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THE STABILITY STUDY OF A PLANE ENGINE

This paper is dedicated to the memory of Prof. W. Szlenk

Abstract. We study the dynamical properties of a plane engine vibrations modelled by a system of ODE.

1. Introduction. This paper is organized in the following way. First we consider a system of ordinary differential equations describing the vibrations of a plane engine. We consider the model given by

$$\begin{aligned}\ddot{\xi} &= -2904\xi - 2976600\xi^3 - 22|\xi|\operatorname{sign}(\dot{\xi}),\\ \ddot{\zeta} &= -2904\zeta - 1488300((\zeta+\phi)^3 + (\zeta-\phi)^3) - 22|\zeta|\operatorname{sign}(\dot{\zeta}),\\ \ddot{\phi} &= -1387\phi - 710678((\zeta+\phi)^3 + (\zeta-\phi)^3)\\ &\quad - 721|\phi|\operatorname{sign}(\dot{\phi}) + \sum_{i=1}^{N} k_i\cos(\omega_i t).\end{aligned}$$

We do not enter the discussion of how this system was deduced, but rather we treat it as given [J]. The model splits into four parts described by four terms which can be interpreted as independent oscillators, smooth interactions, viscosity and external oscillatory forces.

Usually the hardest part of the stability analysis of a mechanical system is to figure out first integrals; the proof of correctness of the guess goes easily by inspection. We found out that the smooth part of the system is integrable and we present some possible ways to see it. In fact the system is Hamiltonian with bounded energy levels, and its perturbations–those considered by engineers (such as viscosity and forcing)–are (mathematically) stable.

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We summarize the results in a theorem.

THEOREM 1. *The system* (*smooth part—independent oscillations with higher order interactions*)

$$\begin{aligned}\ddot{\xi} &= -2904\xi - 2976600\xi^3,\\ \ddot{\zeta} &= -2904\zeta - 1488300((\zeta+\phi)^3 + (\zeta-\phi)^3),\\ \ddot{\phi} &= -1387\phi - 710678((\zeta+\phi)^3 + (\zeta-\phi)^3),\end{aligned} \tag{1}$$

is (*after a linear change of coordinates*) *Hamiltonian. The energy levels* $H = \mathrm{const} > 0$ *are homeomorphic to the sphere* S^3. *The trajectories lie* (*typically*) *densely on tori separating the sphere into different stability regimes.*

The system (*smooth part with viscosity added*)

$$\begin{aligned}\ddot{\xi} =& -2904\xi - 2976600\xi^3 - 22|\xi|\operatorname{sign}(\dot{\xi}),\\ \ddot{\zeta} =& -2904\zeta - 1488300((\zeta+\phi)^3 + (\zeta-\phi)^3) - 22|\zeta|\operatorname{sign}(\dot{\zeta}),\\ \ddot{\phi} =& -1387\phi - 710678((\zeta+\phi)^3 + (\zeta-\phi)^3)\\ & -721|\phi|\operatorname{sign}(\dot{\phi}),\end{aligned}$$

is asymptotically stable and the trajectories tend to zero crossing any two given levels of the Hamiltonian of the smooth system in finite time intervals with speed decreasing proportionally to H *as* H *tends to* 0.

Finally, *the whole system* (*independent oscillations*, *interactions*, *viscosity and external oscillating force*) *given by*

$$\begin{aligned}\ddot{\xi} =& -2904\xi - 2976600\xi^3 - 22|\xi|\operatorname{sign}(\dot{\xi}),\\ \ddot{\zeta} =& -2904\zeta - 1488300((\zeta+\phi)^3 + (\zeta-\phi)^3) - 22|\zeta|\operatorname{sign}(\dot{\zeta}),\\ \ddot{\phi} =& -1387\phi - 710678((\zeta+\phi)^3 + (\zeta-\phi)^3)\\ & -721|\phi|\operatorname{sign}(\dot{\phi}) + \sum_{i=1}^{N} k_i\cos(\omega_i t),\end{aligned}$$

is stable but not asymptotically stable. For large energies H *the forcing oscillations may be seen as small perturbations of the system*, *while the viscosity part presses the trajectories toward zero.*

We can solve the first equation of the system separately; we will show in the next section that it has stable solutions. This textbook case will serve as a point of reference for analysis of the stability of the whole system and for this reason we include the details.

The other two equations may be seen as complicated perturbations of the system of independent linear oscillators. We do not claim that the perturbations are small. We shall consider the system adding the perturbations step by step.

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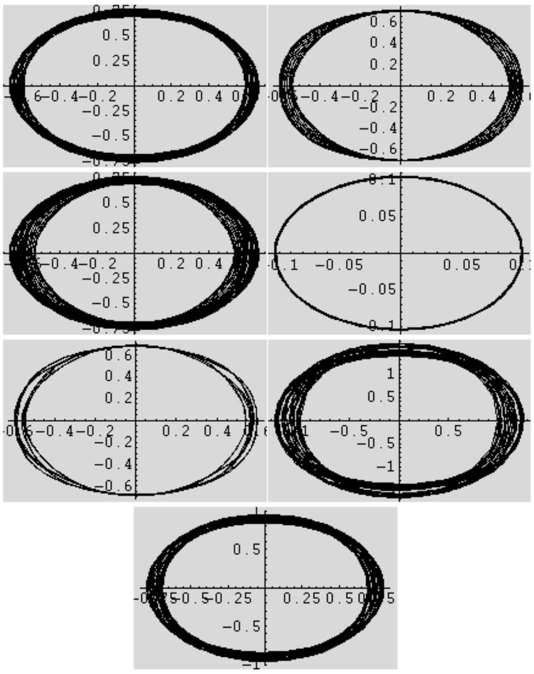
2. Mathematica simulations. The stability of the system (1) has been testified by computer computations. We present scripts and pictures as well.

```
row1=x'[t]==a u[t];
row2=u'[t]==-a x[t]-b x[t]((x[t])^2+3(y[t])^2);
row3=y'[t]==c v[t];
row4=v'[t]==-c y[t]-d y[t]((y[t])^2+3(x[t])^2);
a=2;b=0.8;c=1;d=0.7;
rozw[xx_,yy_,TT_]=NDSolve[{row1,row2,row3,row4,x[0]==xx,
u[0]==0,y[0]==yy,v[0]==0},{x,y,u,v},{t,0,TT},MaxSteps->10000];
ppxu[xx_,yy_,TT_]:=ParametricPlot[Evaluate[{x[t],u[t]}
/.rozw[xx,yy,TT]],{t,0,TT}];
ppyv[xx_,yy_,TT_]:=ParametricPlot[Evaluate[{y[t],v[t]}
/.rozw[xx,yy,TT]],{t,0,TT}];
Show[ppxu[0.6,0.6,100],ppyv[0.6,0.6,100]]
ppxu[0.1,0.3,100]
ppyv[0.5,0.6,50]
ppxu[1,1,100]
ppxu[0.7,0.7,100]
```



Pictures for various parameters show that the orbits behave as those on invariant tori.

3. The first equation. In order to analyze the first equation we write it in the form

$$\ddot{\xi} = -A^2\xi - AB\xi^3 - AC|\xi| \cdot \operatorname{sign}(\dot{\xi})$$

for some $A, B, C > 0$ with $A > C$. We put $\dot{\xi} = A\sigma$ and get

$$\begin{aligned} \dot{\xi} &= A\sigma, \\ \dot{\sigma} &= -A\xi - B\xi^3 - C|\xi| \cdot \operatorname{sign}(A\sigma), \end{aligned}$$

which may be treated as a perturbation of a linear oscillator

$$\begin{aligned} \dot{\xi} &= A\sigma, \\ \dot{\sigma} &= -A\xi, \end{aligned}$$

with integral (Hamiltonian) $H_0 = A(\xi^2 + \sigma^2)/2$. On trajectories (solutions) $\Phi(t)$ we clearly have $\dot{H}_0(\Phi(t)) = (\operatorname{grad} H_0, \dot{\Phi}) = 0$. Now we pass to the perturbed oscillator

$$\begin{aligned} \dot{\xi} &= A\sigma, \\ \dot{\sigma} &= -A\xi - B\xi^3, \end{aligned}$$

with integral $H_B = H_0 + B\xi^4/4$. The levels of H_B are bounded curves (topological circles) and if we introduce the polar coordinates $\varrho^2 = \xi^2 + \sigma^2$ and $\tan(\tau) = \xi/\sigma$ we get for the smooth system

$$\begin{aligned} \dot{\varrho} &= -B\varrho^3 \sin^3(\tau)\cos(\tau), \\ \dot{\tau} &= A + B\varrho^2 \sin^4(\tau), \end{aligned}$$

and for the system with viscosity

$$\begin{aligned} \dot{\varrho} &= -B\varrho^3 \sin^3(\tau)\cos(\tau) - C\varrho^2|\sin(\tau)\cos(\tau)|, \\ \dot{\tau} &= A + \sin^2(\tau)(B\varrho^2 \sin^2(\tau) + C \operatorname{sign}(\sin(\tau)) \operatorname{sign}(\cos(\tau))); \end{aligned}$$

therefore $A - C \le \dot{\tau} \le A + C + H_B$ and the angle grows linearly, which yields almost uniformly oscillatory character of the solutions $(\xi(t), \sigma(t))$.

In particular the solutions on a given level $H_B > 0$ have the property that $|(\xi, \sigma)|$ is separated from zero by a constant depending only on H_B on a definite part of time.

Lemma 2. *Suppose that*

$$\dot{\tau} = A + \eta(t)$$

and $0 \le \eta(t) \le H$. *Then for* $t_0 < t_1$:

1. $A(t_1 - t_0) \le \tau(t_1) - \tau(t_0) \le (A + H)(t_1 - t_0)$.

2. *For fixed* ω *let* $f(t) = |\sin(\omega\tau(t))|$, *and* t', t'' *be two consecutive times with* $f(t) = 0$. *Then for any* $0 < a < 1$ *there exists* $\overline{b} > 0$ *such that for* $t' < s' < s'' < t''$ *with* $f(s') = f(s'') = a$ *we have*

$$|\{t' < t < t'' : f(t) > a\}| = s'' - s' > \overline{b}(t'' - t'),$$

i.e. the time when f *is separated from zero takes a definite proportion of the whole time. This proportion goes to* 1 *as* a *goes to zero.*

3. *For some* $b \ge \overline{b}A/(A+H)$ *if* $t_0 < t' < t'' < t_1$ *then*

$$|\{t_0 < t < t_1 : f(t) > a\}| > b|t_1 - t_0|;$$

here also one may take b *arbitrarily close to* 1 *as* a *is close to zero.*

4. $\int_{t_0}^{t_1} f(t)\,dt \ge ab|t_1 - t_0|$.

5. *Let* $\xi(t), \sigma(t)$ *be the solution of the first equation and* ϱ, τ *be the solution in polar coordinates. Then* $A(\xi^2 + \sigma^2)/2 + B\xi^4/4 = H > 0$ *and*

$$\frac{\sqrt{A^2 + 4BH} - A}{B} \le \varrho^2 \le 2H/A.$$

6. *We have*

$$\varrho^2 \ge \begin{cases} \dfrac{4H}{(\sqrt{2}+1)A} & \text{if } 4BH \le A, \\ \dfrac{4H}{(\sqrt{2}+1)\sqrt{4BH}} & \text{if } 4BH \ge A. \end{cases}$$

Proof. 1. This follows easily by the Mean Value Theorem.

2. By definition t', t'' are such that $\tau(t') = k\pi/\omega$ and $\tau(t'') = (k+1)\pi/\omega$ for some integer k. Let $\alpha = \arcsin(a)$. Then $0 < \alpha < \pi/2$ and as $f(s') = f(s'') = a$ we get $\omega\tau(s') = k\pi + \alpha$ and $\omega\tau(s'') = (k+1)\pi - \alpha$. Hence

$$\begin{aligned} s'' - s' &\ge \frac{\tau(s'') - \tau(s')}{A+H} = \frac{\pi - 2\alpha}{\omega(A+H)}, \\ s' - t' &\le \frac{\tau(s') - \tau(t')}{A} = \frac{\alpha}{\omega A}, \\ t'' - s'' &\le \frac{\tau(t'') - \tau(s'')}{A} = \frac{\alpha}{\omega A}, \\ \frac{s'' - s'}{t'' - t'} &= \frac{s'' - s'}{(t'' - s'') + (s'' - s') + (s' - t')} \ge \frac{A(\pi - 2\alpha)}{A\pi + H \cdot 2\alpha} =: \overline{b}. \end{aligned}$$

3. We can compare the lengths of two consecutive intervals with endpoints defined by $f(t) = 0$.

4. This is an integration of the previous inequality.

5. This an elementary calculation. We have $\varrho^4 \ge \xi^4$.

6. $(\sqrt{A^2 + 4BH} - A)/B = 4H/(\sqrt{A^2 + 4BH} + A)$. ∎

Corollary 3. *The solutions of the first equation are asymptotically stable, with a limit at zero.*

Proof. Consider the Lyapunov function H_B defined as in the first integral of the equation without the viscosity term. Then a solution $\Phi(t) = (\xi(t), \sigma(t))$ of the system with viscosity crosses the levels of $H = H_B$ with the speed

$$\begin{aligned}\dot{H}(t) &= (\operatorname{grad} H, \dot{\Phi})\\ &= (A\xi + B\xi^3)(A\sigma) + (A\sigma)(-A\xi - B\xi^3 - C|\xi|\operatorname{sign}(A\sigma))\\ &= -AC\sigma|\xi|\operatorname{sign}(A\sigma) = -AC|\xi|\cdot|\sigma|.\end{aligned}$$

Clearly $H(t)$ is a positive non-increasing function of time. We pass to the polar coordinates. Using the notation from the previous lemma we obtain

$$\dot{H} = -0.5AC\varrho^2|\sin(2\tau)| \le -0.5AC\frac{\sqrt{A^2+4BH}-A}{B}|\sin(2\tau)|,$$

$$\begin{aligned}H(t_1) - H(t_0) &\le -0.5AC\int_{t_0}^{t_1}|\sin(2\tau(t))|\,dt\cdot\frac{4H(t_1)}{(\sqrt{2}+1)A}\\ &\le \begin{cases} -0.5ACab(t_1-t_0)\cdot\dfrac{4H(t_1)}{(\sqrt{2}+1)A} & \text{if } 4BH < A^2,\\ -0.5ACab(t_1-t_0)\cdot\dfrac{\sqrt{4H(t_1)}}{(\sqrt{2}+1)\sqrt{B}} & \text{if } 4BH > A^2.\end{cases}\end{aligned}$$

So if a trajectory starts at level H_0 at time t_0 then it reaches level $0 < H_1 < H_0$ at some time t_1 such that

$$t_1 - t_0 \le \begin{cases}\operatorname{const}\dfrac{H_0-H_1}{H_1} & \text{if } 4BH < A^2,\\ \operatorname{const}\dfrac{H_0-H_1}{\sqrt{H_1}} & \text{if } 4BH > A^2.\end{cases}$$

In both cases we find that the trajectories cross each level of H in a finite time. ■

4. The smooth system. Now we turn to the interacting part of the system, i.e. the remaining two equations. First we skip the discontinuous (viscosity) and non-autonomous (forced oscillations) parts. In this way we obtain an integrable stable system; by this we mean that there exists a first integral of the system with bounded levels. This may be deduced from the following theorems.

THEOREM 4. *Let $a, b, c, d \in \mathbb{R}^*$, and $F' = f$, $G' = g$, $K' = k$ be real functions. Then the system* [1]

[1] All derivatives with respect to t will be denoted by a dot over the function.

$$\begin{aligned}
\dot{x} &= \quad av,\\
\dot{v} &= -ax - bx(f(x^2) + k(x^2 + y^2)),\\
\dot{y} &= \quad cw,\\
\dot{w} &= -cy - dy(g(y^2) + k(x^2 + y^2)),
\end{aligned}$$

has an integral

$$H(x, v, y, w) = \frac{a}{b}(x^2 + v^2) + F(x^2) + K(x^2 + y^2) + G(y^2) + \frac{c}{d}(y^2 + w^2).$$

Proof. By inspection. Let $\Phi(t) = (x, v, y, w)$ be a solution of the system. Observe that

$$\begin{aligned}
(x^2 \dot{+} v^2) &= 2(x\dot{x} + v\dot{v}) = -2bxv(f(x^2) + k(x^2 + y^2))\\
&= -2\frac{b}{a}x\dot{x}\,(f(x^2) + k(x^2 + y^2))
\end{aligned}$$

and also

$$\begin{aligned}
\dot{F}(x^2(t)) &= 2x\dot{x}\,f(x^2),\\
\dot{K}(x^2(t) + y^2(t)) &= 2(x\dot{x} + y\dot{y})k(x^2 + y^2),\\
\dot{G}(y^2(t)) &= 2y\dot{y}\,g(y^2),
\end{aligned}$$

so all terms cancel in $\dot{H} = dH/dt = (\operatorname{grad} H, \dot{\Phi})$. ∎

THEOREM 5. *With the same assumptions on a, b, c, d as before, $\widetilde{F}' = \widetilde{f}$, $\widetilde{G}' = \widetilde{g}$ and $\lambda \in \mathbb{R}$ the system*

$$\begin{aligned}
\dot{x} &= av,\\
\dot{v} &= -ax - bx(\widetilde{f}(x^2) + \lambda y^2),\\
\dot{y} &= cy,\\
\dot{w} &= -cw - dy(\widetilde{g}(y^2) + \lambda x^2),
\end{aligned}$$

has an integral

$$H(x, v, y, w) = \frac{a}{b}(x^2 + v^2) + \widetilde{F}(x^2) + \lambda(x^2y^2) + \widetilde{G}(y^2) + \frac{c}{d}(y^2 + w^2).$$

Proof. By substitution $\widetilde{f}(z) = f(z) + \lambda z$, $k(z) = \lambda z$ and $\widetilde{g}(z) = g(z) + \lambda z$ in the previous theorem. ∎

We can prove even more:

THEOREM 6. *Suppose that*

$$\begin{aligned}
\dot{x} &= av,\\
\dot{v} &= -ax - f(x) - bk(x^py^q)x^{p-1}y^q,\\
\dot{y} &= cw,\\
\dot{w} &= -cy - g(y) - dk(x^py^q)y^{q-1}x^p.
\end{aligned}$$

Then (*after a linear change of coordinates*) *the system is Hamiltonian.*

Proof. Let $K'(u) = k(u)$ and substitute $(\overline{x}, \overline{v}) = \alpha(x, v)$ and $(\overline{y}, \overline{w}) = \beta(y, w)$ in such a way that $b\alpha^{p-1}\beta^q = p$ and $d\alpha^p\beta^{q-1} = q$. Then the Hamiltonian equals $a(x^2 + v^2)/2 + c(y^2 + w^2)/2 + F(x) + G(y) + K(x^p y^q)$. ∎

THEOREM 7. *If the constants* a, b, c, d (*and* λ) *are positive and the functions* F, G, K *are positive* (*e.g. when* f, g, k *are odd*) *then the solutions of the above systems are bounded.*

Proof. The levels of $H(x, v, y, w) = \text{const} > 0$ are compact (bounded and closed). Moreover if for any point $\overline{u}$ on the sphere S^3 there is a unique $s > 0$ with $H(s \cdot \overline{u}) = H_0$ then the levels of H are homeomorphic to S^3. ∎

COROLLARY 8. *The solutions of the system*

$$\begin{aligned}
\ddot{\xi} &= -2904\xi - 2976600\xi^3,\\
\ddot{\zeta} &= -2904\zeta - 1488300((\zeta + \phi)^3 + (\zeta - \phi)^3),\\
\ddot{\phi} &= -1387\phi - 710678((\zeta + \phi)^3 + (\zeta - \phi)^3),
\end{aligned}$$

are bounded and satisfy $H(\Phi(t)) = H_0 > 0$.

Note that there are solutions with $(x, v) \equiv 0$ and $(y, w) \equiv 0$.

By the previous theorems we see that the system without viscosity and forcing is stable (bounded solutions) but not asymptotically stable (the solutions do not converge to a point). Because of the nature of the problem we may assume that the coefficients are typical, therefore we deal with a typical Hamiltonian perturbation of a Hamiltonian system. By general KAM theory we find that most of the trajectories on the sphere $H = \text{const}$ densely describe two-dimensional (topological) tori forming different patches of quasi-periodic regimes [A], [AKN] and [Z]. The positions of such patches are highly dependent on the choice of the coefficients.

5. The system with viscosity. We write the system in the following form:

$$\begin{aligned}
\dot{x} &= av,\\
\dot{v} &= -ax - bx(x^2 + 3y^2) - \varepsilon|x| \operatorname{sign}(av),\\
\dot{y} &= cw,\\
\dot{w} &= -cy - dy(y^2 + 3x^2) - \delta|y| \operatorname{sign}(cw),
\end{aligned}$$

for some $a, b, c, d, \varepsilon, \delta > 0$ and $\varepsilon < a$ and $\delta < c$. We shall consider the Hamiltonian H for the smooth system as a Lyapunov function for the system with viscosity. For a solution $\Phi(t)$ of the system with viscosity and H we have

$$\begin{aligned}
\dot{H} &= (\operatorname{grad} H, \dot{\Phi}) = -av\varepsilon|x| \operatorname{sign}(av) - cw\delta|y| \operatorname{sign}(cw)\\
&= -(a\varepsilon|x| \cdot |v| + c\delta|y| \cdot |w|).
\end{aligned}$$

We see that the change into polar coordinates (R, T, Q, S) for the smooth system ($\varepsilon = 0 = \delta$) gives

$$\begin{aligned}\dot{R} &= -bR^2 \cos T \sin T\,(R^2 \sin^2 T + 3Q^2 \sin^2 S),\\ \dot{T} &= a + \sin^2 T\,(R^2 \sin^2 T + 3Q^2 \sin^2 S),\end{aligned}$$

and similarly for (Q, S). For the system with viscosity we have to add the discontinuous terms:

$$\begin{aligned}\dot{R} &= -bR^2[\cos T \sin T\,(R^2 \sin^2 T + 3Q^2 \sin^2 S) + \varepsilon|\sin T \cos T|],\\ \dot{T} &= a + \sin^2 T\,(R^2 \sin^2 T + 3Q^2 \sin^2 S + \varepsilon \operatorname{sign}(\sin T)\operatorname{sign}(\cos T)).\end{aligned}$$

Therefore the variables S, T satisfy the assumptions on τ from Lemma 2, are monotone and almost linear, therefore the system is persistently oscillating. The rates of change of T and S are bounded from below by $a - \varepsilon$ and $c - \delta$ and from above by $a + \varepsilon + 4H$ and $c + \varepsilon + 4H$. We also observe that the system cannot stay arbitrarily long near any line with three of the variables x, v, y, w close to zero.

Let $H(\Phi(t_0)) = H_0$ and let $t_1 > t_0$ be such that both $(a-\varepsilon)(t_1-t_0) > K\pi$ and $(c-\delta)(t_1-t_0) > K\pi$. In order to use Lemma 2 we choose a small number $s > 0$ such that the proportion of time $t \in (t_0, t_1)$ with $|\sin 2T(t)| > s$ is larger than 0.9 and similarly for $|\sin 2S(t)| > s$. Therefore

$$|\{t \in (t_0, t_1) : |\sin 2T(t)| > s \text{ and } |\sin 2S(t)| > s\}| \geq 0.8(t_1 - t_0).$$

We have

$$\begin{aligned}H &= aR^2/2 + cQ^2/2 + \alpha R^4 \sin^4(T)\\ &\quad + \beta Q^4 \sin^4(S) + \gamma R^2 \sin^2(T) Q^2 \sin^2(S),\end{aligned}$$

$$aR^2/2 + cQ^2/2 \leq H \leq \begin{cases} (\psi R^2 + \phi Q^2 + \theta)^2 & \text{for } H \text{ large},\\ \psi R^2 + \phi Q^2 & \text{for } H \text{ small},\end{cases}$$

for some $\psi, \phi, \theta > 0$ so at every moment at least one of R^2, Q^2 must be larger than const $\cdot H/(H+1)$. Now

$$\begin{aligned}H_0 - H_1 &= \int_{t_0}^{t_1} (\varepsilon|xv| + \delta|yw|)\,dt = \int_{t_0}^{t_1} \varepsilon|xv|\,dt + \int_{t_0}^{t_1} \delta|yw|\,dt\\ &\geq \int_{t_0}^{t_1} \varepsilon R^2|\sin T \cos T|\,dt + \int_{t_0}^{t_1} \delta Q^2|\sin S \cos S|\,dt\\ &\geq \min(\varepsilon, \delta) \cdot s \cdot (t_1 - t_0) \cdot \text{const}\,\frac{H_1}{H_1 + 1}\end{aligned}$$

and therefore the system is asymptotically stable. All the levels below the starting one will be crossed.

6. The model with forced oscillations. It is enough to consider the levels of the Hamiltonian of the smooth system for large values of H, such that $\sqrt{H}$ is greater than $|k_i|$. Then the same analysis of $\dot{H}$ as for the model with viscosity only holds. We skip the details.

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