APPLICATIONES MATHEMATICAE
27,2 (2000), pp. 219–223

B. GLANC and A. JAKUBOWICZ (Szczecin)

APPLICATION OF THE WEYL CURVATURE TENSOR TO DESCRIPTION OF THE GENERALIZED REISSNER–NORDSTRØM SPACE-TIME

Abstract. The Weyl curvature tensor for the generalized Reissner–Nordstrøm space-time is determined and theorems related to the Penrose conjecture are proved.

1. Introduction. It is known that in standard cosmological models the Weyl curvature tensor vanishes in the neighbourhood of Big Bang (compare [3]). On the other hand the Weyl tensor tends to infinity in the neighbourhood of typical singularities in Black Holes.

Accordingly, R. Penrose has proposed the following conjecture on Weyl curvature ([2], Chapter 2):

(1) the Weyl curvature tensor vanishes for initial-type (P) singularities.

(2) the Weyl curvature tensor tends to infinity for final-type (F) singularities (e.g. Black Holes).

In the present paper we determine the Weyl curvature tensor for the generalized Reissner–Nordstrøm (briefly R–N) space-time and prove theorems analogous to the Penrose conjecture above.

There is a classical fact, conjectured by Albert Einstein, that the presence of matter causes the curvature of space-time. However, even an empty space-time can have non-zero Weyl curvature. Such a situation occurs for example near Black Holes and in regions where gravitation waves radiate.

2. The Weyl curvature tensor of the generalized R–N space-time. The metric tensor of the family of generalized R–N space-times has

2000 *Mathematics Subject Classification*: 53B20, 53B50.
Key words and phrases: Weyl curvature, Reissner–Nordstrøm space-time.

the following form (see [1]):

$$\operatorname{diag}(-E^{a+1}, E^{a-1}, r^2, r^2 \sin^2 \Theta). \tag{2.1}$$

This means that we have a 3-dimensional family depending on the parameters r_0, K and a. The scalar curvature of this space-time has the form (see [1])

$$T = T_a(r) = \frac{ar^2E'' - 2E^a + 2}{r^2E^a} \qquad \left(= \frac{ar^2E'' + 2}{r^2E^a} - \frac{2}{r^2} \right), \tag{2.2}$$

where

$$E = E(r) = 1 - \frac{r_0}{r} + \frac{Kr_0^2}{r^2}. \tag{2.3}$$

Observe that for $a = 0$ this space-time reduces to the ordinary R–N space-time with scalar curvature $T \equiv 0$.

In [1] for the metric tensor (2.1) the following limits for the scalar curvature $T = T_a(r)$ in (2.2) were determined:

$$\lim_{a\to 0^+} \lim_{\substack{\varepsilon\to 0^+ \\ r\neq r_0/2}} T = 0, \qquad \lim_{a\to 0^+} \lim_{\substack{\varepsilon\to 0^+ \\ r= r_0/2}} T = +\infty. \tag{2.4}$$

The limits can be represented in the following form:

$$\delta(r) = \begin{cases} 0 & \text{for } r \neq r_0/2, \\ +\infty & \text{for } r = r_0/2, \end{cases} \tag{2.5}$$

where $\delta(r)$ denotes the special Schwartz distribution, namely the Dirac delta at $r_0/2$.

The following result is an immediate consequence of the results and diagrams of [1].

Theorem 1. *The limit space-time with scalar curvature tensor defined by* (2.4) (*or by* (2.5)) *behaves*, *for points* $r \neq r_0/2$, *like an ordinary R–N space-time with zero scalar curvature* $T \equiv 0$. *Moreover*, *the scalar curvature has a singularity* $T_0 = +\infty$ *at the point* $r = r_0/2$.

The Weyl curvature tensor has the following form:

$$\begin{aligned} C_{hijk} = R_{hijk} &- \frac{1}{n-2}[g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}] \\ &+ \frac{R}{(n-1)(n-2)}(g_{ij}R_{hk} - g_{ik}R_{hj}). \end{aligned} \tag{2.6}$$

In the case of the family of generalized R–N space-times the Weyl curvature tensor has, by (2.6), six independent coordinates of the form

$$\begin{aligned}
C_{2121} &= \frac{sE^a}{6r^2}[2E + 2E^a \cot^2 \Theta - 2E^a \csc^2 \Theta - 2rE' + (a+1)r^2E''],\\
C_{3131} &= \frac{sE}{12}[-2E - 2E^a \cot^2 \Theta + 2E^a \csc^2 \Theta + 2rE' - (a+1)r^2E''],\\
C_{3232} &= \frac{1}{12E}[-2E - 2E^a \cot^2 \Theta + 2E^a \csc^2 \Theta\\
&\qquad + 2rE' - (a+1)r^2E''],\\
C_{4141} &= \frac{sE\sin^2\Theta}{12}[2E^a - 2E + 2rE' - (a+1)r^2E''],\\
C_{4242} &= \frac{\sin^2\Theta}{12E}[2E^a - 2E + 2rE' - (a+1)r^2E''],\\
C_{4343} &= \frac{r^2\sin^2\Theta}{6E^a}[-2E^a + 2E - 2rE' + (a+1)r^2E''].
\end{aligned}\tag{2.7}$$

The 4-dimensional space-time that we consider has a signature $(-,+,+,+)$ (see (2.1)) so the s above has to be taken $s=-1$.

Since $\csc^2\Theta - \cot^2\Theta = 1$, (2.7) yields

$$\begin{aligned}
-\frac{6r^2}{E^a}C_{2121} &= +\frac{12}{E}C_{3131} = -12EC_{3232} = +\frac{12}{E\sin^2\Theta}C_{4141}\\
&= -\frac{12E}{\sin^2\Theta}C_{4242} = -\frac{6E^a}{r^2\sin^2\Theta}C_{4343}\\
&= +\frac{12}{E}C = -12EC = +\frac{12}{E\sin^2\Theta}C\\
&= -\frac{12E}{\sin^2\Theta}C = -\frac{6E^a}{r^2\sin^2\Theta}C,
\end{aligned}$$

where

$$C = C(r) = 2E - 2E^a - 2rE' + (a+1)r^2E''. \tag{2.8}$$

The formula (2.8) gives the following equation:

$$2E - 2E^a - 2rE' + (a+1)r^2E'' = 0.$$

If we substitute in (2.9) E defined by (2.3) we obtain the equation (2.9) depending on the radius r.

For $a=0$ we have from (2.9) the following equation:

$$2E - 2 - 2rE' + r^2E'' = 0. \tag{2.10}$$

The solution of this equation is

$$r = r^*(r_0, K) = 2Kr_0,$$

and in particular, for $K = (1+\varepsilon)/4$, we have

$$r = r^*(r_0, (1+\varepsilon)/4) = (1+\varepsilon)r_0/2.$$

In the limit case $\varepsilon = 0$ one has

$$r = r^*(r_0, 1/4) = r_0/2.$$

The value C in (2.8) for $a = 0$ at the point $r = r_0/2$ is

$$C = 12(4K - 1).$$

In particular for $K = (1+\varepsilon)/4$ one obtains $C = 12\varepsilon$ and for $\varepsilon = 0$ this gives $C = 0$.

Hence we get the following theorems.

Theorem 2. *The Weyl curvature C for the space-time defined in Theorem 1 has, at any point $r \neq r_0/2$, the non-zero value $C = 12\varepsilon$ for $\varepsilon > 0$ close to zero.*

Theorem 3. *The Weyl curvature C for the space-time defined in Theorem 1 has the value $C = 0$ at the singular point $r = r_0/2$.*

Now we want to determine points r for which the tensor C of the Weyl curvature takes, for $a \in (0,1)$ close to zero and arbitrarily small $\varepsilon > 0$, arbitrarily small values.

We do it by means of the following equation (see (2.10)):

$$2E - 2 - 2rE' + r^2E'' = \eta \tag{2.11}$$

where $\eta \neq 0$ is close to zero.

By (2.3) the equation (2.11) has the form

$$\eta r^2 + 6rr_0 - 12Kr_0^2 = 0,$$

and its positive solution is

$$r = \frac{-3 + \sqrt{3(3 + 4K\eta)}}{\eta} r_0.$$

In particular, for $K = (1+\varepsilon)/4$ and $\varepsilon = 0$ we have $K = 1/4$ and

$$r = \frac{-3 + \sqrt{9 + 3\eta}}{\eta} r_0. \tag{2.12}$$

Theorem 4. *The Weyl curvature C takes arbitrarily small values $\eta > 0$ in the small neighbourhood of the point $r = (-3 + \sqrt{9 + 3\eta})/\eta$ of the generalized R–N space-time for $a \in (0,1)$ $(a \neq 0,\ a \neq 1)$ and arbitrarily small $\varepsilon > 0$.*

Remark. The scalar curvature T of this space-time is given in diagram (9) of [1]. For arbitrarily small ε it takes arbitrarily large values.

The formula (2.12) can be written in the form $r = wr_0$, where

$$w = \frac{-3+\sqrt{9+3\eta}}{\eta}, \qquad \eta \neq 0,\ \eta \geq -3, \tag{2.13}$$

is a function of the independent variable $\eta = C$ (i.e. the Weyl curvature). Its value depends, in turn (by virtue of (2.11)), on the radius r.

It follows from the form of the function w that if η increases in the interval $(0, +\infty)$ then the radius r decreases in the interval $r_0/2 > r > 0$.

It follows from (2.13) that for a very small neighbourhood of the radius $r = (1/2 - \alpha)r_0$ (where α is close to zero) the value of the Weyl curvature $C = \eta$ is also very small and it varies according to the formula

$$\eta = \frac{6\alpha}{(1/2-\alpha)^2}. \tag{2.14}$$

The observation above can be summarized in the following.

THEOREM 5. *For $a \in (0,1)$ $(a \neq 0,\ a \neq 1)$ and arbitrarily small $\varepsilon > 0$ the generalized R–N space time has an increasing Weyl curvature C $(C = \eta)$ in the interval $0 < \eta < +\infty$. The radius r decreases from $r_0/2$ to 0.*

COROLLARY. *The value of the Weyl curvature $C = \eta$ in a very small neighbourhood of the point $r_0/2$ (i.e. for $(1/2 - \alpha)r_0$ where α is arbitrarily small) is also very small.*

We would like to thank Dr. S. Ewert-Krzemieniewski for calculating the Weyl curvature using the software MathTensor™.

References

[1] B. Glanc, A. Jakubowicz and H. Kleczewska, *The generalized Reissner–Nordstrøm space-time*, Demonstratio Math. 30 (1997), 635–639.

[2] S. W. Hawking and R. Penrose, *The Nature of Space and Time*, Princeton Univ. Press, 1994.

[3] W. Rindler, *Essential Relativity*, Springer, New York, 1977.

Barbara Glanc and Antoni Jakubowicz
Institute of Mathematics
Technical University of Szczecin
Al. Piastów 17
70-310 Szczecin, Poland
E-mail: glanc@arcadia.tuniv.szczecin.pl

Received on 7.6.1999;
revised version on 24.9.1999