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A UNIFIED APPROACH TO SOME STRATEGIES FOR THE TREATMENT OF BREAKDOWN IN LANCZOS-TYPE ALGORITHMS

Abstract. The Lanczos method for solving systems of linear equations is implemented by using some recurrence relationships between polynomials of a family of formal orthogonal polynomials or between those of two adjacent families of formal orthogonal polynomials. A division by zero can occur in these relations, thus producing a breakdown in the algorithm which has to be stopped. In this paper, three strategies to avoid this drawback are discussed: the MRZ and its variants, the normalized and unnormalized BIORES algorithm and the composite step biconjugate algorithm. We prove that all these algorithms can be derived from a unified framework; in fact, we give a formalism for finding all the recurrence relationships used in these algorithms, which shows that the three strategies use the same techniques.

1. Introduction. Let c be the linear functional on the space of complex polynomials defined by $c(\zeta^i) = c_i$ for $i = 0, 1, \ldots,$ where the c_i's are given complex numbers. The family of formal orthogonal polynomials $\{P_k\}$ with respect to c is defined by

$$c(\zeta^i P_k(\zeta)) = 0 \quad \text{for } i = 0, \ldots, k-1. \tag{1}$$

These polynomials are given by the determinant formula

$$P_k(\zeta) = \frac{\begin{vmatrix} 1 & \zeta & \ldots & \zeta^k \\ c_0 & c_1 & \ldots & c_k \\ \ldots & \ldots & \ldots & \ldots \\ c_{k-1} & c_k & \ldots & c_{2k-1} \end{vmatrix}}{d_k},$$

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where d_k is an arbitrary constant, which is determined by a normalization condition. In the sequel, P_k denotes the formal orthogonal polynomial normalized by the condition $P_k(0) = 1$, and $P_k^{(0)}$ is the monic formal orthogonal polynomial with respect to c.

REMARK 1. P_k exists if and only if

$$H_k^{(1)} = \begin{vmatrix} c_1 & \dots & c_k \\ \dots & \dots & \dots \\ c_k & \dots & c_{2k} \end{vmatrix} \neq 0,$$

moreover, it is of degree k if and only if

$$H_k^{(0)} = \begin{vmatrix} c_0 & \dots & c_{k-1} \\ \dots & \dots & \dots \\ c_{k-1} & \dots & c_{2k-2} \end{vmatrix} \neq 0.$$

Now consider the monic polynomials $P_k^{(1)}$ defined by

$$P_k^{(1)}(\zeta) = \frac{\begin{vmatrix} c_1 & c_2 & \dots & c_{k+1} \\ \dots & \dots & \dots & \dots \\ c_k & c_{k+1} & \dots & c_{2k} \\ 1 & \zeta & \dots & \zeta^k \end{vmatrix}}{\begin{vmatrix} c_1 & \dots & c_k \\ \dots & \dots & \dots \\ c_k & \dots & c_{2k-1} \end{vmatrix}}.$$

P_k and $P_k^{(1)}$ exist under the same condition $H_k^{(1)} \neq 0$ (see [4]). Moreover, $P_k^{(1)}$ satisfies

$$c(\zeta^{i+1} P_k^{(1)}) = 0, \qquad i = 0, 1, \dots, k-1. \tag{2}$$

If we define the linear functional $c^{(1)}$ by

$$c^{(1)}(\zeta^i) = c(\zeta^{i+1}) = c_{i+1}, \qquad i = 0, 1, \dots,$$

then the conditions (2) become

$$c^{(1)}(\zeta^i P_k^{(1)}) = 0 \qquad \text{for } i = 0, \dots, k-1,$$

which shows that the polynomials $P_k^{(1)}$ form a family of formal orthogonal polynomials with respect to $c^{(1)}$. $\{P_k\}$ and $\{P_k^{(1)}\}$ are called *adjacent families* of formal orthogonal polynomials.

It is well known that, when using reccurence relationships between formal orthogonal polynomials (belonging to one family or two adjacent ones), a division by zero can occur in the coefficients of the relation used. Such a division by zero is called a *breakdown*.

2. Lanczos-type algorithms. We consider a system of p linear equations in p unknowns

$$Ax = b, \tag{3}$$

where $A \in \mathbb{C}^{p\times p}$, $b \in \mathbb{C}^p$ and $x \in \mathbb{C}^p$.

Let x_0 be an initial guess, y a non-zero arbitrary vector, and let (x_k) be the sequence of vectors defined by

$$x_k - x_0 \in K_k(A, r_0), \tag{4}$$

and

$$r_k = b - Ax_k \perp K_k(A^*, y), \tag{5}$$

where $K_k(A, r) = \mathrm{span}(r, Ar, \ldots, A^{k-1}r)$, and A^* denotes the conjugate transpose of A.

From (4), $x_k - x_0$ can be written as

$$x_k - x_0 = -\alpha_1 r_0 - \ldots - \alpha_k A^{k-1} r_0,$$

and thus we have

$$r_k = r_0 + \alpha_1 A r_0 + \ldots + \alpha_k A^k r_0 = P_k(A) r_0,$$

where

$$P_k(\zeta) = 1 + \alpha_1 \zeta + \ldots + \alpha_k \zeta^k.$$

The orthogonality condition (5) can be written as

$$(y, A^i r_k) = 0 \quad \text{for } i = 0, \ldots, k-1.$$

If we set

$$c(\zeta^i) = c_i = (y, A^i r_0) \quad \text{for } i = 0, 1, \ldots,$$

we have

$$c(\zeta^i P_k(\zeta)) = 0 \quad \text{for } i = 0, \ldots k-1.$$

These conditions show that P_k is a polynomial of degree at most k belonging to the family of formal orthogonal polynomials with respect to the linear functional c, normalized by the condition $P_k(0) = 1$.

A Lanczos-type method [11, 12] consists in computing P_k recursively, then r_k and finally x_k such that $r_k = b - Ax_k$. Such a method gives the exact solution of the system (3) in at most p iterations; for more details, see [4].

If one of the scalar products appearing in the denominators of the coefficients of the recurrence relations is zero, then a breakdown occurs in the algorithm. This is due to the non-existence of some of the polynomials P_k (*pivot* or *true* breakdown) or to the impossibility of using this relation (*Lanczos* or *ghost* breakdown).

There are many strategies for avoiding a breakdown, for instance, the MRZ and its variants proposed by Brezinski, Redivo Zaglia and Sadok [2] where they jump over the singular polynomials (which does not exist), and they compute only the existing ones. Gutknecht [10] proposes another algorithm, called BIORES (normalized and unnormalized), where he introduces the deficient polynomials (they will be defined in Subsection 3.1) and makes use of a recurrence relation between them and the regular ones. Chan and Bank [5, 6] introduce a simple modification of the BCG algorithm [9], called the composite step bi-conjugate gradient algorithm (CSBCG), which eliminates pivot breakdowns, under the assumption that a Lanczos breakdown does not occur, i.e. $H_k^{(0)} \neq 0$ for all k.

In this paper, a formalism for finding all the recurrence relationships used in these three algorithms is given. It consists in expressing a particular polynomial in a basis formed by the regular polynomials and the deficient ones. This is the subject of the next section.

We note that no new algorithm for solving the breakdown problem is given in this paper. The aim of this work is to give a unified approach to some known breakdown-free Lanczos-type algorithns. This new approach allows us to derive the non-generic BIORES algorithm of Gutknecht in a simpler way than in [10], and to obtain a polynomial interpretation of the composite step bi-conjugate gradient algorithm [5].

3. Choice of basis and recurrence formulas

3.1. *Notations and definitions.* We denote by $0 = n_0 < n_1 < \ldots$ the indices for which the regular polynomials P_{n_k} and $P_{n_k}^{(1)}$ exist, m_k being the jump in the degrees between $P_{n_k}^{(1)}$ and $P_{n_{k+1}}^{(1)}$, that is,

$$n_{k+1} = n_k + m_k.$$

For $n_k < n < n_{k+1}$, we introduce the polynomials

$$P_n(\zeta) = \omega_{n-n_k}(\zeta) P_{n_k}(\zeta), \tag{6}$$

called *deficient* in [10], where ω_{n-n_k} is an arbitrary polynomial of exact degree $n-n_k$. It was proved by Draux [7] that m_k is defined by the conditions

$$c^{(1)}(\zeta^i P_{n_k}^{(1)}) = 0 \quad \text{for } i = 0, \ldots, n_k + m_k - 2, \tag{7}$$

and

$$c^{(1)}(\zeta^i P_{n_k}^{(1)}) \neq 0 \quad \text{for } i = n_k + m_k - 1. \tag{8}$$

If we denote by $0 = \overline{n}_0 < \overline{n}_1 < \ldots$ the indices for which the monic regular polynomials $P_{\overline{n}_k}^{(0)}$ with respect to the functional c exist, the conditions (7) and (8) become

$$c(\zeta^i P_{\overline{n}_k}^{(0)}) = 0 \quad \text{for } i = 0, \ldots, \overline{n}_k + \overline{m}_k - 2, \tag{9}$$

and

(10) $$c(\zeta^i P_{\bar{n}_k}^{(0)}) \neq 0 \quad \text{for } i = \overline{n}_k + \overline{m}_k - 1,$$

where $\overline{m}_k$ is the jump in the degrees between $P_{\bar{n}_k}^{(0)}$ and $P_{\bar{n}_{k+1}}^{(0)}$ and $\overline{n}_{k+1} = \overline{n}_k + \overline{m}_k$. The deficient polynomials corresponding to $P_n^{(0)}$ and $P_n^{(1)}$ are defined by formulas similar to (6).

3.2. *Recurrence formulas.* In the following, we will compute $P_{n_{k+1}}$ in terms of P_{n_k} and $P_{n_k}^{(1)}$. First we suppose that the polynomial P_{n_k} has degree exactly n_k, which is equivalent to saying that the monic polynomial $P_{n_k}^{(0)}$ exists. Thus n_k is equal to some index $\overline{n}_l$.

Consider the family

(11) $$\{P_{\bar{n}_0}^{(0)}, \zeta P_{\bar{n}_0}^{(0)}, \ldots, \zeta^{\bar{n}_0 - 1} P_{\bar{n}_0}^{(0)}, \ldots, P_{\bar{n}_{l-1}}^{(0)}, \zeta P_{\bar{n}_{l-1}}^{(0)}, \ldots, \zeta^{\bar{n}_{l-1}-1} P_{\bar{n}_{l-1}}^{(0)},$$
$$P_{n_k}, \zeta P_{n_k}^{(1)}, \ldots, \zeta^{m_k} P_{n_k}^{(1)}\},$$

where P_{n_i}, $P_{\bar{n}_i}^{(0)}$ and $P_{n_i}^{(1)}$ are the orthogonal polynomials defined previously. The family (11) forms a basis of the vector space of polynomials of degree at most n_{k+1}. Thus the polynomial $P_{n_{k+1}}$ can be expressed as

$$\begin{aligned} P_{n_{k+1}} = {} & \alpha_{\bar{n}_0}^{(0)} P_{\bar{n}_0}^{(0)} + \alpha_{\bar{n}_0}^{(0)} \zeta P_{n_0}^{(0)} + \ldots + \alpha_{\bar{n}_0}^{(\overline{m}_0 - 1)} \zeta^{\bar{n}_0 - 1} P_{\bar{n}_0}^{(0)} + \ldots \\ & + \alpha_{\bar{n}_{l-1}}^{(0)} P_{\bar{n}_{l-1}} + \alpha_{\bar{n}_{l-1}}^{(1)} \zeta P_{\bar{n}_{l-1}}^{(1)} + \ldots + \alpha_{\bar{n}_{l-1}}^{(\overline{m}_{l-1} - 1)} \zeta^{\bar{n}_{l-1} - 1} P_{\bar{n}_{l-1}}^{(1)} \\ & + \alpha_{n_k}^{(0)} P_{n_k} + \alpha_{n_k}^{(1)} \zeta P_{n_k}^{(1)} + \ldots + \alpha_{n_k}^{(m_k)} \zeta^{m_k} P_{n_k}^{(1)}. \end{aligned}$$

Using the orthogonality conditions

$$c(\zeta^{\bar{n}_j + i} P_{n_{k+1}}) = 0 \quad \text{for } i = 0, \ldots, \overline{m}_j - 1 \text{ and } j = 0, \ldots, l-1,$$

we obtain

$$\alpha_{\bar{n}_j}^{(i)} = 0 \quad \text{for } i = 0, \ldots, m_j - 1 \text{ and } j = 0, \ldots, l-1.$$

Finally, the condition $P_{n_{k+1}}(0) = 1$ gives $\alpha_{n_k}^{(0)} = 1$ and we obtain

(12) $$P_{n_{k+1}}(\zeta) = P_{n_k}(\zeta) + \zeta \omega_k(\zeta) P_{n_k}^{(1)}(\zeta),$$

where

$$\omega_k(\zeta) = \alpha_{n_k}^{(1)} + \ldots + \alpha_{n_k}^{(m_k - 1)} \zeta^{m_k - 2} + \alpha_{n_k}^{(m_k)} \zeta^{m_k - 1}.$$

LEMMA 1. *Even if the polynomial P_{n_k} does not have exact degree n_k, the relationship* (12) *holds.*

Proof. It is enough to remark that the coefficients of ω_k are chosen so that the polynomial

$$Q_{n_k}(\zeta) = P_{n_{k+1}}(\zeta) - \zeta \omega_k(\zeta) P_{n_k}^{(1)}(\zeta)$$

has degree at most n_k. Moreover, $c(\zeta^i Q_{n_k}) = 0$ for $i = 0, \ldots, n_k - 1$, thus, since $Q_{n_k}(0) = 1$, Q_{n_k} is identical to P_{n_k}. ■

The recurrence relationship (12) is the first relation used in the MRZ algorithm.

Now we consider the family

$$\begin{aligned}(13)\qquad \{&P^{(1)}_{n_0}, \zeta P^{(1)}_{n_0}, \ldots, \zeta^{m_0-1} P^{(1)}_{n_0},\\ &P^{(1)}_{n_1}, \zeta P^{(1)}_{n_1}, \ldots, \zeta^{m_1-1} P^{(1)}_{n_1}, \ldots P^{(1)}_{n_k}, \zeta P^{(1)}_{n_k}, \ldots, \zeta^{m_k-1} P^{(1)}_{n_k}\}.\end{aligned}$$

For the same reasons as in the case (11), we can prove that (13) forms a basis of the vector space of polynomials of degree at most $n_k + m_k - 1 = n_{k+1} - 1$. Thus we can express the polynomial $P^{(1)}_{n_{k+1}} - \zeta^{m_k} P^{(1)}_{n_k}$ of degree $n_{k+1} - 1$ as

$$\begin{aligned}P^{(1)}_{n_{k+1}} - \zeta^{m_k} P^{(1)}_{n_k} = {} & \alpha^{(0)}_{n_0} P^{(1)}_{n_0} + \alpha^{(1)}_{n_0} \zeta P^{(1)}_{n_0} + \ldots + \alpha^{(m_0-1)}_{n_0} \zeta^{m_0-1} P^{(1)}_{n_0}\\ & + \alpha^{(0)}_{n_1} P^{(1)}_{n_1} + \alpha^{(1)}_{n_1} \zeta P^{(1)}_{n_1} + \ldots + \alpha^{(m_1-1)}_{n_1} \zeta^{m_1-1} P^{(1)}_{n_1} + \ldots\\ & + \alpha^{(0)}_{n_k} P^{(1)}_{n_k} + \alpha^{(1)}_{n_k} \zeta P^{(1)}_{n_k} + \ldots + \alpha^{(m_k-1)}_{n_k} \zeta^{m_k-1} P^{(1)}_{n_k}.\end{aligned}$$

So, we obtain

$$\begin{aligned}P^{(1)}_{n_{k+1}} = {} & \alpha^{(0)}_{n_0} P^{(1)}_{n_0} + \alpha^{(1)}_{n_0} \zeta P^{(1)}_{n_0} + \ldots + \alpha^{(m_0-1)}_{n_0} \zeta^{m_0-1} P^{(1)}_{n_0}\\ & + \alpha^{(0)}_{n_1} P^{(1)}_{n_1} + \alpha^{(1)}_{n_1} \zeta P^{(1)}_{n_1} + \ldots + \alpha^{(m_1-1)}_{n_1} \zeta^{m_1-1} P^{(1)}_{n_1} + \ldots\\ & + \alpha^{(0)}_{n_k} P^{(1)}_{n_k} + \alpha^{(1)}_{n_k} \zeta P^{(1)}_{n_k} + \ldots + \alpha^{(m_k-1)}_{n_k} \zeta^{m_k-1} P^{(1)}_{n_k} + \zeta^{m_k} P^{(1)}_{n_k}.\end{aligned}$$

Using the conditions

$$c^{(1)}(\zeta^{n_j+i} P^{(1)}_{n_{k+1}}) = 0 \quad \text{for } i = 0, \ldots, m_j - 1 \text{ and } j = 0, \ldots, k-2,$$

we get

$$P^{(1)}_{n_{k+1}} = \alpha^{(0)}_{n_{k-1}} P^{(1)}_{n_{k-1}} + \alpha^{(1)}_{n_{k-1}} \zeta P^{(1)}_{n_{k-1}} + \ldots + \alpha^{(m_{k-1}-1)}_{n_{k-1}} \zeta^{m_{k-1}-1} P^{(1)}_{n_{k-1}} + q_k(\zeta) P^{(1)}_{n_k},$$

where q_k is a monic polynomial of degree m_k.

We also have

$$c^{(1)}(\zeta^{n_{k-1}+i} P^{(1)}_{n_{k+1}}) = 0 \quad \text{for } i = 1, \ldots, m_{k-1} - 2,$$

and so

$$\alpha^{(i)}_{n_{k-1}} = 0 \quad \text{for } i = 1, \ldots, m_{k-1} - 1.$$

Thus we recover the second recurrence relationship used in the MRZ algorithm:

$$(14)\qquad P^{(1)}_{n_{k+1}}(\zeta) = \alpha^{(0)}_{n_{k-1}} P^{(1)}_{n_{k-1}}(\zeta) + q_k(\zeta) P^{(1)}_{n_k}(\zeta).$$

We will now see that we can also obtain the recurrence relationship used in the BMRZ (cf. [2]).

In fact, it is enough to choose the coefficient $\alpha_{n_{k+1}}$ such that $P^{(1)}_{n_{k+1}} - \alpha_{n_{k+1}} P_{n_{k+1}}$ has degree $n_{k+1} - 1$ (here we require that $P_{n_{k+1}}$ has degree

n_{k+1}). Expressing this polynomial in the basis (13), we can write

$$\begin{aligned}P^{(1)}_{n_{k+1}} &= \alpha^{(0)}_{n_0}P^{(1)}_{n_0} + \alpha^{(1)}_{n_0}\zeta P^{(1)}_{n_0} + \ldots + \alpha^{(m_0-1)}_{n_0}\zeta^{m_0-1}P^{(1)}_{n_0}\\ &+ \alpha^{(0)}_{n_1}P^{(1)}_{n_1} + \alpha^{(1)}_{n_1}\zeta P^{(1)}_{n_1} + \ldots + \alpha^{(m_1-1)}_{n_1}\zeta^{m_1-1}P^{(1)}_{n_1} + \ldots\\ &+ \alpha^{(0)}_{n_k}P^{(1)}_{n_k} + \alpha^{(1)}_{n_k}\zeta P^{(1)}_{n_k} + \ldots + \alpha^{(m_k-1)}_{n_k}\zeta^{m_k-1}P^{(1)}_{n_k} + \alpha_{n_{k+1}}P_{n_{k+1}}.\end{aligned}$$

From the orthogonality conditions

$$c^{(1)}(\zeta^{n_j+i}P^{(1)}_{n_{k+1}}) = 0 \quad \text{for } i = 0,\ldots,m_j-1 \text{ and } j = 0,\ldots,k-1,$$

we obtain

$$\alpha^{(0)}_{n_j} = \alpha^{(1)}_{n_j} = \ldots = \alpha^{(m_j-1)}_{n_j} = 0 \quad \text{for } j = 0,\ldots,k-1.$$

Using also the fact that

$$c^{(1)}(\zeta^{n_k+i}P^{(1)}_{n_{k+1}}) = 0 \quad \text{for } i = 0,\ldots,m_k-2,$$

we obtain

$$\alpha^{(m_{k-1}-1)}_{n_k} = \ldots = \alpha^{(1)}_{n_k} = 0,$$

and finally

$$P^{(1)}_{n_{k+1}}(\zeta) = \alpha^{(0)}_{n_k}P^{(1)}_{n_k}(\zeta) + \alpha_{n_{k+1}}P_{n_{k+1}}(\zeta). \tag{15}$$

If we set $r_{n_k} = P_{n_k}(A)r_0$ and $z_{n_k} = P^{(1)}_{n_k}(A)r_0$ where $r_0 = Ax_0 - b$ ($n_0 = 0$), then the recurrences (12) and (14) define the MRZ algorithm. Similarly, the recurrences (12) and (15) define the BMRZ.

Since the polynomials $\{P_k\}$ are normalized by the condition $P_k(0) = 1$, the approximations x_{n_k} of the solution of the system (3) can be computed recursively. In fact,

$$r_{n_{k+1}} = r_{n_k} + A\omega_k(A)z_{n_k},$$

so

$$x_{n_{k+1}} = x_{n_k} + \omega_k(A)z_{n_k}.$$

In the MRZ, we express the polynomial $P^{(1)}_{n_{k+1}} - \zeta^{m_k}P^{(1)}_{n_k}$ in the basis (13). The polynomial $P^{(1)}_{n_{k+1}} - \alpha_{n_{k+1}}P_{n_{k+1}}$ can be expressed in the same basis, in order to obtain the BMRZ. However, the polynomial $P_{n_{k+1}}$ does not always have degree n_{k+1}, and, in this case, the BMRZ has to be stopped.

Obviously, the recurrence relationship used in the BMRZ needs less computation than that used in the MRZ. It seems that a combination of these two methods is the best for solving the breakdown problem. It consists of testing the degree of the polynomial $P_{n_{k+1}}$: if it is exactly equal to n_{k+1}, then we use the recurrence relationship of the BMRZ. In the opposite case, we use the MRZ.

Now we study the non-generic BIORES algorithm [10] which is a breakdown-free version of the BIORES algorithm [4]. It is well known that this last algorithm suffers from the ghost breakdown due to the fact that the polynomials $\{P_{n_k}\}$ do not always have exact degree n_k. For curing this drawback we will use the monic formal orthogonal polynomials $P_{n_k}^{(0)}$, and we will show that we can find the recurrence relationships used in [10] by the same techniques as previously.

Consider the family

$$\{P_{\bar{n}_0}^{(0)}, \zeta P_{\bar{n}_0}^{(0)}, \dots, \zeta^{\bar{n}_0 - 1} P_{\bar{n}_0}^{(0)}, P_{\bar{n}_1}^{(0)}, \zeta P_{\bar{n}_1}^{(0)}, \dots, \zeta^{\bar{n}_1 - 1} P_{\bar{n}_1}^{(0)}, \dots, P_{\bar{n}_k}^{(0)}, \zeta P_{\bar{n}_k}^{(0)}, \dots, \zeta^{\bar{n}_k - 1} P_{\bar{n}_k}^{(0)}\}. \tag{16}$$

Obviously, the family (16) forms a basis of the vector space of polynomials of degree $\overline{n}_{k+1} - 1$. Expressing the polynomial $P_{\bar{n}_{k+1}} - \zeta^{\bar{n}_k} P_{\bar{n}_k}$ in this basis, we obtain

$$\begin{aligned} P_{\bar{n}_{k+1}}^{(0)} = {} & \alpha_{\bar{n}_0}^{(0)} P_{\bar{n}_0}^{(0)} + \alpha_{\bar{n}_0}^{(1)} \zeta P_{\bar{n}_0}^{(0)} + \dots + \alpha_{\bar{n}_0}^{(\overline{m}_0 - 1)} \zeta^{\bar{n}_0 - 1} P_{\bar{n}_0}^{(0)} + \dots \\ & + \alpha_{\bar{n}_k}^{(0)} P_{\bar{n}_k}^{(0)} + \alpha_{\bar{n}_k}^{(1)} \zeta P_{\bar{n}_k}^{(0)} + \dots + \alpha_{\bar{n}_k}^{(\overline{m}_k - 1)} \zeta^{\bar{n}_k - 1} P_{\bar{n}_k}^{(0)} + \zeta^{\bar{n}_k} P_{\bar{n}_k}^{(0)}. \end{aligned}$$

Moreover, using the orthogonality conditions, we find

$$P_{\bar{n}_{k+1}}^{(0)}(\zeta) = \alpha_{\bar{n}_{k-1}}^{(0)} P_{\bar{n}_{k-1}}^{(0)}(\zeta) + q_k(\zeta) P_{\bar{n}_k}^{(0)}(\zeta),$$

where q_k is a monic polynomial of degree $\overline{m}_k$. This recurrence relationship is already given in [1], but it was not used to avoid a breakdown.

To obtain all the previous recurrence relationships, we considered the set of regular polynomials and we completed it by particular deficient polynomials which have the form $\zeta^i P_{n_j}^{(1)}$ and/or $\zeta^i P_{\bar{n}_j}^{(0)}$. Now, using the general form of the deficient polynomials, we will find the recurrence relations used in [10].

Thus we consider the family

$$\{P_{\bar{n}_0}^{(0)}, U_1^0 P_{\bar{n}_0}^{(0)}, \dots, U_{\bar{n}_0 - 1}^0 P_{\bar{n}_0}^{(0)}, P_{\bar{n}_1}^{(0)}, U_1^1 P_{\bar{n}_1}^{(0)}, \dots, U_{\bar{n}_1 - 1}^1 P_{\bar{n}_1}^{(0)}, \dots, P_{\bar{n}_k}^{(0)}, U_1^k P_{\bar{n}_k}^{(0)}, \dots, U_{\bar{n}_k - 1}^k P_{\bar{n}_k}^{(0)}\}, \tag{17}$$

where the U_i^j's are arbitrary monic polynomials of degree j. Taking the polynomial $P_{\bar{n}_{k+1}}^{(0)}(\zeta) - \omega_{\bar{n}_k}(\zeta) P_{\bar{n}_k}^{(0)}(\zeta)$, with $\omega_{\bar{n}_k}$ an arbitrary monic polynomial of degree $\overline{m}_k$, and expressing it in the basis (17), we obtain

$$\begin{aligned} P_{\bar{n}_{k+1}}^{(0)} = {} & \alpha_{\bar{n}_0}^{(0)} P_{\bar{n}_0}^{(0)} + \alpha_{\bar{n}_0}^{(1)} U_1^0 P_{\bar{n}_0}^{(0)} + \dots + \alpha_{\bar{n}_0}^{(\overline{m}_0 - 1)} U_{\bar{n}_0 - 1}^0 P_{\bar{n}_0}^{(0)} + \dots \\ & + \alpha_{\bar{n}_k}^{(0)} P_{\bar{n}_k}^{(0)} + \alpha_{\bar{n}_k}^{(1)} U_1^k P_{\bar{n}_k}^{(0)} + \dots + \alpha_{\bar{n}_k}^{(\overline{m}_k - 1)} U_{\bar{n}_k - 1}^k P_{\bar{n}_k}^{(0)} + \omega_{\bar{n}_k} P_{\bar{n}_k}^{(0)}. \end{aligned}$$

The orthogonality conditions give

$$P_{\bar{n}_{k+1}}^{(0)}(\zeta) = (\omega_{\bar{n}_k}(\zeta) - a_k(\zeta)) P_{\bar{n}_k}^{(0)}(\zeta) - \alpha_{\bar{n}_{k-1}} P_{\bar{n}_{k-1}}^{(0)}(\zeta), \tag{18}$$

where

$$a_k(\zeta) = -\sum_{j=1}^{\bar{n}_k-1} \alpha_{\bar{n}_k}^{(j)} U_j^k(\zeta).$$

For $n_k < n < n_{k+1}$, we use the deficient polynomials

$$P_n^{(0)}(\zeta) = \omega_{n-n_k}(\zeta) P_{n_k}^{(0)}(\zeta).$$

If the polynomials ω_m satisfy the three-term recurrence

$$\omega_{m+1}(\zeta) = (\zeta - \alpha_m)\omega_m(\zeta) - \beta_m \omega_{m-1}(\zeta), \tag{19}$$

then the deficient polynomials satisfy

$$P_{n+1}^{(0)}(\zeta) = (\zeta - \alpha_{n-\bar{n}_k}) P_n^{(0)}(\zeta) - \beta_{n-\bar{n}_k} P_{n-1}^{(0)}(\zeta), \quad \overline{n}_k < n < \overline{n}_{k+1}. \tag{20}$$

We can express the polynomials a_k as

$$a_k(\zeta) = \sum_{i=0}^{\bar{n}_k-1} \alpha_i^k \omega_i(\zeta),$$

and the recurrence (18) becomes

$$\begin{aligned} P_{\bar{n}_{k+1}}^{(0)}(\zeta) &= (\zeta - \alpha_{\bar{n}_k-1}^k - \alpha_{\bar{n}_k-1}) P_{\bar{n}_{k+1}-1}^{(0)}(\zeta) \\ &\quad - (\alpha_{\bar{n}_k-2}^k + \beta_{\bar{n}_k-1}) P_{\bar{n}_{k+1}-2}^{(0)}(\zeta) \\ &\quad - \alpha_{\overline{m}_k-3}^k P_{\bar{n}_{k+1}-3}(\zeta) - \ldots - \alpha_0^k P_{\bar{n}_k}^{(0)}(\zeta) - \alpha_k P_{\bar{n}_{k-1}}^{(0)}(\zeta). \end{aligned} \tag{21}$$

We set $r_n = P_n^{(0)}(A) r_0 \mathit{\Gamma}_n$ and $\widetilde{r}_n = \overline{P}_n^{(0)}(A^*) y \overline{\mathit{\Gamma}}_n$, where $\overline{P}_n^{(0)}$ is the polynomial whose coefficients are complex conjugates of those of $P_n^{(0)}$, and $\mathit{\Gamma}_n$, $\overline{\mathit{\Gamma}}_n$ are scale factors. Using the recurences (20) and (21), we recover the non-generic BIORES algorithm of Gutknecht.

To find the approximations x_n of the solution of the problem (3), the scale factors $\mathit{\Gamma}_n$ and $\overline{\mathit{\Gamma}}_n$ are replaced by the relative scale factors

$$\gamma_{n,i} = \mathit{\Gamma}_n / \mathit{\Gamma}_{n-i}, \quad \overline{\gamma}_{n,i} = \overline{\mathit{\Gamma}}_n / \overline{\mathit{\Gamma}}_{n-i}.$$

With a particular choice of $\gamma_{i,j}$, we can eliminate b from both sides of the recurrence satisfied by $r_n = b - Ax_n$, and thus the recurrence relationship between the approximations x_n is established. The corresponding algorithm is called the normalized BIORES. In the unnormalized BIORES algorithm, Gutknecht uses another technique: he introduces two sequences z_n and ϱ_n related by $r_n = b\varrho_n - Az_n$. The second sequence ϱ_n is chosen to eliminate b from both sides of the recurrence satisfied by r_n. Thus the recurrence relationship between the z_n is established and the approximations x_n are given by $x_n = z_n/\varrho_n$.

Now, we are interested in the BCG algorithm. It is well known that it suffers from two kinds of breakdowns. The first one is due to the breakdown of the underlying Lanczos process (*Lanczos* or *ghost* breakdown in [3]), and the second one is due to the fact that some iterates are not well defined by the Galerkin condition on the associated Krylov subspace (*pivot* or *true* breakdown in [3]). Under the condition that Lanczos breakdowns do not occur, i.e. $H_k^{(0)} \neq 0$ for all k, Chan and Bank [5, 6] propose the composite step bi-conjugate gradient algorithm (CSBCG) for eliminating the pivot breakdown. Under this condition, two consecutive Hankel determinants $H_k^{(1)}$ cannot be zero (see [8]), thus $m_k \leq 2$.

Recall that, under the condition $H_k^{(0)} \neq 0$, the monic orthogonal polynomial $P_k^{(0)}$ exists. Now, we assume that, at the kth step, a pivot breakdown occurs in the BCG algorithm. The polynomial P_{k+1} does not exist, and thus $H_k^{(1)} \neq 0$, $H_{k+1}^{(1)} = 0$ and $H_{k+2}^{(1)} \neq 0$.

Remark 2. When $H_{k+1}^{(1)} = 0$, the polynomials P_k and P_{k+2} have exact degree k and $k+2$ respectively.

Obviously, the family

$$\{P_0^{(0)}, P_1^{(0)}, \ldots, P_{k-1}^{(0)}, P_k, \zeta Q_k\} \tag{22}$$

where

$$Q_k = (-1)^k H_k^{(0)}/H_k^{(1)} P_k^{(1)},$$

forms a basis of the vector space of polynomials of degree at most $k+1$. Expressing the polynomial $P_{k+2} + d_{k+2}\zeta P_{k+1}^{(0)}$ of degree $k+1$, where

$$P_{k+2}(\zeta) = -d_{k+2}\zeta^{k+2} + \ldots,$$

in the basis (22), we obtain

$$P_{k+2} = a_k P_k - b_k \zeta Q_k - c_k \zeta P_{k+1}^{(0)},$$

with

$$c_k = d_{k+2}.$$

Finally, using the condition $P_{k+2}(0) = 1$, we obtain

$$P_{k+2} = P_k - b_k \zeta Q_k - c_k \zeta P_{k+1}^{(0)}. \tag{23}$$

We can also express the polynomial $P_{k+1}^{(0)}$ in the basis (22), and we obtain

$$P_{k+1}^{(0)} = \sigma_k P_k + \varrho_k \zeta Q_k. \tag{24}$$

By construction of Q_{k+2}, the polynomial $Q_{k+2} - P_{k+2}$ has degree $k+1$, and we can write

$$Q_{k+2} - P_{k+2} = g_k \zeta^{k+1} + \ldots$$

If we consider the polynomial $Q_{k+2} - P_{k+2} - g_k P_{k+1}^{(0)}$, of degree k, we can express it in the basis $\{Q_0, Q_1, \ldots, Q_k\}$, and we easily obtain the recurrence relationship

$$Q_{k+2} = P_{k+2} + e_k Q_k + g_k P^{(0)}_{k+1}. \tag{25}$$

Setting

$$r_k = P_k(A)r_0, \quad \widetilde{r}_k = \overline{P}_k(A^*)\widetilde{r}_0, \quad p_k = Q_k(A)r_0,$$
$$\widetilde{p}_k = \overline{Q}_k(A^*)\widetilde{r}_0, \quad z_{k+1} = P^{(0)}_{k+1}(A)r_0, \quad \widetilde{z}_{k+1} = \overline{P}^{(0)}_{k+1}(A^*)\widetilde{r}_0,$$

and using the recurrences (23)–(25), we recover the CSBCG algorithm.

From (23), the residuals r_k satisfy the recurrence relation

$$r_{k+2} = r_k - A[b_k p_k - c_k z_{k+1}],$$

thus the approximations x_k can be computed recursively as

$$x_{k+2} = x_k + [b_k p_k - c_k z_{k+1}].$$

4. Conclusion. In the present work we discuss three strategies for treating the breakdown problem in the Lanczos-type algorithms. Theses strategies are derived, using simple arguments, from a unified framework.

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