APPLICATIONES MATHEMATICAE
27,1 (2000), pp. 45–66

M. LACHOWICZ and D. WRZOSEK (Warszawa)

A NONLOCAL COAGULATION-FRAGMENTATION MODEL

Abstract. A new nonlocal discrete model of cluster coagulation and fragmentation is proposed. In the model the spatial structure of the processes is taken into account: the clusters may coalesce at a distance between their centers and may diffuse in the physical space Ω. The model is expressed in terms of an infinite system of integro-differential bilinear equations. We prove that some results known in the spatially homogeneous case can be extended to the nonlocal model. In contrast to the corresponding local models the analysis can be carried out in the $L_1(\Omega)$ setting. Our purpose is to study global (in time) existence, mass conservation and well-posedness of the model.

1. Introduction. In this paper a new model of coagulation and fragmentation is proposed. The model is a generalization of the discrete coagulation-fragmentation model describing the dynamics of cluster growth. Such models arise in polymer science ([HEZ], [Zf] and [ZM]), atmosphere physics ([Dr]), colloidal chemistry ([Sm]), biology and immunology ([DKB]), to give a few examples. An infinite system of equations was originally introduced by Smoluchowski ([Sm]) as a model for coagulation of colloids moving according to a Brownian motion.

The main novelty we introduce is related to the nonlocal (in space) mechanism of cluster interaction. Roughly speaking, we assume that a cluster with center of mass at $y \in \Omega \subset \mathbb{R}^n$ may coalesce with another one with center at $z \in \Omega$. The cluster formed in this way has center at a point $x \in \Omega$ somewhere in the vicinity of y and z. Since the shape and the geometrical structure of clusters may be fairly complicated we assume that the point x is distributed according to a given probability law (the form of this law car-

1991 *Mathematics Subject Classification*: 35K57, 45K05, 92E20.

Key words and phrases: coagulation, fragmentation, diffusion, nonlocal interaction, integro-differential equations, kinetic models.

ries some information on the structure of clusters). Similarly, after cluster fragmentation new clusters have different centers distributed according to a given probability law. Additionally we assume that the clusters may diffuse according to Fick's law.

Consequently, the model of (nonlocal) coagulation-fragmentation process is expressed in terms of an infinite system of integro-differential bilinear equations

$$\partial_t u_j + \mathcal{A}_j \mathbf{u} = J_j[\mathbf{u}], \quad j = 1, 2, \ldots, \quad \text{on }]0,\infty[\, \times \Omega \subset \mathbb{R}^{n+1}, \tag{1.1}$$

with the initial data

$$\mathbf{u}|_{t=0} = \mathbf{U} \tag{1.2}$$

and the boundary condition

$$\mathcal{B}_j \mathbf{u} = 0, \quad j = 1, 2, \ldots \tag{1.3}$$

Here and subsequently

$$\mathbf{u} = (u_1, u_2, \ldots), \quad u_j :]0,\infty[\, \times \Omega \to \mathbb{R}^1, \quad j = 1, 2, \ldots$$

$u_j = u_j(t,x)$ is the number density of the clusters consisting of j elementary objects (particles) at time $t \in]0,\infty[$ with center at $x \in \Omega \subset \mathbb{R}^n$; Ω is a bounded domain in $\mathbb{R}^n$ of class C^2; $(\mathcal{A}_j, \mathcal{B}_j)$, $j = 1, 2, \ldots$, are regular boundary value problems (cf. [A1]),

$$\mathcal{A}_j \mathbf{u}(x) = -\sum_{k,l=1}^{n} \partial_{x_k}(d_{k,l}^{(j)}(x) \partial_{x_l} u_j), \quad j = 1, 2, \ldots, \tag{1.4}$$

$$d_{k,l}^{(j)} = d_{l,k}^{(j)} \in C^1(\overline{\Omega}, \mathbb{R}^1), \tag{1.5a}$$

$$\sum_{k,l=1}^{n} d_{k,l}^{(j)}(x)\xi_k \xi_l > 0 \quad \forall x \in \overline{\Omega},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n - \{0\}, \tag{1.5b}$$

$$\mathcal{B}_j(\mathbf{u}) = \frac{\partial u_j}{\partial \nu} \quad \text{on }]0,\infty[\, \times \partial\Omega, \tag{1.6}$$

and $\nu \in C^1(\Gamma_1, \mathbb{R}^n)$ is the outer normal vector field to $\partial\Omega$. Condition (1.3) is the Neumann condition on $\partial\Omega$ and it corresponds to the no-flux boundary condition.

Remark 1.1. It is possible (cf. [A1] and [A2])—under some additional assumptions—to consider the more general diffusion-convection operator

$$\mathcal{A}_j \mathbf{u}(x) = -\sum_{k,l=1}^{n} \partial_{x_k}(d_{kl}^{(j)}(x) \partial_{x_l} u_j) + \sum_{k=1}^{n} d_k^{(j)}(x) \partial_{x_k} u_j + d_0^{(j)}(x) u_j, \tag{1.7a}$$

with

$$d_k^{(j)}, d_0^{(j)} \in C^0(\overline{\Omega}, \mathbb{R}^1) \tag{1.7b}$$

and the more general boundary condition

$$\mathcal{B}_j(\mathbf{u}) = \begin{cases} u_j & \text{on } \varGamma_0, \\ \dfrac{\partial u_j}{\partial \widetilde{\nu}^{(j)} + \nu_0^{(j)} u} & \text{on } \varGamma_1, \end{cases} \tag{1.7c}$$

where $\partial \varOmega = \varGamma_0 \cup \varGamma_1$, $\varGamma_0$ and $\varGamma_1$ are both open and closed in $\partial \varOmega$, $\varGamma_0 \cap \varGamma_1 = \emptyset$, and $\widetilde{\nu}^{(j)} \in C^1(\varGamma_1, \mathbb{R}^n)$ is an outward pointing, nowhere tangent vector field and $\nu_0^{(j)} \in C^1(\varGamma_1, \mathbb{R})$, but we will not develop this point here.

In this paper we assume that the x-variable is interpreted as the center of clusters, but a more general theory is also possible when x is interpreted as a set of variables describing additionally the cluster shape and/or cluster orientation (cf. [Z1], [Z2], [ZJ]).

$J_j[\mathbf{u}]$ is the following nonlocal coagulation-fragmentation operator:

$$J_1[\mathbf{u}](t,x) = - u_1(t,x) \sum_{k=1}^{\infty} \int_{\varOmega} a_{1,k}(x,y) u_k(t,y)\, dy + \sum_{k=1}^{\infty} \int_{\varOmega} B_{1,k}(x,y) u_{1+k}(t,y)\, dy, \tag{1.8a}$$

$$\begin{aligned} J_j[\mathbf{u}](t,x) = {} & \frac{1}{2} \sum_{k=1}^{j-1} \int_{\varOmega \times \varOmega} A_{j-k,k}(x,y,z) u_{j-k}(t,y) u_k(t,z)\, dy\, dz \\ & - u_j(t,x) \sum_{k=1}^{\infty} \int_{\varOmega} a_{j,k}(x,y) u_k(t,y)\, dy \\ & + \sum_{k=1}^{\infty} \int_{\varOmega} B_{j,k}(x,y) u_{j+k}(t,y)\, dy \\ & - \frac{1}{2} u_j(t,x) \sum_{k=1}^{j-1} b_{j-k,k}(x), \end{aligned} \tag{1.8b}$$

for $j = 2, 3, \ldots$

The coefficient $a_{j,k} = a_{j,k}(x,y)$ is the *coagulation rate*, that is, the number of coalescences, per unit time, of a j-cluster (with center at x) with a k-cluster (with center at y); we assume that

$$a_{j,k}(x,y) = a_{k,j}(y,x) \tag{1.9a}$$

and

$$a_{j,k}(x,y) \geq 0 \tag{1.9b}$$

for $j, k = 1, 2, \ldots$ and almost all (a.a.) $x, y \in \varOmega$;

$$A_{j,k}(x,y,z) = \widetilde{A}_{j,k}(x; y, z) a_{j,k}(y,z) \tag{1.10a}$$

is such that $\widetilde{A}_{j,k} = \widetilde{A}_{j,k}(\,\cdot\,; y, z)$, for a.a. $y \in \Omega$ and a.a. $z \in \Omega$, is the probability density that after the coalescence of a j-cluster (with center at y) with a k-cluster (with center at z), the newly formed $(j+k)$-cluster will have center at x; therefore we assume

$$\int_{\Omega} \widetilde{A}_{j,k}(x; y, z)\, dx = 1 \tag{1.10b}$$

and

$$\widetilde{A}_{j,k}(x; y, z) \geq 0 \tag{1.10c}$$

for $j, k = 1, 2, \ldots$, a.a. $x \in \Omega$, a.a. $y \in \Omega$, a.a. $z \in \Omega$.

The coefficient $b_{j,k} = b_{j,k}(x)$ is the *fragmentation rate*, that is, the number of fragmentations, per unit time, of a $(j+k)$-cluster (with center at x) into j- and k-clusters; we assume that

$$b_{j,k}(x) = b_{k,j}(x) \tag{1.11a}$$

and

$$b_{j,k}(x) \geq 0 \tag{1.11b}$$

for $j, k = 1, 2, \ldots$ and a.a. $x \in \Omega$;

$$B_{j,k}(x, y) = \widetilde{B}_{j,k}(x, y) b_{j,k}(y) \tag{1.12a}$$

is such that $\widetilde{B}_{j,k} = \widetilde{B}_{j,k}(\,\cdot\,, y)$, for a.a. $y \in \Omega$, is the probability density that after the breakup of a $(j+k)$-cluster (with center at y), the newly formed j-cluster will have center at x; therefore we assume

$$\int_{\Omega} \widetilde{B}_{j,k}(x, y)\, dx = 1 \tag{1.12b}$$

and

$$\widetilde{B}_{j,k}(x, y) \geq 0 \tag{1.12c}$$

for $j, k = 1, 2, \ldots$, a.a. $x \in \Omega$, and a.a. $y \in \Omega$.

The model defined by (1.1) has a nonlocal structure. In the case when all functions are independent of the position variable, the model reduces to the usual spatially homogeneous coagulation-fragmentation model considered in [BC]. On the other hand, assuming

$$a_{j,k}(x, y) = \alpha_{j,k}\delta(y - x), \qquad b_{j,k}(x) = \beta_{j,k}, \tag{1.13a}$$

$$A_{j,k}(x; y, z) = \alpha_{j,k}\delta(y - x)\delta(z - y), \quad B_{j,k}(x, y) = \beta_{j,k}\delta(y - x), \tag{1.13b}$$

where the quantities $\alpha_{j,k}$, $\beta_{j,k}$ are constants for all $j, k \in \{1, 2, \ldots\}$, one formally obtains the coagulation-fragmentation local model with diffusion considered in [vD], [Sl], [CP] and [Wr]. In such local models clusters are regarded as point masses.

In the model defined by (1.1) the spatial structure of the processes of coagulation and fragmentation is taken into account: the positions of centers of newly formed clusters are randomly distributed according to the probability densities $\widetilde{A}_{j,k}$ and $\widetilde{B}_{j,k}$.

The idea of nonlocal interactions in the model defined by (1.1) is similar to the one of Jäger and Segel [JS], where the state was however "dominance", characterizing the individuals of a certain population of interacting insects, rather than position in the physical space. Mathematical aspects related to various generalizations of the Jäger and Segel model were studied in [BL], [AL], and [ABL] (see also references therein). Various models describing spatial nonlocal interactions were considered in kinetic theory in the papers [Mo], [Po], [LP] and [BP], to give but a few examples. The model taking into account spatial nonlocal interactions in population dynamics was studied in [BL].

It is worth comparing the nonlocal model (1.1) with the corresponding local version which one obtains formally setting (1.13). Well-posedness and other mathematical properties of solutions to the local model have been recently studied in a number of papers (see [BW], [CP], [LW], [Wr] and [A2]). It turns out that due to the spatial "smearing" of the process of coagulation, the operator J is well defined in the space $L_1(\Omega)$, unlike the corresponding operator for the local model. For the latter, some L_∞-bounds, at least for each component of $\mathbf{u}$, are necessary in order to state the problem correctly in the L_1-setting. This requirement leads, however, to some additional "technical" assumptions on the coagulation and fragmentation coefficients (cf. Assumption (H3) in [Wr]; the optimal assumptions are not known yet), which are not needed in the spatially homogeneous case ([BC]). Moreover, in order to obtain the existence result comparable with that in the spatially homogeneous case (i.e. under the analogous mild growth restrictions (1.17)—see [BC]), one should estimate some finite sums of components of $\mathbf{u}$ in the $L_\infty(]0,T[\times\Omega)$ norm. This is possible under some additional assumptions on the diffusion operator (the diffusion coefficients should be the same at least for large j—cf. [Wr], Assumption (H4)). Indeed, it is well known ([PS]) that for the reaction-diffusion systems one may need some extra assumption on the diffusion coefficients in order to warrant L_∞-bounds.

In this paper, we show that some results known for the spatially homogeneous case may also be proved for the nonlocal model (1.1) under rather mild assumptions. In particular, neither special restrictions for the diffusion operator nor for the coagulation and fragmentation rates are needed.

The operator J, defined by (1.8), formally satisfies

$$\sum_{j=1}^{\infty}\int_{\Omega} jJ_j[\mathbf{u}]\,dx = 0, \tag{1.14}$$

which corresponds to the (total) mass conservation law.

Note, however, that pointwise,

$$\sum_{j=1}^{\infty} j J_j[\mathbf{u}] = 0 \tag{1.15}$$

need not be satisfied, in contrast to the corresponding local model (see e.g. [Sl], [Wr]) in which (1.15) formally holds.

For $r \geq 0$ let X_r be the Banach space

$$X_r = \Big\{ \mathbf{u} = \mathbf{u}(x) : \sum_{j=1}^{\infty} \int_{\Omega} j^r \, |u_j(x)| \, dx < \infty \Big\},$$

equipped with the norm

$$\|\mathbf{u}\|_r = \sum_{j=1}^{\infty} \int_{\Omega} j^r \, |u_j(x)| \, dx.$$

The nonnegative cone of X_r is denoted by X_r^+:

$$X_r^+ = \{ \mathbf{u} \in X_r : \mathbf{u}_j(x) \geq 0, \ \forall j = 1, 2, \ldots, \text{ and a.a. } x \in \Omega \}.$$

Note that the L_1–realization of the boundary value problem $(\mathcal{A}_j, \mathcal{B}_j)$, for $j = 1, 2, \ldots$, given by (1.4)–(1.6), generates a positive contraction semigroup on $L_1(\Omega)$—see [A1].

Throughout the paper, we use two different concepts of solutions. The first one is the standard *mild solution* (cf. [Pa]) in a given Banach space Y on the time interval $[0, T]$—it will be referred to as an *m-solution in Y on* $[0, T]$. If the m-solution in Y is defined on $[0, T]$ for all $T > 0$, it will be referred to as a *global m-solution in Y*. Frequently we will consider an m-solution in X_1. However, we also need a weaker concept of solution (analogous to those of [BC] and [LW]):

Definition 1.1. We say a function $\mathbf{u} = \mathbf{u}(t)$ is a *global w-solution* of the IBVP (1.1)–(1.3) if

(i) $\mathbf{u} : [0, \infty[\, \to X_1^+$,

and if, for each $T > 0$,

(ii) $u_j \in C([0, T]; L_1(\Omega))$ for all $j \geq 1$,

(iii) $u_j = u_j(t)$, for each $j \geq 1$, is an m-solution in $L_1(\Omega)$ on $[0, T]$ to the jth equation of (1.1) with initial data U_j and with boundary conditions (1.3).

In the majority of papers (e.g. [BC], [BW], [CP]) a solution to the infinite system of coagulation-fragmentation equations is constructed as a limit of solutions to some auxiliary truncated systems (we mention however a different approach by Amann [A2]). In Section 2 we study such auxiliary truncated systems corresponding to (1.1).

In Sections 2 and 3 we assume that the coagulation and fragmentation rates satisfy

$$\exists\alpha > 0,\ \forall j,k = 1,2,\ldots,\quad \|A_{j,k}\|_{L_\infty(\Omega^3)} \le \alpha(j+k), \tag{1.17}$$

$$\forall j,k = 1,2,\ldots,\ \exists\beta_{j,k} \in \,]0,\infty[,\quad \|B_{j,k}\|_{L_\infty(\Omega^2)} \le \beta_{j,k}. \tag{1.18}$$

Then, by (1.10a,b) and (1.12a,b), we have

$$\exists\alpha' > 0,\ \forall j,k = 1,2,\ldots,\quad \|a_{j,k}\|_{L_\infty(\Omega^2)} \le \alpha'(j+k), \tag{1.19}$$

$$\forall j,k = 1,2,\ldots,\ \exists\beta'_{j,k} \in \,]0,\infty[,\quad \|b_{j,k}\|_{L_\infty(\Omega)} \le \beta'_{j,k}. \tag{1.20}$$

We will use $\alpha^{(0)}$ and $\beta^{(0)}_{j,k}$ to denote $\max\{\alpha,\alpha'\}$ and $\max\{\beta_{j,k},\beta'_{j,k}\}$, respectively.

In Section 3 we state and prove the main result of the paper: the global existence (Theorem 3.1) of solutions and the property of total mass conservation (3.1). We adopt Ball and Carr's idea of proof ([BC], developed for the spatially homogeneous case). It is worth pointing out that this method does not seem to be applicable to the local model by the reasons already mentioned.

The result we obtain seems to be nearly optimal:

(i) Under some extra (mild) assumption on the initial data (i.e. $\mathbf{U} \in X^+_{1+r}$, for $r > 1$—cf. (I) in Lemma 2.3) we prove the result analogous to that by Ball and Carr regarding the spatially homogeneous case (actually, they assumed a growth condition analogous to (1.17) for the coagulation rates and no restrictions on the fragmentation rates).

(ii) Some additional restriction on the growth of the fragmentation rates (cf. (2.12)) enables us to obtain the results for arbitrary initial data $\mathbf{U} \in X^+_1$ and for coagulation rates satisfying (1.17).

In contrast to the existence theory, the method of proof of uniqueness in [BC] does not seem to be applicable to (1.1).

In Section 4, we assume the following conditions on the coagulation and fragmentation rates: the coagulation rates are uniformly bounded:

$$\exists\alpha^{(1)} > 0,\ \forall j,k = 1,2,\ldots,\quad \|A_{j,k}\|_{L_\infty(\Omega^3)} \le \alpha^{(1)}, \tag{1.21}$$

and the fragmentation rates correspond to weak fragmentation (cf. [BC], [CD]):

$$\exists\beta^{(1)} > 0,\ \forall j,k = 1,2,\ldots,\quad \sum_{k=1}^{j-1} k\|B_{j-k,k}\|_{L_\infty(\Omega^2)} \le \beta^{(1)} j. \tag{1.22}$$

Under these rather restrictive—but physically reasonable—assumptions we prove that the operator J given by (1.8) is locally Lipschitz continuous in the space X_1. Therefore, we obtain the uniqueness of solutions and their continuous dependence on initial data (cf. Theorem 4.1). The result in X_1 is possible due to the nonlocal stucture of the operator J.

2. Approximation of solutions. A solution $\mathbf{u}$ of the infinite system (1.1) is constructed from successive approximations $\{\mathbf{u}^N\}_{N=1,2,\ldots}$, where $\mathbf{u}^N$ is defined as a solution to a suitable truncated system of N equations (cf. [BC]):

$$\partial_t u_j^N + \mathcal{A}_j \mathbf{u}^N = J_j^N[\mathbf{u}^N], \quad j = 1, \ldots, N, \tag{2.1}$$

where

$$J_1^N[\mathbf{u}^N](t,x) = -u_1^N(t,x) \sum_{k=1}^{N-1} \int_{\Omega} a_{1,k}(x,y) u_k^N(t,y)\, dy + \sum_{k=1}^{N-1} \int_{\Omega} B_{1,k}(x,y) u_{1+k}^N(t,y)\, dy, \tag{2.2a}$$

$$\begin{aligned} J_j^N[\mathbf{u}^N](t,x) = {} & \frac{1}{2} \sum_{k=1}^{j-1} \int_{\Omega\times\Omega} A_{j-k,k}(x,y,z) u_{j-k}^N(t,y) u_k^N(t,z)\, dy\, dz \\ & - u_j^N(t,x) \sum_{k=1}^{N-j} \int_{\Omega} a_{j,k}(x,y) u_k^N(t,y)\, dy \\ & + \sum_{k=1}^{N-j} \int_{\Omega} B_{j,k}(x,y) u_{j+k}^N(t,y)\, dy \\ & - \frac{1}{2} u_j^N(t,x) \sum_{k=1}^{j-1} b_{j-k,k}(x) \end{aligned} \tag{2.2b}$$

for $j = 2, \ldots, N$.

System (2.1) may be obtained from (1.1) by setting

$$a_{j,k} \equiv 0, \quad b_{j,k} \equiv 0 \quad \text{for } j + k > N. \tag{2.3}$$

Straightforward computations together with (1.9a), (1.10b), (1.11a) and (1.12b) show that the total mass conservation law is formally satisfied:

$$\sum_{j=1}^{N} \int_{\Omega} j J_j^N[\mathbf{u}]\, dx = 0. \tag{2.4}$$

We adhere to the convention that every finite sequence $(u_1, \ldots, u_N)$ is treated as an infinite one, $(u_1, \ldots, u_N, 0, 0, \ldots)$.

We have

PROPOSITION 2.1. *Fix* $\mathbf{U} \in X_1^+$ *and* $N > 2$. *There exists a unique global* m*-solution* $\mathbf{u}^N = \mathbf{u}^N(t)$ *to the IBVP* (2.1), (1.2), (1.3) *in* X_1. *Moreover,*

$$\mathbf{u}^N(t) \in X_1^+ \tag{2.5}$$

and

$$\|\mathbf{u}^N(t)\|_1 = \|\mathbf{U}\|_1 \quad \textit{for } t \geq 0. \tag{2.6}$$

Proof. The proof is based on the following facts:

(i) the L_1-realization of the boundary value problem $(\mathcal{A}_j, \mathcal{B}_j)$, for $j = 1, 2, \ldots$, generates a positive contraction semigroup on $L_1(\Omega)$ (cf. [A1]),

(ii) the nonlinear operator $(J_1^N, \ldots, J_N^N)$ is locally Lipschitz continuous in X_1 (the proof is straightforward and is based on the nonlocal structure of the operator),

(iii) the solution to (2.1) with nonnegative initial data ($\mathbf{U} \in X_1^+$) is nonnegative (cf. [Ma]),

(iv) (2.4) is satisfied for each $\mathbf{u} \in X_1$, hence (2.6) follows and therefore the local solution can be prolonged onto $]0, \infty[$ by the usual continuation argument. ∎

Throughout the paper, we use the notation

$$\begin{aligned} W_{i,k}[\mathbf{u}](t) = & \int_{\Omega^3} A_{i,k}(x, y, z) u_i(t, y) u_k(t, z)\, dx\, dy\, dz \\ & - \int_{\Omega^2} B_{i,k}(x, y) u_{i+k}(t, y)\, dx\, dy \\ = & \int_{\Omega^2} a_{i,k}(x, y) u_i(t, x) u_k(t, y)\, dx\, dy - \int_{\Omega} b_{i,k}(x) u_{i+k}(t, x)\, dx \end{aligned} \tag{2.7}$$

for $i, k = 1, 2, \ldots$

The following lemma is adapted from [BC]:

LEMMA 2.2. *Let* $\mathbf{u}^N$ *be the solution of* (2.1) *and* $\{g_i\}_{i=1,2,\ldots}$ *be a sequence of real numbers. Then*

$$\begin{aligned} \sum_{i=m}^{N} g_i \int_{\Omega} u_i^N(t, x)\, dx & - \sum_{i=m}^{N} g_i \int_{\Omega} u_i^N(\tau, x)\, dx \\ = & \frac{1}{2} \int_{\tau}^{t} \sum_{\mathfrak{T}_1} (g_{i+k} - g_i - g_k) W_{i,k}[\mathbf{u}^N](s)\, ds \\ & + \frac{1}{2} \int_{\tau}^{t} \sum_{\mathfrak{T}_2} g_{i+k} W_{i,k}[\mathbf{u}^N](s)\, ds + \int_{\tau}^{t} \sum_{\mathfrak{T}_3} (g_{i+k} - g_k) W_{i,k}[\mathbf{u}^N](s)\, ds, \end{aligned} \tag{2.8}$$

where

$$\mathfrak{T}_1 = \{(i,k) : i,k \geq m,\ i+k \leq N\},$$
$$\mathfrak{T}_2 = \{(i,k) : m \leq i+k \leq N,\ i,k < m\},$$
$$\mathfrak{T}_3 = \{(i,k) : 1 \leq i \leq m-1,\ k \geq m,\ i+k \leq N\}.$$

P r o o f. As a consequence of the no-flux boundary conditions (1.3) we obtain

$$\int_{\Omega} u_i^N(t,x)dx - \int_{\Omega} u_i^N(\tau,x)\,dx = \int_{\tau}^{t}\int_{\Omega} J_i^N[\mathbf{u}^N](s,x)\,dx\,ds \tag{2.9}$$

for $i = 1,\ldots,N$.

Multiplying the ith equation of (2.9) by g_i and summing up from m to N we obtain (2.8) proceeding similarly to Lemma 2.1 in [BC]. Note that Assumptions (1.9a) and (1.11a) are essential to the proof of (2.8). ■

Note that setting $g_i = i$, $m = 1$ and $\tau = 0$ we obtain

$$\int_{\Omega}\sum_{i=1}^{N} iu_i^N(t,x)\,dx = \int_{\Omega}\sum_{i=1}^{N} iU_i(x)\,dx \quad \text{for each } t > 0. \tag{2.10}$$

This corresponds to (2.4) and states that the solution $\mathbf{u}^N$ conserves the total mass.

LEMMA 2.3. *Let $\mathbf{u}^N$ be the global m-solution of the IBVP* (2.1), (1.2), (1.3). *Suppose that Assumptions* (1.17) *and* (1.18) *hold.*

(I) *If* $\mathbf{U} \in X_{1+r}^{+}$ *for some* $r > 0$, *then there exist* $\mathbf{u} = \mathbf{u}(t) \in X_1^{+}$, *for* $t > 0$, *and a subsequence again denoted by* $\{\mathbf{u}^N\}_{N=1,2,\ldots}$ *such that*

$$u_j^N \xrightarrow[N\to\infty]{} u_j \quad \text{in } L_q(0,T;L_1(\Omega)) \text{ and a.e. in }]0,T[\times\Omega, \tag{2.11}$$

for all $T > 0$ *and* $1 \leq q < \infty$.

(II) *If* $\mathbf{U} \in X_1^{+}$ *and for each* $j \geq 1$ *there exists* $\beta_j > 0$ *such that*

$$\|B_{j,k-j}\|_{L_\infty(\Omega^2)} \leq \beta_j k,\ k \geq 1, \tag{2.12}$$

then there exists $\mathbf{u} = \mathbf{u}(t) \in X_1^{+}$, *for* $t > 0$, *and a subsequence again denoted by* $\{\mathbf{u}^N\}_{N=1,2\ldots}$ *such that*

$$u_j^N \xrightarrow[N\to\infty]{} u_j \quad \text{in } C([0,T];L_1(\Omega)) \text{ and a.e. in }]0,T[\times\Omega, \tag{2.13}$$

for any $T > 0$.

P r o o f. Let us first give the main idea of the proof. We claim that, for each $j \geq 1$,

$$\|J_j^N[\mathbf{u}^N]\|_{L_1(0,T;L_1(\Omega))} \leq c_j^{(1)} \tag{2.14}$$

and

$$\|J_j^N[\mathbf{u}^N]\|_{L_\infty(0,T;L_1(\Omega))} \le c_j^{(2)} \tag{2.15}$$

in cases (I) and (II), respectively, where $c_j^{(1)}$ and $c_j^{(2)}$ are constants independent of N. Then, in both cases, we may use the compactness argument from [BHV] to each of the equations of (2.1) separately and then apply the Cantor diagonal procedure. Therefore (2.14) will imply (2.11), whereas (2.15) will imply (2.13).

We begin by proving (2.15) in case (II). Fix j. We conclude from (2.10) that, for any $N > 1$,

$$\int_\Omega \sum_{i=1}^N i u_i^N(t,x)\,dx \le \|\mathbf{U}\|_1 \quad \text{for } t \in [0,T]. \tag{2.16}$$

Hence by (1.17),

$$\begin{aligned}\int_{\Omega^3} \sum_{k=1}^{N-j} & A_{j,k}(x,y,z) u_j^N(t,y) u_k^N(t,z)\,dx\,dy\,dz \\ &\le 2\alpha^{(0)}\|\mathbf{u}^N(t)\|_1 \|\mathbf{u}^N(t)\|_0 \le 2\alpha^{(0)}\|\mathbf{U}\|_1^2 \quad \text{for } t \in [0,T],\end{aligned} \tag{2.17}$$

and by (2.12),

$$\int_{\Omega^2} \sum_{k=j+1}^{N} B_{j,k-j}(x,y) u_k^N(t,y)\,dx\,dy \le \beta_j^{(0)}\|\mathbf{u}^N(t)\|_1 \le \beta_j^{(0)}\|\mathbf{U}\|_1 \tag{2.18}$$

for $t \in [0,T]$, where $\beta_j^{(0)} = \beta_j \max\{1, |\Omega|\}$.

Similar considerations can be applied to the remaining terms in the jth equation of (2.1) in order to get (2.15).

In case (I) we proceed in a different way since we do not assume any growth condition on the fragmentation rates.

Setting $m = 1$, $\tau = 0$ and $g_i = i^{1+r}$ in (2.8) we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N & i^{1+r} u_i^N(t,x)\,dx - \int_\Omega \sum_{i=1}^N i^{1+r} U_i(x)\,dx \\ &= \frac{1}{2}\int_0^t \sum_{1<i+k\le N} ((i+k)^{1+r} - i^{1+r} - k^{1+r}) W_{i,k}[\mathbf{u}^N](s)\,ds.\end{aligned} \tag{2.19}$$

Hence using (1.17) and (1.18) yields

$$\begin{aligned}\int_\Omega \sum_{i=1}^N & i^{1+r} u_i^N(t,x)\,dx - \int_\Omega \sum_{i=1}^N i^{1+r} U_i(x)\,dx \\ &+ \frac{1}{2}\int_0^t \int_\Omega \sum_{1<i+k\le N} ((i+k)^{1+r} - i^{1+r} - k^{1+r}) b_{i,k}(x) u_{i+k}^N(s,x)\,dx\,ds\end{aligned} \tag{2.20}$$

$$\le \frac{\alpha^{(0)}}{2}\int_0^t\int_{\Omega^2}\sum_{1<i+k\le N}(i+k)((i+k)^{1+r}-i^{1+r}-k^{1+r}) \times u_i^N(s,x)u_k^N(s,y)\,dx\,dy\,ds.$$

In order to estimate the coagulation terms from above and the fragmentation terms from below, we use the following inequalities (cf. [Ca], Lemma 2.3):

$$(i+k)((i+k)^{1+r}-i^{1+r}-k^{1+r})\le c_r(i\,k^{1+r}+i^{1+r}\,k), \tag{2.21a}$$

for $i,k=1,2,\ldots$, where c_r is a positive constant (depending on r), and

$$l^{1+r}-i^{1+r}-(l-i)^{1+r}\ge 2^{1+r}-2 \quad \text{for } 1\le i\le l-1,\ l\ge 2. \tag{2.21b}$$

By (2.21b) we have

$$\begin{aligned}\frac{1}{2}\int_0^t\int_{\Omega}\sum_{1<i+k\le N}((i+k)^{1+r}-i^{1+r}-k^{1+r})b_{i,k}(x)u_{i+k}^N(s,x)\,dx\,ds \\ =\frac{1}{2}\int_0^t\int_{\Omega}\sum_{l=2}^{N}\sum_{i=1}^{l-1}(l^{1+r}-i^{1+r}-(l-i)^{1+r})b_{i,l-i}(x)u_l^N(s,x)\,dx\,ds \\ \ge (2^r-1)\int_0^t\int_{\Omega}\sum_{l=2}^{N}b_{j,l-j}(x)u_l^N(s,x)\,dx\,ds,\end{aligned} \tag{2.22}$$

for fixed j. Finally, from (2.20)–(2.22) we obtain

$$\|\mathbf{u}^N(t)\|_{1+r}+(2^r-1)\int_0^t\int_{\Omega}\sum_{l=2}^{N}b_{j,l-j}(x)u_l^N(s,x)\,dx\,ds \le \|\mathbf{U}\|_{1+r}+2\alpha^{(0)}c_r\|\mathbf{U}\|_1\int_0^t\|\mathbf{u}^N(s)\|_{1+r}\,ds. \tag{2.23}$$

Hence the Gronwall lemma yields

$$\sup_{t\in[0,T]}\|\mathbf{u}^N(t)\|_{1+r}+\int_0^t\int_{\Omega}\sum_{l=2}^{N}b_{j,l-j}(x)u_l^N(s,x)\,dx\,ds \le \|\mathbf{U}\|_{1+r}(1+\exp(2\alpha^{(0)}c_r\|\mathbf{U}\|_1T)). \tag{2.24}$$

By (2.17) and (2.24) we obtain (2.14), which completes the proof. ∎

3. Existence and conservation of the total mass. We begin by stating the main result of the paper.

THEOREM 3.1. *Let the assumptions of Lemma* 2.3 (*either* (I) *or* (II)) *be satisfied. Then there exists a global w-solution* $\mathbf{u}$ *to the IBVP* (1.1)–(1.3) *with initial data* $\mathbf{U} \in X_1^+$ *such that*

$$\|\mathbf{u}(t)\|_1 = \|\mathbf{U}\|_1 \quad \textit{for all } t \geq 0. \tag{3.1}$$

Proof. The proof is based on the method introduced by Ball and Carr [BC] in the case of the spatially homogeneous coagulation-fragmentation model.

Note that this method cannot be used for the corresponding local model without *a priori* $L_\infty(\Omega)$-bounds for $\sum_{i=1}^N iu_i^N$ uniformly with respect to N (cf. discussion in Section 1).

Fix $T > 0$. By Lemma 2.3 we may assume that there exists a subsequence $\{\mathbf{u}^N\}_{N=1,2,\ldots}$ and $\mathbf{u} \in X_1^+$ such that, for each $j \geq 1$,

$$u_j^N \xrightarrow[N\to\infty]{} u_j \quad \text{in } L_1(0,2T;L_1(\Omega)) \text{ and a.e. in }]0,2T[\, \times \Omega. \tag{3.2}$$

We shall show that

$$J_j^N[\mathbf{u}^N] \xrightarrow[N\to\infty]{} J_j[\mathbf{u}] \quad \text{in } L_1(0,T;L_1(\Omega)) \tag{3.3}$$

for each $j \geq 1$. In consequence, using the continuous dependence on data for the mild solution to the diffusion equation, we obtain

$$u_j^N \xrightarrow[N\to\infty]{} u_j \quad \text{in } C([0,T];L_1(\Omega)). \tag{3.4}$$

In order to show (3.3) we shall find estimates for the tails of the corresponding series. Let

$$\mathfrak{R}_m^N(s) = \int_\Omega \sum_{i=m}^N iu_i^N(s,x)\,dx, \tag{3.5}$$

$$\mathfrak{Q}_m^N(s) = \int_\Omega \Big(\sum_{i=m}^{2m} iu_i^N(s,x) + 2m \sum_{i=2m+1}^N u_i^N(s,x)\Big)\,dx, \tag{3.6}$$

$$\mathfrak{P}_m^N(s) = \int_\Omega \sum_{i=1}^{m-1}\sum_{k=2m}^{N-i} ib_{i,k}u_{i+k}^N(s,x)\,dx \tag{3.7}$$

for $s \in [0,2T]$ and $1 < m < (N-1)/2$. Applying Lemma 2.2 with $g_i = i$ yields

$$\begin{aligned}\mathfrak{R}_m^N(\tau) - \mathfrak{R}_m^N(0) \\ &= \int_0^\tau \sum_{\mathfrak{T}_3} iW_{i,k}[\mathbf{u}^N](s)\,ds + \frac{1}{2}\int_0^\tau \sum_{\mathfrak{T}_2}(i+k)W_{i,k}[\mathbf{u}^N](s)\,ds,\end{aligned} \tag{3.8}$$

for $\tau \in\,]0,2T]$. Then setting in Lemma 2.2

$$g_i = \begin{cases} 0, & 1 \leq i \leq m-1, \\ i, & m \leq i \leq 2m, \\ 2m, & 2m+1 \leq i \leq N, \end{cases}$$

results in

$$
\begin{aligned}
(3.9)\qquad \mathfrak{Q}_m^N(\tau) - \mathfrak{Q}_m^N(0) &= \frac{1}{2}\int_0^\tau \sum_{\mathfrak{T}_1} \mu_{i,k} W_{i,k}[\mathbf{u}^N](s)\,ds \\
&\quad + \frac{1}{2}\int_0^\tau \sum_{\mathfrak{T}_2} (i+k) W_{i,k}[\mathbf{u}^N](s)\,ds \\
&\quad + \int_0^\tau \sum_{\mathfrak{T}_3} \lambda_{i,k} W_{i,k}[\mathbf{u}^N](s)\,ds
\end{aligned}
$$

for $\tau \in\,]0, 2T]$, where $\mu_{i,k} = \mu_{k,i}$ is defined by

$$
(3.10)\qquad \mu_{i,k} = \begin{cases} 2m-(i+k), & i,k \le 2m, \\ -i, & i \le 2m,\ k > 2m, \\ -2m, & i > 2m,\ k > 2m, \end{cases}
$$

and

$$
(3.11)\qquad \lambda_{i,k} = \begin{cases} 0, & k \ge 2m, \\ i, & i+k \le 2m, \\ 2m-k, & i+k \ge 2m+1,\ k < 2m. \end{cases}
$$

Note that

$$
(3.12)\qquad 0 \le -\mu_{i,k} \le 2m \qquad \text{for } (i,k) \in \mathfrak{T}_1,
$$

$$
(3.13)\qquad 0 \le \lambda_{i,k} \le i \qquad \text{for } (i,k) \in \mathfrak{T}_3.
$$

Subtracting (3.9) from (3.8) and then using (3.10)–(3.13) yields

$$
\begin{aligned}
(3.14)\qquad \mathfrak{R}_m^N(\tau) + \int_0^\tau \mathfrak{P}_m^N(s)\,ds &\le \mathfrak{R}_m^N(0) + \mathfrak{Q}_m^N(\tau) \\
&\quad + \int_0^\tau \Big(\sum_{\mathfrak{T}_3} i + 2m \sum_{\mathfrak{T}_1} \Big) \int_{\Omega^2} a_{i,k}(x,y) u_i^N(s,x) u_k^N(s,y)\,dx\,dy\,ds.
\end{aligned}
$$

By the definition of $\mathfrak{T}_1$ and $\mathfrak{T}_3$ and (2.16) it follows that

$$
\begin{aligned}
(3.15)\qquad \sum_{\mathfrak{T}_3} \int_{\Omega^2} i a_{i,k}(x,y) u_i^N(s,x) u_i^N(s,z)\,dx\,dy
&\le \alpha^{(0)} \Big(\sum_{i=1}^{m-1} i \int_\Omega u_i^N(s,x)\,dx \Big) \Big(\sum_{k=m}^{N-i} (i+k) \int_\Omega u_i^N(s,x)\,dx \Big) \\
&\le 2\alpha^{(0)} \|\mathbf{U}\|_1 \mathfrak{R}_m^N(s)
\end{aligned}
$$

and

$$2m \sum_{\mathfrak{T}_1} \int_{\Omega^2} a_{i,k}(x,y) u_i^N(s,x) u_k^N(s,y)\, dx\, dy \le 2\alpha^{(0)} m \sum_{i=m}^{N} \sum_{k=m}^{N} (i+k) \Big(\int_\Omega u_i^N(s,x)\, dx \Big) \Big(\int_\Omega u_k^N(s,x)\, dx \Big) \le 4\alpha^{(0)} \|\mathbf{U}\|_1 \mathfrak{R}_m^N(s). \tag{3.16}$$

We are now in a position to let $N \to \infty$ in the jth equation (j is fixed). To this end we first prove that, for given $\varepsilon > 0$, there exist $M > j$ and $N_0 > 2M$ such that

$$\int_0^\tau \mathfrak{Q}_M^N < \varepsilon \quad \text{for } N \ge N_0. \tag{3.17}$$

Indeed, using (3.2) we can pass to the limit in (2.16) to obtain

$$\int_\Omega \sum_{i=1}^\infty i u_i(s,x)\, dx \le \|\mathbf{U}\|_1 \quad \text{for a.a. } s \in [0,2T]. \tag{3.18}$$

From (3.2) and (3.18) it follows that

$$\sum_{i=l}^\infty \|u_i^N(s,\cdot) - u_i(s,\cdot)\|_{L_1(\Omega)} \le l^{-1} \sum_{i=l}^\infty i(\|u_i^N(s,\cdot)\|_{L_1(\Omega)} + \|u_i(s,\cdot)\|_{L_1(\Omega)}) \le 2l^{-1}\|\mathbf{U}\|_1 \tag{3.19}$$

for a.a. $s \in [0,2T]$. Therefore, for fixed m and a.a. $s \in [0,2T]$,

$$\lim_{N\to\infty} \mathfrak{Q}_m^N(s) = \mathfrak{Q}_m(s) = \sum_{i=m}^{2m} i\|u_i(s,\cdot)\|_{L_1(\Omega)} + 2m \sum_{i=2m+1}^\infty \|u_i(s,\cdot)\|_{L_1(\Omega)}. \tag{3.20}$$

From (3.18) it follows that

$$0 \le \mathfrak{Q}_m^N(s) \le \|\mathbf{U}\|_1 \tag{3.21}$$

for a.a. s in $]0,2T[$; and

$$\mathfrak{Q}_m(s) \le \sum_{i=m}^\infty i\|u_i(s\cdot)\|_{L_1(\Omega)} \tag{3.22}$$

for a.a. s in $]0,2T[$. Hence

$$\lim_{m\to\infty} \mathfrak{Q}_m(s) \to 0 \tag{3.23}$$

for a.a. s in $]0,2T[$; and for given $\varepsilon > 0$ there exists $M > j$ such that

$$\int_0^\tau \mathfrak{Q}_M(s)\,\mathrm{d}s < \frac{\varepsilon}{2} \quad \text{for all } \tau \in [0,2T], \tag{3.24}$$

and

$$\mathfrak{R}_M^N(0) = \sum_{i=M}^{\infty} iU_i(x) < \frac{\varepsilon}{T} \quad \forall N > 2M. \tag{3.25}$$

From (3.21) and (3.23), by the Lebesgue dominated convergence theorem it follows that

$$\lim_{N\to\infty} \int_0^\tau |\mathfrak{Q}_M^N(s) - \mathfrak{Q}_M(s)|\,ds = 0. \tag{3.26}$$

Hence, using (3.24) there exists $N_0 > 2M$ such that

$$\int_0^\tau \mathfrak{Q}_M^N(s)\,ds \le \int_0^\tau |\mathfrak{Q}_M^N(s) - \mathfrak{Q}_M(s)|\,ds + \int_0^\tau \mathfrak{Q}_M(s)\,ds < \varepsilon \quad \text{for } N > N_0,$$

which shows (3.17).

Integrating (3.14) on $[0,t]$, where $0 < t \le 2T$, and using (3.15), (3.17) together with (3.24), and (3.25) yields

$$\int_0^t \mathfrak{R}_M^N(\tau)\,d\tau + \int_0^t\int_0^\tau \mathfrak{P}_M^N(s)\,ds\,d\tau \le 2\varepsilon + \mathcal{H}_0 \int_0^t\int_0^\tau \mathfrak{R}_M^N(s)\,ds\,d\tau, \tag{3.27}$$

where $\mathcal{H}_0 = 6\alpha^{(0)}\|\mathbf{U}\|_1$. By the Gronwall lemma

$$\int_0^t \mathfrak{R}_M^N(\tau)\,d\tau \le \varepsilon\mathcal{H}_1, \tag{3.28}$$

where $\mathcal{H}_1 = 3\exp(12\alpha^{(0)}\|\mathbf{U}\|_1 T)$, and by (3.27) it follows that

$$\int_0^t\int_0^\tau \mathfrak{P}_M^N(s)\,ds\,d\tau \le \varepsilon\mathcal{H}_2 \quad \text{for } 0 < t \le 2T, \tag{3.29}$$

where $\mathcal{H}_2 = 3 + \mathcal{H}_1$. Since

$$\int_0^t\int_0^\tau \mathfrak{P}_M^N(s)\,ds\,d\tau = \int_0^t (t-\tau)\mathfrak{P}_M^N(\tau)\,d\tau, \tag{3.30}$$

we conclude from (3.29) that for $N > N_0$ and $N_0 > 2M$,

$$\int_0^T \mathfrak{P}_M^N(s)\,ds < \frac{\varepsilon\mathcal{H}_2}{T}. \tag{3.31}$$

Note that

$$\sum_{k=1}^{\infty}\int_0^T\int_{\Omega^2} a_{j,k}(x,y)u_j(t,x)u_k(t,y)\,dx\,dy\,dt<\infty, \tag{3.32}$$

$$\sum_{k=1}^{\infty}\int_0^T\int_{\Omega} b_{j,k}(x)u_{j+k}(t,x)\,\mathrm{d}x\,dt<\infty. \tag{3.33}$$

Indeed, (3.32) follows from (1.17) and (3.18), whereas (3.33) results from the uniform bounds (2.18) and (2.24), under Assumptions (I) and (II) in Lemma 2.3, respectively. Note also that, letting $N\to\infty$ in (3.31) gives

$$\int_0^T\int_{\Omega}\sum_{i=1}^{M-1}\sum_{k=2M}^{\infty} ib_{i,k}(x)u_k(t,x)\,dx\,dt<\frac{\varepsilon\mathcal{H}_2}{T}. \tag{3.34}$$

It follows from (3.28) and (1.17) that

$$\begin{aligned}
&\int_0^T\Big|\int_{\Omega^2}\sum_{k=1}^{\infty}a_{j,k}(x,y)u_j(t,x)u_k(t,y)\,dx\,dy \\
&\qquad-\int_{\Omega^2}\sum_{k=1}^{N-j}a_{j,k}(x,y)u_j^N(t,x)u_k^N(t,y)\,dx\,dy\Big|\,dt \\
&\quad\le\alpha^{(0)}\int_0^T\Big(\|u_j(t)\|_{L_1(\Omega)}\Big(\sum_{k=1}^{M-1}(j+k)\|\mathbf{u}_k(t)-\mathbf{u}_k^N(t)\|_{L_1(\Omega)} \\
&\qquad+(j+1)\Big(\sum_{k=M}^{N-j}k\|\mathbf{u}_k^N(t)\|_{L_1(\Omega)}+k\|\mathbf{u}_k(t)\|_{L_1(\Omega)}\Big) \\
&\qquad+\sum_{k=N-j+1}^{\infty}k\|\mathbf{u}_k(t)\|_{L_1(\Omega)}\Big) \\
&\qquad+\Big\|\sum_{k=1}^{N-j}(j+k)\mathbf{u}_k^N(t)\Big\|_{L_1(\Omega)}\|\mathbf{u}_j(t)-\mathbf{u}_j^N(t)\|_{L_1(\Omega)}\Big)\,dt \\
&\quad\le\alpha^{(0)}\int_0^T\|\mathbf{U}\|_1\sum_{k=1}^{M-1}(j+k)\|\mathbf{u}_k(t)-\mathbf{u}_k^N(t)\|_{L_1(\Omega)}\,dt+(2j+3)\mathcal{H}_1\varepsilon \\
&\qquad+(j+1)\|\mathbf{U}\|_1\int_0^T\|\mathbf{u}_j(t)-\mathbf{u}_j^N(t)\|_{L_1(\Omega)}.
\end{aligned} \tag{3.35}$$

Using (3.31) and (3.34), the fragmentation terms may be handled in a similar way:

$$\int_0^T \Big| \int_\Omega \sum_{k=1}^{\infty} b_{j,k}(x) u_{j+k}(t,x)\,dx - \int_\Omega \sum_{k=1}^{N-j} b_{j,k}(x) u^N_{j+k}(t,x)\,dx \Big|\,dt \tag{3.36}$$
$$\le \int_0^T \sum_{k=1}^{2M-1} \|b_{j,k}(u_j(t) - u^N_j(t))\|_{L_1(\Omega)} + \frac{2\mathcal{H}_2}{T}\varepsilon.$$

Letting $N \to \infty$ in (3.35) and (3.36) and using (3.2) we conclude that

$$u^N_j \sum_{l=1}^{N-j} \int_\Omega a_{j,l}(\cdot,y) u^N_l(\cdot,y)\,dy \xrightarrow[N\to\infty]{} u^N_j \sum_{l=1}^{\infty} \int_\Omega a_{j,k}(\cdot,y) u_l(\cdot,y)\,dy$$

and

$$\sum_{l=1}^{N-j} \int_\Omega b_{j,k}(\cdot,y) u^N_{j+k}(\cdot,y)\,dy \xrightarrow[N\to\infty]{} \sum_{l=1}^{\infty} \int_\Omega b_{j,k}(\cdot,y) u_{j+k}(\cdot,y)\,dy,$$

in the space $L_1(0,T\,;\,L_1(\Omega))$. By (3.2) we let $N \to \infty$ in the remaining sums in the jth equation. Thus (3.3) and (3.4) hold, which completes the proof of the existence of solutions.

In order to prove the total mass conservation (3.1) it is sufficient to show that, for fixed $\tau > 0$ and given $\varepsilon > 0$, there exist M_1 and $N_1 > M_1$ such that

$$\mathfrak{R}^N_{M_1}(\tau) < \varepsilon \quad \text{for } N \ge N_1. \tag{3.37}$$

To this end we use (3.15)–(3.17) to obtain

$$\mathfrak{R}^N_m(\tau) \le \mathfrak{R}^N_m(0) + \mathfrak{Q}^N_m(\tau) + \mathcal{H}_0 \int_0^\tau \mathfrak{R}^N_m(s)\,ds. \tag{3.38}$$

By (3.4) we deduce that (3.20)–(3.23) hold for all $s > 0$. In particular,

$$\lim_{m\to\infty} \mathfrak{Q}_m(\tau) = 0 \quad \text{for each } \tau \ge 0.$$

Thus for given $\varepsilon_1 > 0$ there exist M_1 and $N_1 > M_1$ such that

$$\mathfrak{Q}^N_{M_1}(\tau) \le |\mathfrak{Q}^N_{M_1}(\tau) - \mathfrak{Q}_{M_1}(\tau)| + \mathfrak{Q}_{M_1}(\tau) < \varepsilon_1$$

and

$$\mathfrak{R}^N_{M_1}(0) < \varepsilon_1$$

for $N > N_1$. Then by (3.38) and the Gronwall lemma we obtain

$$\mathfrak{R}^N_{M_1}(\tau) \le 2(1 + \exp(\mathcal{H}_0))\varepsilon_1,$$

and hence (3.37) follows.

Finally, for $N > N_1 > M_1$ we have

$$\begin{aligned}\Big\| \sum_{i=1}^{\infty} iu_i(\tau) - \sum_{i=1}^{N} iu_i^N(\tau)\Big\|_{L_1(\Omega)} \\ &\le \sum_{i=1}^{M-1} i\|u_i(\tau) - u_i^N(\tau)\|_{L_1(\Omega)} + \sum_{i=M}^{N} \|iu_i^N(\tau)\|_{L_1(\Omega)} \\ &\quad + \sum_{i=M}^{\infty} \|iu_i(\tau)\|_{L_1(\Omega)} \\ &\le \sum_{i=1}^{M-1} i\|u_i(\tau) - u_i^N(\tau)\|_{L_1(\Omega)} + 4(1+\exp(\mathcal{H}_0\tau))\varepsilon.\end{aligned}$$

Now we may let $N \to \infty$ and then using (2.10) we obtain (3.1), which completes the proof. ∎

4. Well-posedness. In this section we consider the IBVP (1.1)–(1.3) under rather restrictive assumptions (1.21) and (1.22) on the coagulation and fragmentation rates.

THEOREM 4.1. *Let the assumptions* (1.21) *and* (1.22) *be satisfied. Then there exists a unique global m-solution $\mathbf{u} = \mathbf{u}(t)$ in X_1^+ to the IBVP* (1.1)–(1.3) *with initial data $\mathbf{U} \in X_1^+$. Moreover, if $\widetilde{\mathbf{U}} \in X_1^+$ and if $\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}(t)$ is the global m-solution of* (1.1)–(1.3) *with initial data $\widetilde{\mathbf{U}}$, then*

$$\sup_{t\in[0,T]} \|\mathbf{u}(t) - \widetilde{\mathbf{u}}(t)\|_1 \le c_T \|\mathbf{U} - \widetilde{\mathbf{U}}\|_1 \tag{4.1}$$

for all $T \ge 0$, where c_T is a constant (*depending on T*).

Proof. Note that (1.21) and (1.22) imply (1.17) and (2.12), respectively. Therefore the assumptions of Theorem 3.1 are satisfied and the existence of a global w-solution together with (3.1) follow.

Furthermore, note that the operator $\mathcal{A}$ given by

$$\mathcal{A}\mathbf{u} = (\mathcal{A}_1\mathbf{u}, \mathcal{A}_2\mathbf{u}, \ldots)$$

generates a C^0 semigroup of contractions in the space X_1.

We claim that the operator J given by $J[\mathbf{u}] = (J_1[\mathbf{u}], J_2[\mathbf{u}], \ldots)$ is locally Lipschitz continuous in X_1:

$$\|J[\mathbf{u}] - J[\widetilde{\mathbf{u}}]\|_1 \le 2(\alpha^{(2)}(\|\mathbf{u}\|_1 + \|\widetilde{\mathbf{u}}\|_1) + \beta^{(2)})\|\mathbf{u} - \widetilde{\mathbf{u}}\|_1, \tag{4.2}$$

where $\alpha^{(2)} = \alpha^{(1)} \max\{1, |\Omega|\}$, $\beta^{(2)} = \beta^{(1)} \max\{1, |\Omega|\}$ and $\alpha^{(1)}$, $\beta^{(1)}$ are given in (1.21) and (1.22). Indeed,

$$\tag{4.3} \sum_{i=1}^{\infty} i \int_{\Omega} \Big| \sum_{k=1}^{\infty} \int_{\Omega} B_{i,k}(x,y)(u_{i+k}(y) - \widetilde{u}_{i+k}(y))\,dy \Big|\,dx$$
$$\le \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} i \int_{\Omega^2} B_{i,k-i}(x,y)|u_k(y) - \widetilde{u}_k(y)|\,dx\,dy$$
$$= \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} i \int_{\Omega^2} B_{i,k-i}(x,y)|u_k(y) - \widetilde{u}_k(y)|\,dx\,dy \le \beta^{(1)}|\Omega| \cdot \|\mathbf{u} - \widetilde{\mathbf{u}}\|_1,$$

$$\tag{4.4} \frac{1}{2} \sum_{i=1}^{\infty} i \int_{\Omega} \Big| (u_i(x) - \widetilde{u}_i(x)) \sum_{k=1}^{i-1} b_{i-k,k}(x) \Big|\,dx$$
$$\le \sum_{i=1}^{\infty} \int_{\Omega} \sum_{k=1}^{i-1} k b_{i-k,k}(x)|u_i(x) - \widetilde{u}_i(x)|\,dx \le \beta^{(1)}|\Omega| \cdot \|\mathbf{u} - \widetilde{\mathbf{u}}\|_1$$

and the detailed estimation of the remaining terms is left to the reader.

The above mentioned properties of the operator $\mathcal{A}$ together with (4.2) imply the local existence of an m-solution, its uniqueness and (4.1) for some $T > 0$. On the other hand, the local m-solution satisfies (3.1) and therefore it can be prolonged onto $]0, \infty[$ by the usual continuation argument, which completes the proof. ■

Acknowledgments. This work was supported by the Polish State Committee for Scientific Research under GRANT 2P03A00717.

References

[A1] H. Amann, *Dual semigroups and second order linear elliptic boundary value problems*, Israel J. Math. 45 (1983), 225–254.

[A2] —, *Coagulation-fragmentation processes*, to appear.

[ABL] L. Arlotti, N. Bellomo and M. Lachowicz, *Kinetic equations modelling population dynamics*, Transport Theory Statist. Phys., to appear.

[AL] L. Arlotti and M. Lachowicz, *Qualitative analysis of an equation modelling tumor-host dynamics*, Math. Comput. Modelling 23 (1996), 11–29.

[BC] J. M. Ball and J. Carr, *The discrete coagulation-fragmentation equations: existence, uniqueness and density conservation*, J. Statist. Phys. 61 (1990), 203–234.

[BHV] P. Baras, J. C. Hassan et L. Véron, *Compacité de l'opérateur définissant la solution d'une équation d'évolution non homogène*, C. R. Acad. Sci. Paris Sér. I 284 (1977), 799–802.

[BL] N. Bellomo and M. Lachowicz, *Mathematical biology and kinetic theory: Evolution of the dominance in a population of interacting organisms*, in: *Nonlinear Kinetic Theory and Hyperbolic Systems*, V. Boffi *et al.* (eds.), World Sci., Singapore, 1992, 11–20.

[BP] N. Bellomo and J. Polewczak, *The generalized Boltzmann equation, existence and exponential trend to equilibrium*, C. R. Acad. Sci. Paris Sér. I 319 (1994), 893–898.

[BW] P. Bénilan and D. Wrzosek, *On an infinite system of reaction-diffusion equations*, Adv. Math. Sci. Appl. 7 (1997), 349–364.

[Ca] J. Carr, *Asymptotic behaviour of solutions to the coagulation-fragmentation equations. I. The strong fragmentation case*, Proc. Roy. Soc. Edinburgh Sect. A 121 (1992), 231–244.

[CD] J. Carr and F. P. da Costa, *Asymptotic behavior of solutions to the coagulation-fragmentation equations. II. Weak fragmentation*, J. Statist. Phys. 77 (1994), 89–123.

[CP] J. F. Collet and F. Poupaud, *Existence of solution to coagulation-fragmentation system with diffusion*, Transport Theory Statist. Phys. 25 (1996), 503–513.

[DKB] E. B. Dolgosheina, A. Yu. Karulin and A. V. Bobylev, *A kinetic model of the agglutination process*, Math. Biosciences 109 (1992), 1–10.

[vD] P. G. J. van Dongen, *Spatial fluctuations in reaction-limited aggregation*, J. Statist. Phys. 54 (1989), 221–271.

[Dr] R. Drake, *A general mathematical survey of the coagulation equation*, in: *Topics in Current Aerosol Research*, G. M. Hidy and J. R. Brock (eds.), Pergamon Press, Oxford, 1972, 202–376.

[HEZ] E. M. Hendriks, M. H. Ernst and R. M. Ziff, *Coagulation equations with gelation*, J. Statist. Phys. 31 (1983), 519–563.

[JS] E. Jäger and L. Segel, *On the distribution of dominance in a population of interacting anonymous organisms*, SIAM J. Appl. Math. 52 (1992), 1442–1468.

[LP] M. Lachowicz and M. Pulvirenti, *A stochastic particle system modeling the Euler equation*, Arch. Rational Mech. Anal. 109 (1990), 81–93.

[LW] P. Laurençot and D. Wrzosek, *Fragmentation-diffusion model. Existence of solutions and asymptotic behaviour*, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), 759–774.

[Ma] R. H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York, 1976.

[Mo] D. Morgenstern, *Analytical studies related to the Maxwell–Boltzmann equation*, Arch. Rational Mech. Anal. 4 (1955), 533–555.

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1982.

[PS] M. Pierre and D. Schmitt, *Blowup in reaction-diffusion system with dissipation of mass*, SIAM J. Math. Anal. 28 (1997), 259–269.

[Po] A. Y. Povzner, *The Boltzmann equation in the kinetic theory of gases*, Amer. Math. Soc. Transl. 47 (1962), 193–216.

[Sl] M. Slemrod, *Coagulation-diffusion systems: derivation and existence of solution for the diffuse interface structure equations*, Phys. D 46 (1990), 351–366.

[Sm] M. Smoluchowski, *Versuch einer mathematischen Theorie der kolloiden Lösungen*, Z. Phys. Chem. 92 (1917), 129–168.

[Wr] D. Wrzosek, *Existence of solution for the discrete coagulation-fragmentation model with diffusion*, Topol. Methods Nonlinear Anal. 9 (1997), 279–296.

[Z1] A. Ziabicki, *Generalized theory of nucleation kinetics. IV. Nucleation as diffusion in the space of cluster dimensions, positions, orientations, and internal structure*, J. Chem. Phys. 85 (1986), 3042–3057.

[Z2] —, *Configurational space for clusters in the theory of nucleation*, Arch. Mech. 42 (1990), 703–715.

[ZJ] A. Ziabicki and L. Jarecki, *Cross sections for molecular aggregation with positional and orientational restrictions*, J. Chem. Phys. 101 (1993), 2267–2272.

[Zf] R. M. Ziff, *Kinetics of polymerization*, J. Statist. Phys. 23 (1980), 241–263.

[ZM] R. M. Ziff and E. D. McGrady, *The kinetics of cluster fragmentation and depolymerisation*, J. Phys. A 18 (1985), 3027–3037.

Mirosław Lachowicz
Institute of Applied Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: lachowic@mimuw.edu.pl

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland

Dariusz Wrzosek
Institute of Applied Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: darekw@mimuw.edu.pl

Received on 15.1.1999