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LINEARIZATION OF ARBITRARY PRODUCTS OF CLASSICAL ORTHOGONAL POLYNOMIALS

Abstract. A procedure is proposed in order to expand $w = \prod_{j=1}^{N} P_{i_j}(x) = \sum_{k=0}^{M} L_k P_k(x)$ where $P_i(x)$ belongs to a classical orthogonal polynomial sequence (Jacobi, Bessel, Laguerre and Hermite) ($M = \sum_{j=1}^{N} i_j$). We first derive a linear differential equation of order 2^N satisfied by w, from which we deduce a recurrence relation in k for the linearization coefficients L_k. We develop in detail the two cases $[P_i(x)]^N$, $P_i(x)P_j(x)P_k(x)$ and give the recurrence relation in some cases ($N = 3, 4$), when the polynomials $P_i(x)$ are monic Hermite orthogonal polynomials.

1. Introduction.

Let $\{P_k\}$ be a system of polynomials of degree exactly k. The traditional linearization problem [1, 2, 3, 5] consists in expanding the product P_iP_j in the $\{P_k\}$ basis ($P_r(x) \equiv P_r$):

$$P_iP_j = \sum_{k=0}^{i+j} L_{i,j,k} P_k. \tag{1}$$

When $\{P_k\}$ is an orthogonal family (with respect to some positive measure $d\mu(x)$), many results concerning the positivity of the coefficients $L_{i,j,k}$ [1, 6, 7] and the recurrence relation satisfied by $L_{i,j,k}$ [1, 2] are known; in some cases (classical orthogonal polynomials) the coefficients $L_{i,j,k}$ are given explicitly, very often in terms of hypergeometric functions.

In a recent paper [17], we proved that for a family of classical orthogonal polynomials, the coefficients $L_{i,j,k}$ satisfy a linear second-order recurrence

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relation involving only the index k. More recently, Lewanowicz [9], rewriting the fourth order differential equation for the product P_iP_j (P_i classical) given in [17], has obtained the explicit coefficients $A_i(k)$, $i = 0, 1, 2$, of this second order recurrence relation:

$$A_0(k)L_{i,j,k-1} + A_1(k)L_{i,j,k} + A_2(k)L_{i,j,k+1} = 0. \tag{2}$$

A first extension of relation (1) was obtained in [10] for products $P_i\overline{P}_j$ where now $\overline{P}_j$ belongs to a classical orthogonal family different from that of the P_i.

The aim of this work is to generalize further, considering now the linearization problem for the product $w = P_{i_1} \ldots P_{i_N}$. The algorithm we developed and applied in [12–14] requires to search first for a differential equation for the product w. This differential equation of order $N + 1$ is given in Section 2 when all indices i_j are equal: $w = [P_i]^N$, and in Section 3 for three different indices: $w = P_iP_jP_k$ (order 8). The technique used in the case $i \neq j \neq k$ is easily extended to the general case $w = P_{i_1} \ldots P_{i_N}$ in Section 4 (order 2^N). Some of these differential equations enter our algorithm in Section 5 giving explicitly the recurrence relation satisfed by L_k for $[H_n(x)]^3$ and $H_i(x)H_j(x)H_k(x)$ where $H_n(x)$ are the monic Hermite orthogonal polynomials.

For completeness, let us mention that for classical *discrete* orthogonal polynomials: Hahn, Krawtchouk, Meixner, Charlier, for which a recurrence relation in k for L_k also exists, the cases of $w = P_iP_j$ and $w = [P_i]^N$ are already covered in respectively [14] and [15] from the difference equation satisfied by these products.

2. Differential equations satisfied by $[P_j(x)]^N$. In order to obtain the differential equation satisfied by $[P_i(x)]^N$, where $P_i(x)$ are classical orthogonal polynomials, we extend the technique already developed in [10] giving the 4th order differential equation satisfied by the product $P_i(x)\overline{P}_j(x)$ where P_i and $\overline{P}_j$ are any distinct families among the four classical families of Jacobi, Bessel, Laguerre, and Hermite.

The following notations will simplify the writing.

Let us denote $P_i(x)$ by i, $[P_i(x)]'$ by i', $[P_i(x)]''$ by i'', and the number λ_i by $\bar{\imath}$; $\sigma \equiv \sigma(x)$, $\tau \equiv \tau(x)$. The basic differential equation for i now reads [16]

$$(\sigma D^2 + \tau D + \bar{\imath} I_d)[i] = 0, \tag{3}$$

where $\sigma = \sigma(x)$ is a polynomial of degree smaller than or equal to 2, $\tau = \tau(x)$ is a polynomial of degree 1 and $\lambda_i = \bar{\imath} = -\frac{1}{2}i[(i-1)\sigma'' + 2\tau']$. This allows us to eliminate i'' and all other higher derivatives of i (by iteration) from

$$\sigma i'' = -\tau i' - \bar{\imath} i. \tag{4}$$

The first step is to compute $(i^N)''$ and after multiplication by σ we obtain the first operator R^0:

$$R^0[i^N] \equiv (\sigma D^2 + \tau D + \bar{\imath} N I_d)[i^N] = \sigma N(N-1)(i')^2 i^{N-2} = A_2^0 (i')^2 i^{N-2}. \tag{5}$$

The second step is again peculiar because the right hand side contains again the σ terms and the derivative of relation (5) can be written easily in the following form:

$$R^1[i^N] \equiv (DR^0 + 2(N-1)\bar{\imath} D)[i^N] = A_2^1 (i')^2 i^{N-2} + A_3^1 (i')^3 i^{N-3} \tag{6}$$

where

$$\begin{cases} A_2^1 \equiv A_2^1(x,N) = N(N-1)(\sigma' - 2\tau), \\ A_3^1 \equiv A_3^1(x,N) = N(N-1)(N-2)\sigma. \end{cases} \tag{7}$$

Now the derivatives of $(i')^2 i^{N-2}$ generate a linear combination (after elimination of i'') of terms $(i')^k i^{N-k}$, $2 \le k \le N$.

So let us build recursively differential operators R^j, $1 < j \le N-1$, acting on i^N, which will allow writing the $(N+1)$th differential equation using an N by N determinant containing differential operators up to R^{N-1} of order $N+1$.

In order to do that let us start with

$$R^j[i^N] = \sum_{k=2}^{j+2} A_k^j (i')^k i^{N-k}, \tag{8}$$

and generate a relation between the A_k^j from the link between R^{j+1} and R^j:

$$R^{j+1}[i^N] \equiv \left(\sigma D R^j + 2\frac{\bar{\imath}}{N} A_2^j D\right)[i^N] \quad (j \ge 1). \tag{9}$$

The following two relations come from identification of the highest terms in k:

$$\begin{cases} A_{j+3}^{j+1} = (N-2-j)\sigma A_{j+2}^j, \\ A_{j+2}^{j+1} = (N-1-j)\sigma A_{j+1}^j + \sigma(A_{j+2}^j)' - (j+2)\tau A_{j+2}^j. \end{cases} \tag{10}$$

The coefficients A_k^{j+1} for $2 \le k \le j+1$, $j \ge 1$, are now controlled by

$$A_k^{j+1} = (N+1-k)\sigma A_{k-1}^j + \sigma(A_k^j)' - k\tau A_k^j - (k+1)\bar{\imath} A_{k+1}^j. \tag{11}$$

It should be emphasized that A_k^j being a function of x and N, $A_{j+2}^j(x,N) = 0$ when $j = N-1$ (see A_3^1 for instance).

These three relations allow us to write the required determinant generalizing the situation examined in [21]:

$$\tag{12} \begin{vmatrix} R^0[i^N] & A_2^0 & 0 & 0 & \dots & 0 \\ R^1[i^N] & A_2^1 & A_3^1 & 0 & \dots & 0 \\ R^2[i^N] & A_2^2 & A_3^2 & A_4^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ R^j[i^N] & A_2^j & A_3^j & A_4^j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ R^{N-1}[i^N] & A_2^{N-1} & A_3^{N-1} & A_4^{N-1} & \dots & A_N^{N-1} \end{vmatrix} = 0.$$

In the cases $j = 2$, $j = 3$ (for $j = 0$ and $j = 1$ see equations (5) and (6)) elementary computation gives, from the previous recurrences in j,

$$\tag{13} j = 2 : \begin{cases} A_2^2 = \sigma(A_2^1)' - 2\tau A_2^1 - 3\bar{\imath} A_3^1, \\ A_3^2 = (N-2)\sigma A_2^1 + \sigma(A_3^1)' - 3\tau A_3^1, \\ A_4^2 = (N-3)\sigma A_3^1, \end{cases}$$

$$\tag{14} j = 3 : \begin{cases} A_2^3 = \sigma(A_2^2)' - 2\tau A_2^2 - 3\bar{\imath} A_3^2, \\ A_3^3 = (N-2)\sigma A_2^2 + \sigma(A_3^2)' - 3\tau A_3^2 - 4\bar{\imath} A_4^2, \\ A_4^3 = (N-3)\sigma A_3^2 + \sigma(A_4^2)' - 4\tau A_4^2, \\ A_5^3 = (N-4)\sigma A_4^2. \end{cases}$$

The third order differential equation satisfied by $i^2 = [P_i(x)]^2$ is now

$$\tag{15} [\sigma^2 D^3 + 3\sigma\tau D^2 + (4\sigma\bar{\imath} + \sigma\tau' - \sigma'\tau + 2\tau^2)D + 2\bar{\imath}(2\tau - \sigma')I_d][P_i(x)]^2 = 0,$$

which coincides with the equation given in [17].

When $N = 3$, the fourth order differential equation reduces to

$$\tag{16} \begin{vmatrix} R^0 & \sigma & 0 \\ DR^0 + 4\bar{\imath}D & \sigma' - 2\tau & 1 \\ \begin{matrix}\sigma D[DR^0 + 4\bar{\imath}D] \\ +4\bar{\imath}(\sigma' - 2\tau)D\end{matrix} & \begin{matrix}\sigma(\sigma'' - 2\tau' - 3\bar{\imath}) \\ -2\tau(\sigma' - 2\tau)\end{matrix} & 2\sigma' - 5\tau \end{vmatrix} = 0$$

with

$$R^0[\omega(x)] = (\sigma D^2 + \tau D + 3\bar{\imath} I_d)\omega(x) \qquad (\bar{\imath} = \lambda_i).$$

The third power of the Hermite polynomial $H_i(x) = i$ is a solution of the scalar equation ($\sigma = 1,\ \tau = -2x,\ \bar{\imath} = 2i,\ R^0 = D^2 - 2xD + 2i$)

$$\tag{17} (D^4 - 12xD^3 + 4(11x^2 + 5i - 2)D^2 - 4x(12x^2 + 30i - 7)D + 12i(12x^2 + 3i - 2)I_d)[i^3] = 0.$$

Of course for N larger than 3, a computer algebra package like Mathematica or Mapple must be used in order to compute the determinant and to simplify the differential equation.

3. Differential equation satisfied by $P_i(x)P_j(x)P_k(x)$. The same technique can be applied in order to find the differential equation satisfied by the product of three classical orthogonal polynomials denoted in the spirit of Section 2 by $w = ijk$ for $w = P_i(x)P_j(x)P_k(x)$.

The second derivative of w gives, after elimination of i'', j'' and k'' from the three differential equations (3) respectively for i, j and k,

$$S^0[w] \equiv \sigma w'' + \tau w' + (\bar{\imath} + \bar{\jmath} + \bar{k})w = 2\sigma(i'j'k + i'jk' + ij'k'), \tag{18}$$

which obviously generalizes the $R^0[i^3]$ relation given in (5). When $i \neq j \neq k$, the relevant differential equation is of order 8 and could be given by a determinant 7 by 7 containing an operator of order 8. But working that way we are losing the symmetry in the i, j, k variables as shown easily from the next step computing the derivative of $S^0[w]$.

Three one term derivatives: $i'jk$, $ij'k$ and ijk' appear after elimination of i'', j'' and k'' but they are functionally dependent. For instance, we could eliminate ijk' using

$$ijk' = w' - i'jk - ij'k. \tag{19}$$

The strategy is therefore to keep the symmetry, adding this equation in symmetric form.

We therefore consider the seven quantities to be eliminated:

$$i'jk, \quad ij'k, \quad ijk', \quad i'j'k, \quad i'jk', \quad ij'k', \quad i'j'k'. \tag{20}$$

The coefficients generated after s derivations of relation (18) will be denoted respectively:

$$B_i^s, \quad B_j^s, \quad B_k^s, \quad B_{ij}^s, \quad B_{jk}^s, \quad B_{ijk}^s. \tag{21}$$

For $s = 1$, we obtain

$$\begin{aligned} S^1[w] = {} & B_i^1 i'jk + B_j^1 ij'k + B_k^1 ijk' \\ & + B_{ij}^1 i'j'k + B_{ik}^1 i'jk' + B_{jk}^1 ij'k' + B_{ijk}^1 i'j'k' \end{aligned} \tag{22}$$

where all B^1 depend on x in general and are explicitly given by

$$\begin{cases} B_i^1 = -2(\bar{\jmath} + \bar{k}), \quad B_j^1 = -2(\bar{\imath} + \bar{k}), \quad B_k^1 = -2(\bar{\imath} + \bar{\jmath}), \\ B_{ij}^1 = B_{jk}^1 = B_{ik}^1 = 2(\sigma' - 2\tau), \quad B_{ijk}^1 = 6\sigma. \end{cases} \tag{23}$$

The successive differential operators are now built in the following way:

$$S^{s+1}[w] = (\sigma D S^s + \bar{\imath} B_i^s + \bar{\jmath} B_j^s + \bar{k} B_k^s)[w], \qquad 1 \le s \le 5. \tag{24}$$

The new relation reads

$$\begin{aligned} S^{s+1}[w] = {} & B_i^s i'jk + B_j^s ij'k + B_k^s ijk' \\ & + B_{ij}^s i'j'k + B_{ik}^s i'jk' + B_{jk}^s ij'k' + B_{ijk}^s i'j'k'. \end{aligned} \tag{25}$$

The seven relations generated by $S^0, \dots, S^6$ with the addition of the obvious one, already indicated,

$$w' = i'jk + ij'k + ijk', \tag{26}$$

give therefore an 8 by 8 determinant for the eighth order differential equation satisfied by $P_i(x)P_j(x)P_k(x)$. Of course, if any 2 indices are equal, the equation is still valid but can be reduced to a sixth order one, and if the three indices are equal, the solution is given in Section 2. Again computer algebra cannot be avoided but yields easily the required differential equation. This kind of elimination technique can be applied in several other situations and that is why there is no special interest to write out the larger determinant corresponding to this particular simple case.

Let us mention some other linearization problems tackled in the same way:

1) $P_i(x)P_j^*(x)P_k^{**}(x)$, when $P_j^*(x)$ (and $P_k^{**}(x)$) belong to a family different from $P_i(x)$ [10], which means that P_j^* for instance is a solution, like P_i in equation (3), of

$$(\sigma^* D^2 + \tau^* D + \bar{j}^* I_d)[P_j^*] = 0.$$

2) $P_i(a_ix + b_i)P_j(a_jx + b_j)P_k(a_kx + b_k)$, where the 3 families are now identical (the same σ and τ) but with different arguments which define new coefficients $(\sigma_i, \tau_i, \bar{\imath}_i)$ for $\overline{P}_i$ coming from the differential equation satisfied by $\overline{P}_i \equiv P_i(a_ix + b_i)$:

$$\sigma(a_ix + b_i)a_i^2\overline{P}_i'' + \tau(a_ix + b_i)a_i\overline{P}_i' + \bar{\imath}_iP_i = 0.$$

3) $x_iP_j(x)P_k(x)$, where the monomial family x^i is a solution of $x(x^i)' - ix^i = 0$ ($\sigma = 0$, $\tau = x$, $\lambda_i = -i$).

4. Differential equation satisfied by $P_{i_1}(x)\dots P_{i_N}(x)$. This differential equation of order 2^N if all i_j are distinct will again be obtained from a determinant now of size 2^N by 2^N by eliminating $2^N - 1$ quantities generalizing the seven ones appearing in equation (20).

It is easy to realize that after the elimination of second and higher derivatives from equation (4) in all derivatives of the product

$$i_1 \dots i_N \tag{27}$$

there are $\binom{N}{1}$ terms containing one first derivative, $\binom{N}{2}$ terms containing two first derivatives, etc.

The number of distinct terms to be eliminated is therefore

$$\sum_{j=1}^{N} \binom{N}{j} = 2^N - 1 \tag{28}$$

and the procedure explained in Section 3 can be extended in a straightforward way.

This combinatorial argument is another way to recover the maximal order (2^N) of the differential equation satisfied by the general arbitrary products considered, and elimination by hand without a computer becomes very quickly impractical.

But products of 4 terms occur anyway in the literature. The following integral is explicitly given in [4]:

$$\int_{-\infty}^{\infty} e^{-x^2}[H_m(x)]^2[H_n(x)]^2\,dx = m!n!\sqrt{\pi}\sum_{r=1}^{n}\binom{m}{r}\binom{n}{r}\binom{2r}{r}, \qquad m \ge n, \tag{29}$$

where H_m, H_n are monic Hermite polynomials.

This result comes directly from the product of the separate linearization of $[H_m(x)]^2$ and $[H_n(x)]^2$ as given also in [4].

The full linearization expansion of $[H_m(x)]^2[H_n(x)]^2 = \sum_{k=0}^{2(m+n)} L_k H_k(x)$ gives of course more information than the previous integral which involves only L_0, L_k, being connected with 5 products in the integral

$$\int_{-\infty}^{\infty} e^{-x^2}[H_m(x)]^2[H_n(x)]^2 H_k(x)\,dx. \tag{30}$$

The differential equation satisfied by $[H_nH_m]^2$ is still tractable by computer but the previous integral is already solved by the linearization coefficients L_k ($k = n$) in the linearization problem

$$[H_m(x)]^2[H_n(x)]^2 = \sum_{k=0}^{2(m+n)} L_k H_k(x).$$

5. Generalized linearization problems. The algorithm developed and applied to many situations in [9, 17, 21] allows us to expand any polynomial $w(x)$ of degree M which is a solution of a differential equation or difference equation [12–15] in the form

$$w(x) = \sum_k L_k P_k(x) \tag{31}$$

where $P_k(x)$ belongs to a family of classical, continuous or discrete, orthogonal polynomials.

When $P_k(x)$ is the monic Hermite family, the recurrence relation for L_k is particularly easy to derive from the differential equation for w of order K written $\mathcal{L}^K[w] = 0$.

In the relation

$$\mathcal{L}^K[w] = \sum_k L_k \mathcal{L}^K[H_k(x)] = 0 \tag{32}$$

we replace DH_k by kH_{k-1}, xH_k by $H_{k+1} + \frac{k}{2}H_{k-1}$ and we iterate as many times as we need until we reach $D^K H_k$ and $x^r H_k$ where r is the highest power in the polynomial coefficient of the differential equation $\mathcal{L}^K[w] = 0$. Equation (30) is now transformed into a linear constant coefficient combination of Hermite polynomials, and by collecting the coefficients L_j of H_k, we obtain immediately the required recurrence relation for L_k and the initial conditions starting with L_N if N is the degree of w. The length of this recurrence is obviously finite and depends only on both K and the degree of the polynomials involved in $\mathcal{L}^K$.

This process gives a recurrence relation for L_k which is not in general the "minimal" one but works in all cases. Several strategies described in [9, 17] allow finding the minimal relation but these strategies are not universal (see examples).

Example 1. From the differential equation (15) applied to Hermite polynomials $w = (H_i(x))^2$,

$$\sum_{k=0}^{2i} L_k[D^3 - 6xD^2 + 2(4i - 1 + 4x^2)D - 16ixI_d]H_k = 0, \tag{33}$$

we generate from the algorithm [5, 12, 16, 21] a recurrence relation for L_k of order 2 already obtained in [8, 9]:

$$4(k - 2i - 1)L_{k-1} + (k + 1)^2 L_{k+1} = 0 \qquad (L_k = L_{i,i,k}). \tag{34}$$

The Feldheim result [4] in monic form

$$[H_i(x)]^2 = \frac{i!}{2^i}\sum_{r=0}^{i}\binom{i}{r}\frac{2^r H_{2r}}{r!} = \sum_{r=0}^{i} L_{2r}H_{2r}$$

gives $2(n - r + 1)L_{2r-2} = r^2 L_{2r}$ equivalent to (32) with $2r = k + 1$.

Example 2. With $w = [H_i(x)]^3 = \sum_{k=0}^{3i} L_k H_k(x)$, we get from equation (17) the recurrence relation

$$\begin{aligned}(k + 1)(k + 2)(k + i + 2)L_{k+2} + (7k^2 + 4k + 6ki - 9i^2 - 12i)L_k\\ - 12(3i + 2 - k)L_{k-2} = 0.\end{aligned} \tag{35}$$

The integrals of Nth powers of classical orthogonal polynomials $P_n(x)$ multiplied by the orthogonality weight have an important combinatorial interpretation as indicated for instance by Askey [1].

6. Final remarks. The knowledge of the linearization coefficient L_k gives by orthogonality the integrals of a product of $N + 1$ $P_i(x)$'s:

$$J(i_1, \ldots, i_N, k) = \int_a^b P_{i_1}(x)\ldots P_{i_N}(x)P_k(x)\varrho(x)\,dx = L_k(i_1\ldots i_N)d_k^2$$

where $\varrho(x)$ is the orthogonality weight of the $P_i(x)$ on the interval (a,b) and

$$d_k^2 = \int_a^b [P_k(x)]^2 \varrho(x)\,dx.$$

For $k = M-1,\ M-2,\ldots$ the coefficients L_k are easy to compute from the initial condition $L_M = 1$.

This means that the corresponding integrals $J(i_1,\ldots,i_N,M-1)$, $J(i_1,\ldots,i_N,M-2),\ldots$ are relatively easy to compute ($M=\sum_{j=1}^N i_j$) from the first terms of the recurrence.

Integrals containing products of three classical orthogonal polynomials appear frequently in many domains of physics [11, 18, 19] but are trivially controlled by relation (2). Integrals containing products of four polynomials or more can be computed from the approach developed in Sections 2–4 and using the full algorithm mentioned before [9, 17]. Integrals involving products of N identical polynomials also appear in statistical mechanics.

As last comment, let us mention that a new mathematical question arises from the generalized linearization problem. When $N=2$, there exist criteria ensuring positivity of the linearization coefficients [1, 6, 7, 20], and when they are positive for $N=2$, they also are for arbitrary N (by iteration). But the converse may not be true. Positive L_k for some $N>2$ could be not necessarily positive for $N=2$. Criteria and possible counterexamples would be interesting.

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