Probab. Lett. 20, 117–123.

J. N o r r i s (1995), *Twisted sheets*, J. Funct. Anal. 132, 273–334.

C. R o v i r a and M. S a n z - S o l é (1995), *A nonlinear hyperbolic SPDE*: *Aproximations and support*, in: London Math. Soc. Lecture Note Ser. 216, Cambridge Univ. Press, 241–261.

C. R o v i r a and M. S a n z - S o l é (1996), *The law of the solution to a nonlinear hyperbolic SPDE*, J. Theoret. Probab. 9, 863–901.

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