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GLOBAL EXISTENCE AND BLOW-UP FOR A COMPLETELY COUPLED FUJITA TYPE SYSTEM

Abstract. The Fujita type global existence and blow-up theorems are proved for a reaction-diffusion system of m equations ($m > 1$) in the form

$$u_{it} = \Delta u_i + u_{i+1}^{p_i}, \qquad i = 1, \ldots, m-1,$$
$$u_{mt} = \Delta u_m + u_1^{p_m}, \qquad x \in \mathbb{R}^N,\ t > 0,$$

with nonnegative, bounded, continuous initial values and positive numbers p_i. The dependence on p_i of the length of existence time (its finiteness or infinitude) is established.

1. Introduction. The main concern of this paper is to examine the following system of reaction-diffusion equations:

$$\text{(1.1)} \qquad \begin{cases} u_{it} - \Delta u_i = u_{i+1}^{p_i}, & i = 1, \ldots, m-1, \\ u_{mt} - \Delta u_m = u_1^{p_m}, \end{cases}$$

for $x \in \mathbb{R}^N$, $t > 0$, $p_i > 0$; and

$$u_i(0, x) = u_{0i}(x), \qquad i = 1, \ldots, m,\ x \in \mathbb{R}^N,$$

where u_{0i} are nonnegative, continuous, bounded functions.

We consider the behaviour of classical nonnegative solutions for (1.1) as regards their maximal existence time. We describe the cases of global existence and finite blow-up time for every solution of (1.1) (in the second situation, a nontrivial solution) in terms of p_i and N.

This work generalizes results of Fujita [F1], [F2] (for the scalar Cauchy problem), Escobedo and Herrero [EH] (for the system (1.1) with $m = 2$) and the author [R] (for the system (1.1) with $m = 3$). We want to mention that the method used here is strictly connected with the special form of the

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system (1.1). In the case of weakly coupled systems proofs are much more complicated (we refer to [EL] for the case of two equations).

We introduce some notations to formulate the main theorems. Let A_m be a matrix of the form

$$A_m = \begin{bmatrix} 0 & p_1 & & & & \\ & 0 & p_2 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & & \\ & & & & 0 & p_{m-1} \\ p_m & & & & & 0 \end{bmatrix}. \tag{1.2}$$

We denote by $\alpha = (\alpha_1, \ldots, \alpha_m)$ the unique solution of

$$(A_m - I)\alpha^t = (1, \ldots, 1)^t. \tag{1.3}$$

Let

$$\delta = \det(A_m - I) = (-1)^{m+1}\Big(\prod_{j=1}^{m} p_j - 1\Big). \tag{1.4}$$

Whenever $\delta \neq 0$, we have

$$\alpha_1 = \frac{1 + p_1 + p_1 p_2 + \ldots + \prod_{j=1}^{m-1} p_j}{\prod_{j=1}^{m} p_j - 1}, \tag{1.5}$$

$$\alpha_i = \frac{1 + p_i + p_i p_{i+1} + \ldots + \prod_{j\neq i-1} p_j}{\prod_{j=1}^{m} p_j - 1}, \qquad i = 2, \ldots, m.$$

Theorem 1.1. *Suppose* $\det(A_m - I) = 0$. *Then every solution of* (1.1) *is global.*

Theorem 1.2. *Suppose* $\det(A_m - I) \neq 0$ *and* $\max_{i=1,\ldots,m} \alpha_i < 0$. *Then all the solutions of* (1.1) *are global.*

Theorem 1.3. *Suppose* $\det(A_m - I) \neq 0$ *and* $\max_{i=1,\ldots,m} \alpha_i \geq N/2$. *Then every nontrivial solution of* (1.1) *blows up in a finite time.*

The plan of this paper is the following: we prove some auxiliary lemmas in Section 2, global existence theorems is the content of the last section, whereas the global nonexistence theorem is proved in Section 3.

2. Preliminaries. Let $S(t)$ be the semigroup operator for the heat equation, i.e.

$$S(t)\xi_0(x) = \int_{\mathbb{R}^N} (4\pi t)^{-N/2} \exp\Big(-\frac{|x-y|^2}{4t}\Big)\xi_0(y)\,dy.$$

A classical solution of (1.1) satisfies the following variation of constants formulas:

$$
\begin{aligned}
u_i(t) &= S(t)u_{0i} + \int_0^t S(t-s)u_{i+1}^{p_i}(s)\,ds, \qquad i = 1,\ldots,m-1, \\
u_m(t) &= S(t)u_{0m} + \int_0^t S(t-s)u_1^{p_m}(s)\,ds.
\end{aligned} \tag{2.1}
$$

It follows that

$$
\begin{aligned}
u_i(\tau) &= S(\tau - t_i)u_i(t_i) + \int_0^{\tau - t_i} S(\tau - t_i - s)u_{i+1}^{p_i}(s)\,ds \\
&\geq S(\tau - t_i)u_i(t_i), \qquad i = 1,\ldots,m-1, \\
u_m(\tau) &= S(\tau - t_m)u_m(t_m) + \int_0^{\tau - t_m} S(\tau - t_m - s)u_1^{p_m}(s)\,ds \\
&\geq S(\tau - t_m)u_m(t_m).
\end{aligned}
$$

We remark that if (x_i, t_i) are such that $u_i(x_i, t_i) > 0$, $i = 1,\ldots,m$, then by positivity of $S(t)$, $u_i(\tau) > 0$ for $\tau > t_i$. Taking $t_0 = \max_{i=1,\ldots,m} t_i$ we have $u_i(x,\tau) > 0$ for $x \in \mathbb{R}^N$, $\tau > t_0$, $i = 1,\ldots,m$.

LEMMA 2.1. *Let $u = (u_1,\ldots,u_m)$ with $u_0 = (u_{01},\ldots,u_{0m}) \not\equiv 0$ be a solution of (1.1). Then we can choose $\tau = \tau(u_{01},\ldots,u_{0m})$ and some constants $c > 0$, $a > 0$ such that $\min_i(u_i(\tau)) \geq ce^{-a|x|^2}$.*

Proof. We can assume that for instance $u_{01} \neq 0$. Then there exists $R > 0$ satisfying

$$\nu_1 = \inf\{u_{01}(x) : |x| < R\} > 0.$$

By formula (2.1) we have

$$u_1(t) \geq S(t)u_{01} \geq \nu_1(4\pi t)^{-N/2} \exp\left(\frac{-|x|^2}{4t}\right) \int_{|y|\leq R} \exp\left(\frac{-|y|^2}{4t}\right) dy.$$

We put

$$
\begin{gathered}
\overline{u}_1(t) = u_1(t + \tau_0) \qquad \text{for some } \tau_0 > 0, \\
a_1 = \frac{1}{4\tau_0}, \qquad c_1 = \nu_1(4\pi\tau_0)^{-N/2} \int_{|y|\leq R} \exp\left(\frac{-|y|^2}{4\tau_0}\right) dy.
\end{gathered}
$$

Hence

$$\overline{u}_1(0) = u_1(\tau_0) > c_1 \exp(-a_1|x|^2).$$

To obtain the assertion we use an inductive argument. By (2.1),

$$u_i(\tau_0) \geq \int_0^{\tau_0} S(\tau_0 - s)(S(s)u_{0(i+1)})^{p_i}\,ds.$$

If $p_i \ge 1$ then

$$u_i(\tau) \ge \int_0^\tau (S(\tau-s)S(s)u_{0(i+1)})^{p_i}\,ds = \tau(S(\tau)u_{0(i+1)})^{p_i}.$$

Otherwise,

$$u_i(\tau) \ge \int_0^\tau S(\tau-s)S(s)(u_{0(i+1)})^{p_i}\,ds = \tau S(\tau)u_{0(i+1)}^{p_i}.$$

It is clear that if we assume that $u_{0(i+1)}$ satisfies $u_{0(i+1)} \ge c\exp(-a|x|^2)$ for some constants c and a, then an analogous estimate holds true for $u_i(\tau)$. Replacing $u_{0(i+1)}$ by $u_{i+1}(\tau_0)$ for some τ_0 if necessary, we get the conclusion. ■

To fix ideas, we assume henceforth that

$$\max_i \alpha_i = \alpha_1 \quad \text{if} \quad \prod_{j=1}^m p_j > 1.$$

LEMMA 2.2. *Let $(u_1(t),\ldots,u_m(t))$ be a bounded solution of* (1.1) *in some strip $[0,T)\times\mathbb{R}^N$, $0<T\le\infty$. Assume that $\prod_{j=1}^m p_j > 1$. Let n be an integer such that*

$$p_i \ge 1 \quad \text{for } 1\le i\le n, \qquad p_i<1 \quad \text{for } n<i\le m. \tag{2.2}$$

Then there exists a positive constant C, depending on p_i only, such that

$$\begin{aligned} t^{\prod_{j=n+1}^m p_j\alpha_1}\|S(t)u_{01}^{\prod_{j=n+1}^m p_j}\|_\infty &\le C \quad \text{if } n<m,\\ t^{\alpha_1}\|S(t)u_{01}\|_\infty &\le C \quad \text{if } n=m,\ t\in[0,T). \end{aligned} \tag{2.3}$$

Proof. Using (2.1) for $i=1$ in (2.1) for $i=m$ we get

$$u_m(t) \ge \int_0^t S(t-s)(S(s)u_{01})^{p_m}\,ds.$$

Consider first $n=m$. Then by the Jensen inequality for $p_m\ge1$ we have

$$u_m(t) \ge \int_0^t (S(t)u_{01})^{p_m}\,ds = t(S(t)u_{01})^{p_m}. \tag{2.4}$$

Using (2.4) in (2.1) for $i=m-1$ and the Jensen inequality for $p_mp_{m-1}\ge1$ we get

$$\begin{aligned} u_{m-1}(t) &\ge \int_0^t S(t-s)(s(S(s)u_{01})^{p_m})^{p_{m-1}}\,ds\\ &\ge \int_0^t s^{p_{m-1}}(S(t)u_{01})^{p_mp_{m-1}}\,ds\\ &\ge \frac{1}{p_{m-1}+1}(S(t)u_{01})^{p_mp_{m-1}}t^{p_{m-1}+1}. \end{aligned}$$

We now reason by induction. Assume that

$$u_{m-j}(t) \ge c_j (S(t)u_{01})^{\pi_j} t^{r_j}, \tag{2.5}$$

where

$$\begin{aligned} \pi_j &= p_{m-j}\pi_{j-1}, & \pi_0 &= p_m, \\ r_j &= p_{m-j} r_{j-1} + 1, & r_0 &= 1, \\ c_j &= \frac{1}{r_j} c_{j-1}^{p_{m-j}}, & c_0 &= 1. \end{aligned} \tag{2.6}$$

Substituting (2.5) in (2.1) for $i = m - j - 1$, by the Jensen inequality we obtain

$$\begin{aligned} u_{m-(j+1)}(t) &\ge \int_0^t S(t-s)[c_j (S(s)u_{01})^{\pi_j} s^{r_j}]^{p_{m-(j+1)}}\, ds \\ &\ge c_j^{p_{m-(j+1)}} \int_0^t s^{r_j p_{m-j-1}} (S(t)u_{01})^{p_{m-j-1}\pi_j}\, ds \\ &\ge \frac{c_j^{p_{m-j-1}}}{r_j p_{m-j-1} + 1} (S(t)u_{01})^{p_{m-j-1}\pi_j} t^{r_j p_{m-j-1}+1}, \end{aligned}$$

whence, by (2.6),

$$u_{m-j-1}(t) \ge c_{j+1} (S(t)u_{01})^{\pi_{j+1}} t^{r_{j+1}}$$

and for $j = m-1$, (2.5) gives

$$u_1(t) \ge c_{m-1} (S(t)u_{01})^{\pi_{m-1}} t^{r_{m-1}}, \tag{2.7}$$

where

$$\begin{aligned} \pi_{m-1} &= \prod_{j=1}^m p_j, \\ r_{m-1} &= (\ldots(((p_{m-1}+1)p_{m-2}+1)p_{m-3}+1)\ldots)p_1 + 1 \\ &= \alpha_1\Big(\prod_{j=1}^m p_j - 1\Big), \\ c_{m-1}^{-1} &= (p_{m-1}+1)^{p_1\cdots p_{m-2}} (p_{m-2}(p_{m-1}+1)+1)^{p_1\cdots p_{m-3}} \ldots \\ &\qquad \ldots [(\ldots((p_{m-1}+1)p_{m-2}+1)\ldots)p_1 + 1]. \end{aligned} \tag{2.8}$$

Iterating this scheme and setting $p = \pi_{m-1}$, $r = r_{m-1}$, we obtain an estimate for $u_1(t)$ from below:

$$u_1(t) \ge \prod_{i=1}^m A_i^k (S(t)u_{01})^{p^k} t^{r(1+p+p^2+\ldots+p^{k-1})}, \tag{2.9}$$

where

$$\begin{aligned} 1/A_1^k = r^{p^{k-1}} [r(p+1)]^{p^{k-2}} [r(p^2+p+1)]^{p^{k-3}} \ldots \\ \ldots [r(p^{k-1} + \ldots + p^2 + p + 1)], \end{aligned} \tag{2.10}$$

$$\begin{aligned}
1/A_m^k &= (p_m r+1)^{p^{k-1}/p_m}[p_m r(p+1)+1]^{p^{k-2}/p_m}\\
&\quad\times[p_m r(p^2+p+1)+1]^{p^{k-3}/p_m}\ldots\\
&\quad\ldots[p_m r(p^{k-2}+\ldots+p^2+p+1)+1]^{p/p_m},\\
1/A_{m-1}^k &= (p_{m-1}+1)^{p^k/(p_{m-1}p_m)}(p_{m-1}p_m r+p_{m-1}+1)^{p^{k-1}/(p_{m-1}p_m)}\\
&\quad\times[p_{m-1}p_m r(p+1)+p_{m-1}+1]^{p^{k-2}/(p_{m-1}p_m)}\ldots\\
&\quad\ldots[p_{m-1}p_m r(p^{k-2}+\ldots+p^2+p+1)+p_{m-1}+1]^{p/(p_{m-1}p_m)},\\
1/A_{m-i}^k &= [p_{m-i}(\ldots(p_{m-1}+1)+\ldots+1)]^{p^k/(p_{m-i}\cdots p_m)}\\
&\quad\times[p_{m-i}(p_{m-i+1}(\ldots(p_m r+1)\\
&\qquad\qquad+\ldots+1)+1)+1]^{p^{k-1}/(p_{m-i}\cdots p_m)}\\
&\quad\times[p_{m-i}(\ldots(p_m r(p+1)+1)+\ldots+1)+1]^{p^{k-2}/(p_{m-i}\cdots p_m)}\ldots\\
&\quad\ldots[p_{m-i}(\ldots(p_m r(p^{k-2}+\ldots+p+1)+1)\\
&\qquad\qquad+\ldots+1)+1]^{p/(p_{m-i}\cdots p_m)}.
\end{aligned}\tag{2.10 [cont.]}$$

We can rewrite $1/A_1^k$ in (2.10) as

$$1/A_1^k = r^{(p^k-1)/(p-1)}\prod_{j=1}^{k-1}\left(\frac{p^{j+1}-1}{p-1}\right)^{p^{k-j-1}}.\tag{2.11}$$

By assumption and $p>1$, we have $\max\alpha_i=\alpha_1$, so the following inequalities remain true for $j\ge 0$:

$$\begin{aligned}
&r(1+p+\ldots+p^j)=\alpha_1(p-1)(1+p+\ldots+p^j)>1,\\
&r(1+p+\ldots+p^j)>1+p_{m-1},\\
&r(1+p+\ldots+p^j)>1+p_{m-2}p_{m-1}+p_{m-2},\ \ldots,\\
&r(1+p+\ldots+p^j)>1+p_{m-i}\ldots p_{m-1}+p_{m-i}\ldots p_{m-2}+\ldots+p_{m-i},
\end{aligned}$$

for $i=1,\ldots,m-2$.

Using this in (2.10) for A_2^k to A_m^k we obtain

$$\begin{aligned}
1/A_m^k &\le [r(p_m+1)]^{p^{k-1}/p_m}[r(p+1)(p_m+1)]^{p^{k-2}/p_m}\ldots\\
&\quad\ldots[r(p^{k-2}+\ldots+p^2+p+1)(p_m+1)]^{p/p_m}\\
&= [r(p_m+1)]^{\frac{1}{p_m}\frac{p^k-p}{p-1}}\left[\prod_{j=1}^{k-2}\left(\frac{p^{j+1}-1}{p-1}\right)^{p^{k-j-1}}\right]^{1/p_m},\\
1/A_{m-1}^k &\le [r(p_{m-1}p_m+1)]^{p^{k-1}/(p_{m-1}p_m)}\\
&\quad\times[r(p+1)(p_{m-1}p_m+1)]^{p^{k-2}/(p_{m-1}p_m)}\ldots\\
&\quad\ldots[r(p^{k-2}+\ldots+p^2+p+1)\\
&\quad\times(p_{m-1}p_m+1)]^{p/(p_{m-1}p_m)}r^{p^k/(p_{m-1}p_m)}
\end{aligned}\tag{2.12}$$

$$= r^{p^k/(p_{m-1}p_m)}[r(p_{m-1}p_m+1)]^{\frac{1}{p_{m-1}p_m}\frac{p^k-p}{p-1}}$$
$$\times\Big[\prod_{j=1}^{k-2}\Big(\frac{p^{j+1}-1}{p-1}\Big)^{p^{k-j-1}}\Big]^{1/(p_{m-1}p_m)}, \ \ldots,$$

$$(A^k_{m-i})^{-\prod_{j=0}^{i}p_{m-j}} \tag{2.13}$$
$$\le \Big[\Big(\prod_{j=0}^{i}p_{m-j}+1\Big)r\Big]^{p^{k-1}}\Big[r(p+1)\Big(\prod_{j=0}^{i}p_{m-j}+1\Big)\Big]^{p^{k-2}}\ldots$$
$$\ldots\Big[r(p^{k-2}+\ldots+p+1)\Big(\prod_{j=0}^{i}p_{m-j}+1\Big)\Big]^{p}r^{p^k}$$
$$= r^{p^k}\Big[\Big(\prod_{j=0}^{i}p_{m-j}+1\Big)\Big]^{\frac{p^k-p}{p-1}}\prod_{j=1}^{k-2}\Big(\frac{p^{j+1}-1}{p-1}\Big)^{p^{k-j-1}},$$

for $i = 1,\ldots,m-2$.

Substituting (2.11), (2.12) and (2.13) for $i = 1,\ldots,m-2$ into (2.9) we infer

$$(S(t)u_{01})^{p^k}t^{r\frac{p^k-1}{p-1}} \le r^{\frac{p^k-1}{p-1}[1+\frac{p}{p_m}+\frac{p}{p_{m-1}p_m}+\ldots+\frac{p}{\prod_{j=0}^{m-2}p_{m-j}}]} \tag{2.14}$$
$$\times\prod_{i=0}^{m-2}\Big[\Big(1+\prod_{j=0}^{i}p_{m-j}\Big)^{\frac{p^k-p}{p-1}(\prod_{j=0}^{i}p_{m-j})^{-1}}\Big]u(t)$$
$$\times\Big[\prod_{j=1}^{k-1}\Big(\frac{p^{j+1}-1}{p-1}\Big)^{p^{k-j-1}}\Big]^{1+\frac{1}{p_m}+\frac{1}{p_{m-1}p_m}+\ldots+(\prod_{j=0}^{m-2}p_{m-j})^{-1}}.$$

Using (1.5) and (2.2) we observe that

$$1+\frac{p}{p_m}+\ldots+\frac{p}{p_m\ldots p_2} = 1+p_1+p_1p_2+\ldots+p_1\ldots p_{m-1} = r,$$
$$1+\frac{1}{p_m}+\ldots+\frac{1}{p_m\ldots p_2} = 1+\frac{r-1}{p},$$
$$\prod_{j=0}^{i}p_{m-j} \le \prod_{j=0}^{m-1}p_{m-j} = p \quad \text{for all } i = 0,1,\ldots,m-2,$$

whence by (2.14) we obtain

$$(S(t)u_{01})^{p^k}t^{r\frac{p^k-1}{p-1}} \tag{2.15}$$
$$\le r^{\frac{p^k-1}{p-1}r}(1+p)^{\frac{p^k-p}{p-1}\frac{r-1}{p}}\Big[\prod_{j=1}^{k-1}\Big(\frac{p^{j+1}-1}{p-1}\Big)^{k-j-1}\Big]^{1+\frac{r-1}{p}}u(t).$$

Now, we have to estimate

$$B_k = \prod_{j=1}^{k-1} \left(\frac{p^{j+1}-1}{p-1}\right)^{k-j-1} = \prod_{j=1}^{k} \left(\frac{p^{j}-1}{p-1}\right)^{k-j}.$$

Thus

$$\ln B_k = \sum_{j=1}^{k} p^{k-j}[\ln(p^j-1)-\ln(p-1)] \tag{2.16}$$
$$\le p\ln p\frac{p^k-1}{(p-1)^2} - \ln(p-1)\frac{p^k-1}{p-1} = \frac{p^k-1}{p-1}\ln\left(\frac{p^{\frac{p}{p-1}}}{p-1}\right)$$
$$= \ln\left[\left(\frac{p^{\frac{p}{p-1}}}{p-1}\right)^{\frac{p^k-1}{p-1}}\right] = \ln[b^{\frac{p^k-1}{p-1}}],$$

where $b = p^{p/(p-1)}/(p-1) > 0$.

Employing this in (2.15) we finally have

$$S(t)u_{01}t^{r\frac{p^k-1}{p^k(p-1)}} \le r^{\frac{(p^k-1)r}{p^k(p-1)}}(1+p)^{\frac{p^k-p}{p^k(p-1)}\frac{r-1}{p}}$$
$$\times b^{\frac{p^k-1}{p^k(p-1)}(1+\frac{r-1}{p})}\|u(t)\|_\infty^{1/p^k}.$$

Using $\|u(t)\|_\infty < \infty$ and letting $k\to\infty$, we get

$$t^{r/(p-1)}\|S(t)u_{01}\|_\infty \le c < \infty, \tag{2.17}$$

where $c = c(p,r,b) = c(p_i)$, $i = 1,\ldots,m$.

Noting $r = \alpha_1(p-1)$ we get the assertion.

Now we assume that $n < m$, i.e. $p_i < 1$ for $n < i \le m$. Then instead of (2.4) we get, using the Jensen inequality for $p_m < 1$,

$$u_m(t) \ge \int_0^t S(t)u_{01}^{p_m}\,ds = tS(t)u_{01}^{p_m}$$

and, analogously, for $p_{m-j} < 1$ (i.e. for $m-j > n$) we can prove inductively, exactly as (2.5),

$$u_{m-j}(t) \ge c_j S(t)u_{01}^{\pi_j}t^{r_j}, \tag{2.18}$$

where c_j, π_j, r_j are given by (2.6).

Using (2.18) for $j = m-n-1$ in (2.1) for $i = n$, and the Jensen inequality for $p_n \ge 1$, we get

$$u_n(t) \ge c_{m-n}(S(t)u_{01}^{\pi_{m-n-1}})^{p_n}t^{r_{m-n}} \quad \text{where} \quad \pi_{m-n-1} = \prod_{j=n+1}^{m} p_j.$$

Again, arguing by induction as in the proof of formula (2.5), we conclude

$$u_{m-j}(t) \ge c_j(S(t)u_{01}^{\pi_{m-n-1}})^{p_n\ldots p_{m-j}}t^{r_j}$$

for $j \geq m-n$, and

$$u_1(t) \geq c_{m-1}(S(t)u_{01}^{\pi_{m-n-1}})^{p_n \cdots p_1} t^{r_{m-1}} = c_{m-1}(S(t)u_{01}^{\pi_{m-n-1}})^{\pi_{m-1}/\pi_{m-n-1}} t^{r_{m-1}}, \tag{2.19}$$

where $\pi_{m-n-1} = \prod_{i=n+1}^{m} p_i$ by (2.6) and π_{m-1}, r_{m-1}, c_{m-1} are given by (2.8).

Let us remark that $\pi_{m-1}/\pi_{m-n-1} = \prod_{i=1}^{n} p_i \geq 1$ and since $\pi_{m-1} > 1$, also $\prod_{i=1}^{l} p_i \geq \pi_{m-1} > 1$ for any $n \leq l \leq m$. It follows that we apply henceforth the Jensen inequality with an exponent greater than 1. Therefore, repeating the above considerations, we get instead of (2.9),

$$u_1(t) \geq \prod_{i=1}^{m} A_i^k (S(t)u_{01}^{\pi_{m-n-1}})^{p^k/\pi_{m-n-1}} t^{r\frac{p^k-1}{p-1}}, \tag{2.20}$$

where the A_i^k are defined by (2.10). To estimate $\prod_{i=1}^{m} A_i^k$, we proceed in the same way as before (because we have not used n there), whence

$$\|S(t)u_{01}^{\pi_{m-n-1}}\|_\infty t^{\pi_{m-n-1}\alpha_1} \leq c < \infty$$

after letting $k \to \infty$. Thus the proof is complete. ∎

LEMMA 2.3. *Let the assumptions of Lemma 2.2 be satisfied. Then we can find a constant $c > 0$, $c = c(p_i)$, $i = 1, \ldots, m$, such that for $t > 0$,*

$$\|S(t)u_1^{\pi_{m-n-1}}\|_\infty t^{\pi_{m-n-1}\alpha_1} \leq c < \infty. \tag{2.21}$$

Proof. For $\tau, t \geq 0$ we have, by (2.1) for $i = 1$,

$$u_1(t+\tau) = S(t+\tau)u_{01} + \int_0^{t+\tau} S(t+\tau-s)u_2^{p_1}(s)\,ds$$
$$= S(t)u_1(\tau) + \int_0^{t} S(t-s)(u_2(\tau+s))^{p_1}\,ds.$$

It now follows that we can replace u_{01} by $u_1(\tau)$ in (2.3), whence we have the conclusion by Lemma 2.1 and setting $t = \tau$. ∎

3. Proof of Theorem 1.3. We prove Theorem 1.3 by contradiction. Assuming that (2.2) holds and $\max \alpha_i \geq N/2$ we see by (1.5) that $\alpha_1 \geq N/2$ and $p = \prod_{j=1}^{m} p_j > 1$. We shall derive a lower bound for a solution which contradicts the bounds obtained in Lemmas 2.2 and 2.3.

By Lemma 2.1 we have, for nontrivial initial data of (1.1),

$$u_{01}(x) \geq c_* e^{-a|x|^2}$$

for some constants $c_* > 0$, $a > 0$. Employing this in (2.1) for $i = 1$, since

$$S(t)e^{-a|x|^2} = (1+4at)^{-N/2} \exp\left(\frac{-a|x|^2}{1+4at}\right),$$

we get

$$u_1(t) \ge S(t)u_{01} \ge c_*(1+4at)^{-N/2}\exp\left(\frac{-a|x|^2}{1+4at}\right). \tag{3.1}$$

Using this bound in (2.1) for $i=m$, we have

$$\begin{aligned} u_m(t) &\ge \int_0^t S(t-s)(u_1(s))^{p_m}\,ds \\ &\ge c_*^{p_m}\int_0^t S(t-s)(1+4as)^{-Np_m/2}\exp\left(\frac{-ap_m|x|^2}{1+4as}\right)ds \\ &= c_*^{p_m}\int_0^t (1+4as)^{-N(p_m-1)/2}(1+4as+4ap_m(t-s))^{-N/2} \\ &\quad\times\exp\left(\frac{-ap_m|x|^2}{1+4as+4ap_m(t-s)}\right)ds. \end{aligned} \tag{3.2}$$

Recall the definition of the number n in Lemma 2.2. By induction, we prove that the following estimate holds for $m-j>n$ and some constant C:

$$u_{m-j}(t) \ge c(1+4at)^{-N/2}(4at-\varrho_j)^{\beta_j}\exp\left(\frac{-a\pi_j|x|^2}{1+4a\pi_j t}\right), \tag{3.3}$$

where $\varrho_j = \varrho^{2^{j-1}}$ for $j>0$, $\varrho_0=0$, $\varrho>1$,

$$\begin{aligned} \pi_j &= p_{m-j}\pi_{j-1} = \prod_{k=m-j}^{m} p_j, \qquad \pi_0 = p_m, \\ \beta_j &= p_{m-j}\beta_{j-1} + \frac{N}{2}(1-p_{m-j})+1, \qquad \beta_{-1}=0, \end{aligned} \tag{3.4}$$

and $t>\varrho_j/(4a)$.

Let $n<m$. Then $p_m<1$, $p_{n+1}<1$ and $p_n\ge 1$ by definition, and for the function $f_m(s)=1+4as+4ap_m(t-s)$, we have $f'(s)=4a(1-p_m)>0$. Thus $f_m(0)\le f_m(s)\le f_m(t)$ for $s\in[0,t]$. Using this, we estimate (3.2) as follows:

$$\begin{aligned} u_m(t) &\ge c_*^{p_m}(1+4at)^{-N/2}\exp\left(\frac{-ap_m|x|^2}{1+4ap_m t}\right)\int_0^t (1+4as)^{-N(p_m-1)/2}\,ds \\ &\ge c(1+4at)^{-N/2}(4at)^{1-N(p_m-1)/2}\exp\left(\frac{-ap_m|x|^2}{1+4ap_m t}\right), \end{aligned}$$

so we have (3.3) for $j=0$.

Substituting (3.3) in (2.1) for $i = m-j-1$, we have

$$\begin{aligned}
u_{m-(j+1)}(t) &\\
\geq& \int_{\varrho_j/(4a)}^{t} S(t-s)(u_{m-j}(s))^{p_{m-j-1}}\,ds\\
\geq& c\int_{\varrho_j/(4a)}^{t} S(t-s)(1+4as)^{-Np_{m-j-1}/2}(4as-\varrho_j)^{p_{m-j-1}\beta_j}\\
&\times \exp\left(\frac{-ap_{m-j-1}\pi_j|x|^2}{1+4a\pi_j s}\right)ds\\
=& c\int_{\varrho_j/(4a)}^{t} (1+4as)^{-Np_{m-j-1}/2}(1+4a\pi_j s)^{N/2}\\
&\times (1+4a\pi_j s+4a\pi_j p_{m-j-1}(t-s))^{-N/2}\\
&\times \exp\left(\frac{-ap_{m-j-1}\pi_j|x|^2}{1+4a\pi_j s+4a\pi_j p_{m-j-1}(t-s)}\right)(4as-\varrho_j)^{p_{m-j-1}\beta_j}\,ds.
\end{aligned}$$

Consider $f_{m-j-1}(s) = 1+4a\pi_j s+4a\pi_{j+1}(t-s)$. From $f'_{m-j-1}(s) = 4a\pi_j(1-p_{m-j-1}) > 0$, we can conclude

$$\begin{aligned}
u_{m-(j+1)}(t) &\\
\geq& c_j^{p_{m-j-1}}(1+4a\pi_j t)^{-N/2}\exp\left(\frac{-a\pi_{j+1}|x|^2}{1+4a\pi_{j+1}t}\right)\\
&\times \int_{\varrho_j/(4a)}^{t} (1+4as)^{-Np_{m-j-1}/2}(4as-\varrho_j)^{p_{m-j-1}\beta_j}(1+4a\pi_j s)^{N/2}\,ds.
\end{aligned}$$

To estimate the integral, notice that

$$(3.5)\qquad (1+4a\pi_j t)^{N/2} \geq \pi_j^{N/2}(1+4at)^{N/2} \quad \text{since } \pi_j < 1,$$

$$(3.6)\qquad (1+4a\pi_j t)^{-N/2} \geq (1+4a\pi_{m-1}t)^{-N/2} \geq \pi_{m-1}^{-N/2}(1+4at)^{-N/2},$$

and

$$(3.7)\qquad 4at-\varrho_j \geq c_0(1+4at) \quad \text{for } t > \varrho_j^2/(4a) \text{ with } c_0 = \varrho_j\frac{\varrho_j-1}{\varrho_j^2+1}.$$

Thus, integrating we get

$$\begin{aligned}
u_{m-j-1}(t) \geq c(1+4at)^{-N/2}&\exp\left(\frac{-a\pi_{j+1}|x|^2}{1+4a\pi_{j+1}t}\right)\\
&\times (4at-\varrho_j^2)^{1+N(1-p_{m-j-1})/2+p_{m-j-1}\beta_j}
\end{aligned}$$

for $t > \varrho_j^2/(4a)$. This, in view of (3.4), gives (3.3) with j replaced by $j+1$.

Analogously, for $m-j \le n$ we have the following estimate for any $t > \varrho_j/(4a)$:

$$u_{m-j}(t) \ge c(1+4at)^{-N/2}(4at-\varrho_j)^{\beta_j} \exp\left(\frac{-a\pi_j|x|^2}{1+4a\pi_{m-n-1}t}\right), \tag{3.8}$$

with ϱ_j, β_j, π_j defined in (3.4).

We proceed in a similar way. Assuming (3.8) for some $j \ge m-n$ and using (2.1) for $i = m-(j+1)$ we obtain

$$\begin{aligned} u_{m-j-1}(t) \ge c \int\limits_{\varrho_j/(4a)}^{t} & (1+4as)^{-Np_{m-j-1}/2}(4as-\varrho_j)^{p_{m-j-1}\beta_j} \\ & \times (1+4a\pi_{m-n-1}s)^{N/2} \\ & \times (1+4a\pi_{m-n-1}s + 4a\pi_j p_{m-j-1}(t-s))^{-N/2} \\ & \times \exp\left(\frac{-ap_{m-j-1}\pi_j|x|^2}{1+4a\pi_{m-n-1}s+4a\pi_j p_{m-j-1}(t-s)}\right) ds. \end{aligned}$$

Therefore, we have $g_{m-j-1}(s) = 1 + 4a(\pi_{m-n-1}s + \pi_j p_{m-j-1}(t-s))$ so

$$g'_{m-j-1}(s) = 4a\pi_{m-n-1}\left(1 - \frac{\pi_{j+1}}{\pi_{m-n-1}}\right) = 4a\pi_{m-n-1}\left(1 - \prod_{k=m-j-1}^{n} p_k\right)$$

and since $p_k \ge 1$ for $k \le n$ we get $g'_{m-j-1}(s) \le 0$. Thus

$$\begin{aligned} u_{m-j-1}(t) &\ge c(1+4a\pi_{j+1}t)^{-N/2} \exp\left(\frac{-a\pi_{j+1}|x|^2}{1+4a\pi_{m-n-1}t}\right) \\ &\times \int\limits_{\varrho_j/(4a)}^{t} (1+4as)^{-Np_{m-j-1}/2}(4as-\varrho_j)^{p_{m-j-1}\beta_j}(1+4a\pi_{m-n-1}s)^{N/2}\, ds. \end{aligned}$$

Using (3.5) with π_j replaced by $\pi_{m-n-1} < 1$, (3.6) with π_{j+1} and integrating we obtain (3.8) with j replaced by $j+1$ and $t > \varrho_{j+1}/(4a)$.

We need a lower bound for $u_1(t)$. By assumption $\prod_{j=1}^m p_j > 1$, whence by (2.2), $p_1 > 1$, but we can have $p_2 < 1$ or $p_2 \ge 1$. In the former case we have $m-j = 2 > n = 1$, therefore (3.3) gives

$$u_2(t) \ge c(1+4at)^{-N/2}(4at-\varrho_{m-2})^{\beta_{m-2}} \exp\left(\frac{-a\pi_{m-2}|x|^2}{1+4a\pi_{m-2}t}\right). \tag{3.9}$$

Otherwise, from $m-j = 2 \le n$ and (3.8) we get

$$u_2(t) \ge c(1+4at)^{-N/2}(4at-\varrho_{m-2})^{\beta_{m-2}} \exp\left(\frac{-a\pi_{m-2}|x|^2}{1+4a\pi_{m-n-1}t}\right). \tag{3.10}$$

Defining $M = \pi_{m-n-1}$ (with $\pi_{m-2} = M$ for $n = 1$), we conclude

$$u_2(t) \ge c(1+4at)^{-N/2}(4at - \varrho_{m-2})^{\beta_{m-2}} \exp\left(\frac{-a\pi_{m-2}|x|^2}{1+4aMt}\right). \tag{3.11}$$

Using (2.1) for $i = 1$ we obtain, as $p_1 > 1$,

$$\begin{aligned} u_1(t) &\ge \int_{\varrho_{m-2}/(4a)}^{t} S(t-s)(u_2(s))^{p_1}\,ds \\ &\ge c \int_{\varrho_{m-2}/(4a)}^{t} (1+4as)^{-Np_1/2}(1+4aMs)^{N/2} \\ &\quad \times (1+4aMs+4a\pi_{m-1}(t-s))^{-N/2} \\ &\quad \times \exp\left(\frac{-a\pi_{m-1}|x|^2}{1+4aMs+4a\pi_{m-1}(t-s)}\right)(4as-\varrho_{m-2})^{p_1\beta_{m-2}}\,ds \\ &\ge c(1+4a\pi_{m-1}t)^{-N/2}\exp\left(\frac{-a\pi_{m-1}|x|^2}{1+4aMt}\right) \\ &\quad \times \int_{\varrho_{m-2}/(4a)}^{t} (4as-\varrho_{m-2})^{p_1\beta_{m-2}}(1+4aMs)^{N/2}(1+4as)^{-Np_1/2}\,ds \\ &\ge c(1+4at)^{-N/2}\exp\left(\frac{-a\pi_{m-1}|x|^2}{1+4aMt}\right)\int_{\varrho_{m-1}/(4a)}^{t}(1+4as)^{\beta_{m-1}-1}\,ds. \end{aligned} \tag{3.12}$$

By recursive definition (3.4) and a routine inductive calculation we get

$$\begin{aligned} \beta_j = 1 &+ p_{m-j} + p_{m-j}p_{m-j-1} + \ldots \\ &\ldots + p_{m-j}\ldots p_{m-1} + \frac{N}{2}(1 - p_{m-j}\ldots p_m), \end{aligned}$$

therefore, using the definition of p, r, α_1 (see (1.5), (2.8)), we obtain

$$\begin{aligned} \beta_{m-1} &= 1 + p_1 + p_1p_2 + \ldots + p_1\ldots p_{m-1} + \frac{N}{2}(1 - p_1\ldots p_m) \\ &= r + \frac{N}{2}(1-p). \end{aligned}$$

Since $\alpha_1 \ge N/2$, $p > 1$ and $r = \alpha_1(p-1)$ we see that $\beta_{m-1} \ge 0$, so (3.12) gives

$$u_1(t) \ge (1+4at)^{-N/2}\exp\left(\frac{-ap|x|^2}{1+4aMt}\right)\log\left(\frac{1+4at}{1+\varrho_{m-1}}\right)$$

for $t > \varrho_{m-1}/(4a)$. It follows that

$$S(t)(u_1(t))^M \geq (1+4at)^{-NM/2}\left[\log\left(\frac{1+4at}{1+\varrho_{m-1}}\right)\right]^M S(t)\exp\left(\frac{-apM|x|^2}{1+4aMt}\right)$$

$$= c(1+4at)^{-NM/2}\left[\log\left(\frac{1+4at}{1+\varrho_{m-1}}\right)\right]^M$$

$$\times (1+4aM(p+1)t)^{-N/2}(1+4aMt)^{N/2}\exp\left(\frac{-apM|x|^2}{1+4aM(p+1)t}\right)$$

$$\geq c(1+4at)^{-NM/2}\left[\frac{1+4aMt}{(1+4aMt)(p+1)}\right]^{N/2}\exp\left(\frac{-apM|x|^2}{1+4aM(p+1)t}\right)$$

$$\times \left[\log\left(\frac{1+4at}{1+\varrho_{m-1}}\right)\right]^M.$$

Putting $x=0$ in the last estimate, we have

$$(1+4at)^{NM/2}S(t)(u_1(t,0))^M \geq c\left[\log\left(\frac{1+4at}{1+\varrho_{m-1}}\right)\right]^M.$$

Therefore, for $t > \max\{1, \varrho_{m-1}/(4a)\}$ we obtain, using $\alpha_1 \geq N/2$ and $M = \pi_{m-n-1}$,

$$t^{\pi_{m-n-1}\alpha_1}S(t)(u_1(t,0))^{\pi_{m-n-1}} \geq c\left[\log\left(\frac{1+4at}{1+\varrho_{m-1}}\right)\right]^{\pi_{m-n-1}}. \tag{3.13}$$

It is clear that for t large enough (3.13) is incompatible with the bound (2.21). This implies that $u_1(t)$ blows up in a finite time, which concludes the proof of the theorem in this case.

If $n=m$, i.e. $p_i \geq 1, i=1,\ldots,m$, then instead of (3.8) we obtain

$$u_{m-j}(t) \geq c(1+4at)^{-N/2}(4at-\varrho_j)^{\beta_j}\exp\left(\frac{-a\pi_j|x|^2}{1+4at}\right) \tag{3.14}$$

for $t > \varrho_j/(4a)$ and with $\varrho_j, \beta_j, \pi_j$ defined as above. Then we can repeat our considerations concerning the lower bound for $u_1(t)$ in the case $n < m$, simply using (3.14) instead of (3.3) and (3.8) and, starting from (3.11), replacing everywhere $M = \pi_{m-n-1}$ by 1. Finally, we conclude that for $t > \max\{1, \varrho_{m-1}/(4a)\}$

$$t^{\alpha_1}S(t)u_1(t,0) \geq c\log\left(\frac{1+4at}{1+\varrho_{m-1}}\right). \tag{3.15}$$

This contradicts (2.21) in the case $n=m$ and the proof is complete. ∎

4. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. By assumption, $\prod_{j=1}^{m} p_j = 1$. We are looking for a global supersolution to (1.1) of the form

$$\begin{pmatrix} \overline{u}_1 \\ \vdots \\ \overline{u}_m \end{pmatrix} = \begin{pmatrix} A_1 e^{\theta_1 t} \\ \vdots \\ A_m e^{\theta_m t} \end{pmatrix}. \tag{4.1}$$

We choose A_i, $i = 1, \ldots, m$, so large that $\|u_{0i}\|_{L^\infty} \leq A_i$. Then (4.1) is a supersolution to (1.1) if for all $t \geq 0$,

$$\begin{gathered} \overline{u}_{it} - \Delta \overline{u}_i \geq \overline{u}_{i+1}^{p_i}, \quad i = 1, \ldots, m-1, \\ \overline{u}_{mt} - \Delta \overline{u}_m \geq \overline{u}_1^{p_m}. \end{gathered} \tag{4.2}$$

This is equivalent to

$$\begin{gathered} \theta_i > A_i^{-1} A_{i+1}^{p_i} \exp((p_i \theta_{i+1} - \theta_i)t), \quad i = 1, \ldots, m-1, \\ \theta_m > A_m^{-1} A_1^{p_m} \exp((p_m \theta_1 - \theta_m)t). \end{gathered} \tag{4.3}$$

If we take $\theta_{i+1} = \theta_i / p_i$, $i = 1, \ldots, m-1$, then from $\prod_{j=1}^{m} p_j = 1$ we get $\theta_m = \theta_1 / \prod_{j=1}^{m-1} p_j = \theta_1 p_m$, so (4.3) holds for θ_1 large enough. Thus, the solution of (1.1) is global. ∎

Proof of Theorem 1.2. In this case, from $p_j \geq 0$, (1.4), (1.5), we have $\prod_{j=1}^{m} p_j < 1$. We assert that $(\overline{u}_i)_{i=1}^m$ of the form

$$\begin{pmatrix} \overline{u}_1 \\ \vdots \\ \overline{u}_m \end{pmatrix} = \begin{pmatrix} A_1 (t + t_0)^{\theta_1} \\ \vdots \\ A_m (t + t_0)^{\theta_m} \end{pmatrix} \tag{4.4}$$

is a global supersolution to (1.1) for some positive constants A_i, θ_i. We have to choose t_0 such that $\overline{u}_i(x, 0) \geq u_{0i}$ for $x \in \mathbb{R}^N$. The inequalities (4.2) imply that

$$\begin{gathered} \theta_i - p_i \theta_{i+1} \geq 1, \quad i = 1, \ldots, m-1, \\ \theta_m - p_m \theta_1 \geq 1. \end{gathered} \tag{4.5}$$

Noting that (4.5) has the form $(A - I)(-\theta) \geq \mathbf{1}$ where $\theta = (\theta_1, \ldots, \theta_m)$, $\mathbf{1} = (1, \ldots, 1)$, A is given by (1.2), we assume $\theta_i = -\alpha_i$, so $\theta_i > 0$ as $\max \alpha_i < 0$. Then (4.4) satisfies (4.2) provided that

$$\begin{gathered} A_i \theta_i \geq A_{i+1}^{p_i}, \quad i = 1, \ldots, m-1, \\ A_m \theta_m \geq A_1^{p_m}, \end{gathered} \tag{4.6}$$

and t_0 is large enough. Thus, the proof is complete. ∎

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