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MARKOV OPERATORS: APPLICATIONS TO DIFFUSION PROCESSES AND POPULATION DYNAMICS

To the memory of Wiesław Szlenk

Abstract. This note contains a survey of recent results concerning asymptotic properties of Markov operators and semigroups. Some biological and physical applications are given.

1. Introduction. Dynamical systems and dynamical systems with stochastic perturbations can be effectively studied using Markov operators and Markov semigroups. Semigroups of Markov operators are generated by partial differential equations (transport equations). Equations of this type appear in the theory of diffusion processes and in population dynamics. In this note we present new results in the theory of Markov operators and illustrate them by some biological and physical applications. The results presented are based on the papers [16–18, 22].

The organization of the paper is as follows. Section 2 contains the definitions of a Markov operator and a Markov semigroup and some examples of them. In the next section we study asymptotic properties of Markov operators and semigroups: asymptotic stability and sweeping. Theorems concerning asymptotic stability and sweeping allow us to formulate the Foguel alternative. This alternative says that under suitable conditions a Markov operator (semigroup) is asymptotically stable or sweeping. Then we define a new notion called a Hasminskiĭ function. This notion is very useful in proofs of asymptotic stability of Markov semigroups. In Section 4 we give some applications of the general results to differential equations connected with diffusion and jump processes.

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2. Markov operators and semigroups

2.1. *Definitions.* Let the triple (X, Σ, m) be a σ-finite measure space. Denote by D the subset of the space $L^1 = L^1(X, \Sigma, m)$ which contains all densities

$$D = \{f \in L^1 : f \geq 0,\ \|f\| = 1\}.$$

A linear mapping $P : L^1 \to L^1$ is called a *Markov operator* if $P(D) \subset D$. A family $\{P(t)\}_{t\geq 0}$ of Markov operators which satisfies conditions

(a) $P(0) = \mathrm{Id}$,
(b) $P(t+s) = P(t)P(s)$ for $s, t \geq 0$,
(c) for each $f \in L^1$ the function $t \mapsto P(t)f$ is continuous

is called a *Markov semigroup*.

2.2. *Markov operators.* Now we give some examples of Markov operators.

1. *Frobenius–Perron operator.* This operator describes statistical properties of simple point to point transformations [10]. Let (X, Σ, m) be a σ-finite measure space and let S be a transformation of X. If a measure μ describes the distribution of points in the phase space X, then the measure ν given by $\nu(A) = \mu(S^{-1}(A))$ describes the distribution of points after S. Assume that S is non-singular, that is, if $m(A) = 0$ then $m(S^{-1}(A)) = 0$. If μ is absolutely continuous with respect to m, then ν is also absolutely continuous. If f is the density of μ and if g is the density of ν then we define the operator P_S by $P_S f = g$. This operator can be extended to a linear operator $P_S : L^1 \to L^1$. In this way we obtain a Markov operator which is called the *Frobenius–Perron operator* for the transformation S.

2. *Iterated function system.* Let $S_1, \ldots, S_n$ be non-singular transformations of the space X. Let $P_1, \ldots, P_n$ be the Frobenius–Perron operators corresponding to the transformations $S_1, \ldots, S_n$. Let $p_1(x), \ldots, p_n(x)$ be non-negative measurable functions defined on X such that $p_1(x) + \ldots + p_n(x) = 1$ for all $x \in X$. We consider the following process. Take a point x. We choose a transformation S_i with probability $p_i(x)$ and $S_i(x)$ describes the position of x after the action of the system. The evolution of densities of the distribution is described by the Markov operator

$$Pf = \sum_{i=1}^{n} P_i(p_i f).$$

3. *Integral operator.* If $k : X \times X \to [0, \infty)$ is a measurable function such that

$$\int_X k(x, y)\, m(dx) = 1$$

for each $y \in X$, then

$$Pf(x) = \int_X k(x,y) f(y)\, m(dy)$$

is a Markov operator.

Now we give an example of integral Markov operator which appears in a model of cell cycle proposed by J. Tyrcha [26] which generalizes the model of Lasota–Mackey [11] and the tandem model of Tyson–Hannsgen [27].

In the Tyrcha model it is assumed that the cell cycle consists of two phases A and B. Phase A begins at birth and lasts until the occurrence of a critical event which is necessary for mitosis. Then cell enters phase B. The end of phase B coincides with cell division. The duration t_B of phase B is constant, while the length t_A of phase A is random. More precisely, the probability that the critical moment occurs in the interval $[t, t+\Delta t]$ equals

$$\mathrm{Prob}(t \le t_A \le t + \Delta t \,|\, t_A \ge t) = \varphi(x(t))\Delta t + o(\Delta t)$$

where $x(t)$ is the size (or amount of mitogen) of cell at time t and φ is a given non-negative function. Further, it is assumed that the cell size grows according to the equation

$$\frac{dx}{dt} = g(x), \qquad x(0) = r \tag{2.1}$$

where $g(x) > 0$ for $x > 0$ and $g(0) = 0$. Denote by x_n the initial size of cell in the nth generation. Evidently, x_n can be considered as a random variable. Using the above assumptions it can be shown that

$$x_{n+1} = \lambda^{-1}\{Q^{-1}[\,Q(x_n) + \xi_n\,]\} \tag{2.2}$$

where

$$Q(x) = \int_0^x \frac{\varphi(y)}{g(y)}\, dy, \qquad \lambda(x) = \pi(-t_B, 2x)$$

and $\pi(t,r)$ is the solution of equation (2.1). The random variables ξ_n are independent and have the common distribution function $\mathrm{Prob}(\xi_n < x) = H(x)$. An elementary calculation shows that the transition operator for the dynamical system (2.2) has the form

$$Pf(x) = \int_0^{\lambda(x)} \frac{\partial}{\partial x}\{H(Q(\lambda(x)) - Q(y))\} f(y)\, dy. \tag{2.3}$$

We assume that Q, H, and λ are absolutely continuous, non-decreasing and

$$\lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty, \qquad \lambda(0) = 0.$$

These conditions imply that P is an integral Markov operator on $L^1([0,\infty))$.

2.3. *Markov semigroups*

4. *Fokker–Planck equation.* In the d-dimensional space $\mathbb{R}^d$ the Fokker–Planck equation has the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij}(x)u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i(x)u)}{\partial x_i}, \qquad u(x,0) = v(x). \tag{2.4}$$

We assume that the functions a_{ij} and b_i are sufficiently smooth and

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \ge \alpha|\lambda|^2$$

for some $\alpha > 0$ and every $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. The solution of this equation describes the distribution of a diffusion process. This equation generates a Markov semigroup given by $P(t)v(x) = u(x,t)$, where $v(x) = u(x,0)$.

5. *Liouville equation.* If we assume that $a_{ij} \equiv 0$ in (2.4), then we obtain the *Liouville equation*

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i(x)u\big). \tag{2.5}$$

As in the previous example, (2.5) generates a Markov semigroup given by $P(t)v(x) = u(x,t)$, where $v(x) = u(x,0)$. This equation has the following interpretation. In the space $\mathbb{R}^d$ we consider the movement of points given by the differential equation

$$x'(t) = b(x(t)), \qquad x(0) = x_0.$$

We assume that this problem has a unique solution defined for all t. We look at this movement statistically, that is, we consider the evolution of densities of the distribution of points. Then this evolution is described by (2.5).

6. *Randomly flashing diffusion.* Consider the stochastic equation

$$dX_t = (Y_t\sigma(X_t))dW_t + b(X_t)dt, \tag{2.6}$$

where Y_t is a homogeneous Markov process with values 0 and 1 independent of W_t and X_0. Equation (2.6) describes the process which randomly jumps between stochastic and deterministic states. Such processes appear in transport phenomena in sponge-type structures [1, 4, 12]. This process also generates a Markov semigroup but on the space $L^1(\mathbb{R} \times \{0,1\})$. The density of the distribution of this process satisfies the following system of equations:

$$\text{(2.7)} \qquad \begin{cases} \dfrac{\partial u_1}{\partial t} = -pu_1 + qu_0 + \dfrac{\partial^2}{\partial x^2}(a(x)u_1) - \dfrac{\partial}{\partial x}(b(x)u_1), \\ \dfrac{\partial u_0}{\partial t} = pu_1 - qu_0 - \dfrac{\partial}{\partial x}(b(x)u_0). \end{cases}$$

In a similar way we can introduce the notion of a multistate diffusion process [22] and check that it generates a Markov semigroup. The density of the distribution of a two-state diffusion process corresponds to the following system of equations:

$$\text{(2.8)} \qquad \begin{cases} \dfrac{\partial u_1}{\partial t} = -p(x)u_1 + q(x)u_0 + A_1 u_1, \\ \dfrac{\partial u_0}{\partial t} = p(x)u_1 - q(x)u_0 + A_0 u_0, \end{cases}$$

where the operators A_1 and A_0 are the right-hand sides of a Fokker–Planck or a Liouville equation.

7. *Transport equations.* If the equation $\frac{\partial u}{\partial t} = Au$ generates a Markov semigroup, P is a Markov operator, and $\lambda > 0$, then the equation

$$\text{(2.9)} \qquad \frac{\partial u}{\partial t} = Au - \lambda u + \lambda Pu$$

also generates a Markov semigroup. Equations of this type appear in such diverse areas as population dynamics [13, 15], in the theory of jump processes [19, 25], and in astrophysics—where they describe the fluctuations in the brightness of the Milky Way [5]. For instance, the operator P can be the Frobenius–Perron operator corresponding to some transformation S. If we have a movement of points described by an ordinary differential equation $x' = b(x)$ and we assume that points can randomly jump from x to $S(x)$, with probability Δt in a time interval of length Δt, then a density of distribution of these points satisfies (2.9) with the operator A given by

$$Au = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i(x)u).$$

In this case P is a Markov operator which describes the jumps of points.

Time and size dependent models of populations can be described by a transport equation of the form (2.9), namely

$$\text{(2.10)} \qquad \frac{\partial u}{\partial t} + \frac{\partial (V(x)u)}{\partial x} = -u(x,t) + Pu(x,t).$$

Here the function $V(x)$ is the velocity of the growth of the size of a cell and P is a Markov operator describing the process of replication. If we assume that the size of a daughter cell is exactly half of the size of the mother cell, then $Pf(x) = 2f(2x)$. If we consider unequal division then P is some integral operator.

It is interesting that more advanced models of population dynamics lead to equations similar to (2.10), but instead of the operator $P - I$ on the right-hand side of (2.10) appears a nonbounded linear operator Q (see e.g. [6]). Also these equations often generate Markov semigroups [24].

3. Asymptotic properties of Markov operators and semigroups. Now we introduce some notions which characterize the behaviour of Markov semigroups $\{P(t)\}_{t\ge 0}$ as $t \to \infty$ and powers of Markov operators P^n as $n \to \infty$. Since the powers of Markov operators also form a (discrete time) semigroup we will consider only Markov semigroups.

3.1. *Asymptotic stability.* Consider a Markov semigroup $\{P(t)\}_{t\ge 0}$. A density f_* is called *invariant* if $P(t)f_* = f_*$ for each $t > 0$. The Markov semigroup $\{P(t)\}_{t\ge 0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for } f \in D.$$

If the semigroup $\{P(t)\}_{t\ge 0}$ is generated by some differential equation then asymptotic stability means that all solutions of the equation starting from a density converge to the invariant density.

In order to formulate the main result of this section we need an auxiliary definition. A Markov semigroup $\{P(t)\}_{t\ge 0}$ is called *partially integral* if there exist $t_0 > 0$ and a measurable nonnegative function $q(x, y)$ such that

$$\int_X \int_X q(x,y)\, m(dx)\, m(dy) > 0 \tag{3.1}$$

and

$$P(t_0)f(x) \ge \int q(x,y) f(y)\, m(dy) \quad \text{for every } f \in D. \tag{3.2}$$

The main result of this part is

THEOREM 1 [22]. *Let $\{P(t)\}_{t\ge 0}$ be a partially integral Markov semigroup. Assume that $\{P(t)\}_{t\ge 0}$ has an invariant density f_* and has no other periodic points in the set of densities. If $f_* > 0$ then $\{P(t)\}_{t\ge 0}$ is asymptotically stable.*

Now we formulate a corollary which is often used in applications. Let f be a measurable function. The *support* of f is defined up to a set of measure zero by the formula

$$\operatorname{supp} f = \{x \in X : f(x) \neq 0\}.$$

We say that a Markov semigroup $\{P(t)\}_{t\ge0}$ *spreads supports* if for every set $A\in\Sigma$ and for every $f\in D$ we have

$$\lim_{t\to\infty} m(\operatorname{supp} P(t)f\cap A)=m(A).$$

COROLLARY 1 [22]. *A partially integral Markov semigroup which spreads supports and has an invariant density is asymptotically stable.*

This corollary generalizes some earlier results [2, 14, 20, 21] for integral Markov semigroups. The proof bases on the abstract theory of Markov semigroups given in the book [7]. Another proof of Theorem 1 is given in [3].

Corollary 1 remains true also for the Frobenius–Perron operators. Precisely, let S be a double-measurable transformation of a probabilistic measure space (X,Σ,m). If S preserves the measure m and the Frobenius–Perron operator P_S spreads supports, then the powers of P_S are asymptotically stable [22]. It is interesting that if we assume only that a Markov operator (or semigroup) P has an invariant density f_* and spreads supports, then P is weakly asymptotically stable (*mixing*). It means that for every $f\in D$ the sequence P^nf converges weakly to f_*. One can expect that we can omit in Corollary 1 the assumption that the semigroup is partially integral. But then it is not longer true. Indeed, in [23] we construct a Markov operator $P: L^1[0,1]\to L^1[0,1]$ which spreads supports and $P\mathbf{1}=\mathbf{1}$ but which is not asymptotically stable.

3.2. *Sweeping.* A Markov semigroup $\{P(t)\}_{t\ge0}$ is called *sweeping* with respect to a set $A\in\Sigma$ if for every $f\in D$,

$$\lim_{t\to\infty}\int_A P(t)f(x)\,m(dx)=0. \tag{3.3}$$

The notion of sweeping was introduced by Komorowski and Tyrcha [9]. The crucial role in theorems concerning sweeping is played by the following condition. We say that a Markov semigroup $\{P(t)\}_{t\ge0}$ and a set $A\in\Sigma$ satisfy condition (KT) if there exists a measurable function f_* such that: $0<f_*<\infty$ a.e., $P(t)f_*\le f_*$ for $t\ge0$, $f_*\notin L^1$ and $\int_A f_*\,dm<\infty$.

THEOREM 2 [9]. *Let $\{P(t)\}_{t\ge0}$ be an integral Markov semigroup which has no invariant density. Assume that the semigroup $\{P(t)\}_{t\ge0}$ and a set $A\in\Sigma$ satisfy condition* (KT). *Then $\{P(t)\}_{t\ge0}$ is sweeping with respect to A.*

In the paper [22] it was shown that Theorem 2 holds for a wider class of operators than integral ones. In particular, the following result was proved:

THEOREM 3. *Let $\{P(t)\}_{t\ge0}$ be a Markov semigroup which has no invariant density and spreads supports. Assume that $\{P(t)\}_{t\ge0}$ and a set $A\in\Sigma$ satisfy condition* (KT). *Then $\{P(t)\}_{t\ge0}$ is sweeping with respect to A.*

The main difficulty in applying Theorems 2 and 3 is to prove that a Markov semigroup satisfies condition (KT). Now we formulate a new criterion for sweeping which will be useful in applications.

THEOREM 4 [22]. *Let X be a metric space, and Σ be the σ-algebra of Borel sets. We assume that a Markov semigroup $\{P(t)\}_{t\ge 0}$ spreads supports and for every $y_0 \in X$ there exist $\varepsilon > 0$ and a measurable function $\eta \ge 0$ such that $\int \eta \, dm > 0$ and*

$$q(x,y) \ge \eta(x)\mathbf{1}_B(y_0,\varepsilon)(y),$$

where q is a function satisfying (3.1) and (3.2). If $\{P(t)\}_{t\ge 0}$ has no invariant density then it is sweeping with respect to compact sets.

3.3. *Foguel alternative.* We say that a Markov semigroup $\{P(t)\}_{t\ge 0}$ satisfies the *Foguel alternative* if it is asymptotically stable or sweeping from a sufficiently large family of sets. For example, this family can be all compact sets.

From Corollary 1 and Theorem 4 we immediately get

THEOREM 5. *Let X be a metric space, and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\ge 0}$ be a Markov semigroup. We assume that there exist $t > 0$ and a continuous function $q : X \times X \to (0,\infty)$ such that*

$$P(t)f(x) \ge \int_X q(x,y)f(y)\, m(dy) \quad \text{for } f \in D. \tag{3.4}$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

Using Theorem 5 one can check that the Foguel alternative holds for multistate diffusion processes [12, 17, 22], diffusion with jumps [18] and transport equations (2.9) [16].

More general results concerning the Foguel alternative can be found in [22]. These results were applied to the Markov operator P from the cell cycle model (2.3). We find that if $H'(x)$ is a continuous and positive function then P satisfies the Foguel alternative: it is asymptotically stable or sweeping from bounded sets [21, 22].

3.4. *Hasminskiĭ function.* Now we consider only continuous time Markov semigroups. Sometimes we know that a given semigroup satisfies the Foguel alternative. We want to prove that this semigroup is asymptotically stable. In order to exclude sweeping we introduce a new notion called a Hasminskiĭ function.

Consider a Markov semigroup $\{P(t)\}_{t\ge 0}$ and let A be the infinitesimal generator of $\{P(t)\}_{t\ge 0}$. Let $\mathcal{R} = (I - A)^{-1}$ be the resolvent operator at the point 1. A measurable function $V : X \to [0,\infty)$ is called a *Hasminskiĭ* *function* for the Markov semigroup $\{P(t)\}_{t\ge 0}$ and a set $Z\in\Sigma$ if there exist $M>0$ and $\varepsilon>0$ such that

$$\int_X V(x)\mathcal{R}f(x)\,dm(x) \le \int_X (V(x)-\varepsilon)f(x)\,dm(x) + \int_Z M\mathcal{R}f(x)\,dm(x). \tag{3.5}$$

THEOREM 6. *Let $\{P(t)\}$ be the Markov semigroup generated by the equation*

$$\frac{\partial u}{\partial t} = Au.$$

Assume that there exists a Hasminskiĭ function for the semigroup $\{P(t)\}_{t\ge 0}$ and a set Z. Then $\{P(t)\}$ is not sweeping with respect to Z.

In application we take V such that the function A^*V is "well defined" and it satisfies the condition $A^*V(x)\le -c<0$ for $x\notin Z$. Then we check that V satisfies inequality (3.5). This method was applied to multistate diffusion processes [17] and diffusion with jumps [18], where (3.5) was proved by using some generalization of the maximum principle. This method was also applied to transport equations (2.9) in [16] but the proof of (3.5) is different and it bases on approximation of V by a sequence of elements from the domain of the operator A^*.

The function V was called a Hasminskiĭ function because he showed [8] that the semigroup generated by the Fokker–Planck equation (2.4) has an invariant density if there exists a positive function V such that $A^*V(x)\le -c<0$ if $\|x\|\ge r$.

4. Applications

4.1. *The Fokker–Planck equation.* The Markov semigroup generated by the Fokker–Planck equation is an integral semigroup. That is,

$$P(t)f(x) = \int_{\mathbb{R}^d} q(t,x,y)f(y)\,dy, \qquad t>0,$$

and the kernel q is continuous and positive. The Foguel alternative implies

COROLLARY 2. *Let $\{P(t)\}_{t\ge 0}$ be a Markov semigroup generated by the Fokker–Planck equation. Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.*

It is easy to check that if this semigroup is not asymptotically stable, then it is sweeping with respect to the family of sets with finite Lebesgue measures.

In the case of the Fokker–Planck equation the operator A^* is given by

$$A^*V = \sum_{i,j=1}^d a_{ij}\frac{\partial^2 V}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i\frac{\partial V}{\partial x_i}.$$

If there exist a non-negative C^2-function V, $\varepsilon > 0$ and $r \ge 0$ such that

$$A^*V(x) \le -\varepsilon \quad \text{for } \|x\| \ge r$$

then the Markov semigroup generated by the Fokker–Planck equation is asymptotically stable.

Since a lot of transport equations generate a partially integral semigroup which spreads supports we can obtain similar results for these equations.

4.2. *Diffusion with jumps.* Consider the equation

$$\frac{\partial u}{\partial t} = Au - \lambda u + \lambda P u \tag{4.1}$$

where $\lambda > 0$,

$$Au = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij} u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i u)}{\partial x_i}$$

and P is a Markov operator corresponding to the iterated function system

$$(S_1(x), \ldots, S_N(x), p_1(x), \ldots, p_N(x)).$$

The probabilistic interpretation of (4.1) is similar to that of (2.9). We assume that for each j we have

$$\lim_{\|x\| \to \infty} \|S_j(x)\| = \infty.$$

Assume that

$$\lim_{\|x\| \to \infty} 2\langle x, b(x)\rangle + \lambda \sum_{j=1}^{n} p_j(x)(\|S_j(x)\|^2 - \|x\|^2) = -\infty,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. Then a Markov semigroup $\{P(t)\}_{t \ge 0}$ generated by equation (4.1) is asymptotically stable [18].

4.3. *Transport equation.* Consider a partial differential equation with an integral perturbation

$$\frac{\partial u}{\partial t} + \lambda u = -\sum_{i=1}^{d} \frac{\partial (b_i u)}{\partial x_i} + \lambda \int k(x,y) u(t,y)\, dy. \tag{4.2}$$

If $k(x,y)$ is a continuous and strictly positive function and there exists a C^1-function $V : X \to [0,\infty)$ such that

$$\sum_{i=1}^{d} b_i \frac{\partial V}{\partial x_i} - \lambda V(x) + \lambda \int k(y,x) V(y)\, dy \le -c < 0$$

for $\|x\| \ge r$, $r > 0$, then the Markov semigroup $\{P(t)\}_{t \ge 0}$ generated by equation (4.2) is asymptotically stable [16].

4.4. *Randomly interrupted diffusion.* This process was described by the following system of equations:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = -pu_1 + qu_0 + A_1 u_1, \\ \dfrac{\partial u_0}{\partial t} = pu_1 - qu_0 + A_0 u_0. \end{cases}$$

The semigroup generated by this system satisfies the Foguel alternative. In order to prove asymptotic stability it is sufficient to construct a proper Hasminskiĭ function. One can check that if there exist non-negative C^2-functions V_1 and V_2 such that

$$\begin{aligned} -p(x)V_1(x) + p(x)V_2(x) + A_1^* V_1(x) &\le -\varepsilon, \\ q(x)V_1(x) - q(x)V_2(x) + A_2^* V_2(x) &\le -\varepsilon \end{aligned}$$

for $\|x\| \ge r$, then the corresponding Markov semigroup is asymptotically stable [17].

4.5. *Population dynamics equation.* Some models of size-structured cell populations lead to transport equations similar to (2.9), but these equations do not generate Markov semigroups. Also in these cases we can often apply results presented in Section 3. We consider here a model derived in [24], which generalized some earlier models of cell populations (e.g. [6]).

We assume that a cell is fully characterized by its size x which ranges from $x = a$ to $x = 1$. The cell size grows according to equation (2.1). Cells can die or divide with rates $\mu(x)$ and $b(x)$. We assume that the cells cannot divide before they have reached a minimal size $a_0 \in (a, 1)$. Since the cells have to divide before they reach the maximal size $x = 1$, we assume that $\lim_{x\to 1} \int_a^x b(\xi)\, d\xi = \infty$. If $x \ge a_0$ is the size of a mother cell at the point of cytokinesis, then a newly born daughter cell has the size which is randomly distributed in the interval $(a, x-h]$, where h is a positive constant. We denote by $\mathcal{P}(x, [x_1, x_2])$ the probability for a daughter cell born from a mother cell of size x to have a size between x_1 and x_2.

The function $N(x, t)$ describing the distribution of the size satisfies the equation

$$\frac{\partial N}{\partial t} = -\frac{\partial (gN)}{\partial x} - (\mu + b)N + 2P(bN), \tag{4.3}$$

where $P : L^1(a, 1) \to L^1(a, 1)$ is a Markov operator such that $P^* \mathbf{1}_B(x) = \mathcal{P}(x, B)$. The main result concerning (4.3) is the following

THEOREM 7. *There exist $\lambda \in \mathbb{R}$ and continuous and positive functions f_* and w defined on the interval $(a, 1)$ such that $e^{-\lambda t} N(\cdot, t)$ converges to $f_* \Phi(N)$ in $L^1(a, 1)$, where $\Phi(N) = \int_a^1 N(x, 0) w(x)\, dx$.*

The proof of Theorem 7 goes as follows. (4.3) can be written as an evolution equation $N'(t) = AN$. First we show that A is an infinitesimal generator of a continuous semigroup $\{T(t)\}_{t\geq 0}$ of linear operators on $L^1(a,1)$. Then we prove that there exist $\lambda \in \mathbb{R}$ and continuous and positive functions v and w such that $Av = \lambda v$ and $A^*w = \lambda w$. From this it follows that the semigroup $\{P(t)\}_{t\geq 0}$ given by $P(t) = e^{-\lambda t}T(t)$ is a Markov semigroups on the space $L^1(X, \Sigma, m)$, where m is a Borel measure on the interval $[a,1]$ given by $m(B) = \int_B w(x)\,dx$. Moreover, for some $c > 0$ the function $f_* = cv$ is an invariant density with respect to $\{P(t)\}_{t\geq 0}$. Finally, from Theorem 1 we conclude that this semigroup is asymptotically stable. Since the Lebesgue measure and the measure m are equivalent we deduce that $e^{-\lambda t}N(\cdot,t)$ converges to $f_*\Phi(N)$ in $L^1(a,1)$.

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