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CHARACTERIZATIONS OF POWER DISTRIBUTIONS VIA MOMENTS OF ORDER STATISTICS AND RECORD VALUES

Abstract. Power distributions can be characterized by equalities involving three moments of order statistics. Similar equalities involving three moments of k-record values can also be used for such a characterization. The case of samples with random sizes is also considered.

1. Introduction. Too and Lin [8] have given a characterization of the uniform distribution by an equality involving only two moments of order statistics. We extend that result to power distributions. Moreover, we give a characterization of power distributions in terms of moments of k-record values. In Sections 4 and 5 we treat the characterization problem when sample sizes are random (cf. [1], [6], [9]).

2. A characterization of power distributions. Let $X_{k:n}$ be the kth smallest order statistic of a random sample $(X_1, \ldots, X_n)$ from a distribution F. Let m be a negative integer. We start with the problem of characterizing the power distribution function F defined as follows (cf. [1]):

$$F(x) = 1 - (1 + mx)^{-1/m}, \quad x \in (0, -1/m). \tag{2.1}$$

THEOREM 1. *With the above notation suppose that $EX^2_{k:n} < \infty$ for some pair (k, n). Then the equality*

$$EX^2_{k:n} - \frac{2}{m}\left[\frac{n_{[k]}}{(n-m)_{[k]}} EX_{k:n-m} - EX_{k:n}\right] + \frac{1}{m^2}\left[\frac{n_{[k]}}{(n-2m)_{[k]}} - \frac{2n_{[k]}}{(n-m)_{[k]}} + 1\right] = 0, \tag{2.2}$$

where $n_{[k]} = n(n-1)\ldots(n-k+1)$, holds iff F is given by (2.1).

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Proof. Let $F^{-1}(t)=\inf\{x: F(x)\ge t,\ t\in(0,1)\}$. Taking into account that

$$EX^l_{k:n}=\frac{n!}{(k-1)!(n-k)!}\int_0^1 (F^{-1}(t))^l t^{k-1}(1-t)^{n-k}dt,\qquad l\ge 1, \tag{2.3}$$

we see that $E|X_{k:n}|<\infty$, and $E|X_{k:n-m}|<\infty$. Furthermore, when F is given by (2.1), we find that

$$\begin{aligned} EX_{k:s}=\frac{k}{m}\binom{s}{k}[&B(k,s-m-k+1)\\ &-B(k,n-k+1)],\qquad n\le s\le n-m. \end{aligned}\tag{2.4}$$

and

$$\begin{aligned} EX^2_{k:n}=\frac{k}{m^2}\binom{n}{k}[&B(k,n-k-2m+1)-2B(k,n-k-m+1)\\ &+B(k,n-k+1)], \end{aligned}\tag{2.5}$$

where $B(a,b)$ is the Beta function, and so (2.2) holds true.

Conversely, assume that (2.2) holds. Applying (2.3) we see that (2.2) can be written as

$$\int_0^1\left(F^{-1}(t)-\frac{(1-t)^{-m}-1}{m}\right)^2 t^{k-1}(1-t)^{n-k}\,dt=0,$$

which implies that $F(x)$ is given by (2.1).

When $m=-1$ Theorem 1 reduces to the following characterization of the uniform distribution.

Corollary 1 (cf. [8]). *Let $EX^2_{k:n}<\infty$ for some pair (k,n). Then*

$$EX^2_{k:n}-\frac{2k}{n+1}EX_{k+1:n+1}+\frac{k(k+1)}{(n+1)(n+2)}=0 \tag{2.2$'$}$$

iff $F(x)=x$ on $(0,1)$.

In proving (2.2′) we use the equality

$$(n-k)EX_{k:n}+kEX_{k+1:n}=nEX_{k:n-1}$$

with $k=1$ (cf. [2]).

Using (2.4) and (2.5) we obtain the following characterizing conditions.

Theorem 1′. *Under the assumptions of Theorem 1 the distribution function F is given by (2.1) iff*

$$EX_{k:s}=\frac{1}{m}\left[\frac{s_{[k]}}{(s-m)_{[k]}}-1\right],\qquad s=n,\ n-m,$$

$$EX^2_{k:n}=\frac{1}{m^2}\left[\frac{n_{[k]}}{(n-2m)_{[k]}}-2\frac{n_{[k]}}{(n-m)_{[k]}}+1\right].$$

The same results can be derived for the distribution

$$F(x) = 1 - (1 + mx)^{-1/m}, \qquad x > 0, \tag{2.6}$$

where m is a positive integer (cf. [1]). In this case (2.2) holds with $n - 2m - k > 0$.

3. Characterizations in terms of moments of k-record values. Let $\{X_n,\ n \ge 1\}$ be a sequence of i.i.d. random variables with a common distribution function F. For a fixed integer $k \ge 1$ we define (cf. [3]) the sequence of *k-record values* as follows:

$$Y_n^{(k)} = X_{L_k(n):L_k(n)+k-1}, \qquad n \in \mathbb{N},$$

where the sequence $\{L_k(n),\ n \ge 1\}$ of *k-record times* is given by $L_k(1) = 1$, $L_k(n+1) = \min\{j : j > L_k(n),\ X_{j:j+k-1} > X_{L_k(n):L_k(n)+k-1}\}$, $n \in \mathbb{N}$.

A characterization of F in (2.1) is contained in the following theorem.

THEOREM 2. *Let $\{X_n,\ n \ge 1\}$ be a sequence of i.i.d. random variables with a common distribution function F such that $E|\min(X_1, \ldots, X_k)|^{2p} < \infty$ for a fixed $k \ge 1$ and some $p > 1$. Then F is given by* (2.1) *iff*

$$E(Y_n^{(k)})^2 - \frac{2}{m}\left[\left(\frac{k}{k-m}\right)^n EY_n^{(k-m)} - EY_n^{(k)}\right] + \frac{1}{m^2}\left(1 - 2\left(\frac{k}{k-m}\right)^n + \left(\frac{k}{k-2m}\right)^n\right) = 0 \tag{3.1}$$

for $n = 1, 2, \ldots$

Proof. Suppose that F is given by (2.1). Then we have

$$\begin{aligned} EY_n^{(k)} &= \frac{k^n}{(n-1)!}\int_0^1 F^{-1}(t)[-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt \\ &= \frac{1}{m}\left[\left(\frac{k}{k-m}\right)^m - 1\right] \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} E(Y_n^{(k)})^2 &= \frac{k^n}{(n-1)!}\int_0^1 (F^{-1}(t))^2[-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt \\ &= \frac{k^n}{(n-1)!}\int_0^1 [(1-t)^{-m} - 1]^2[-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt \\ &= \frac{k^n}{(n-1)!m^2}\left[\frac{\Gamma(n)}{(k-2m)^n} - \frac{2\Gamma(n)}{(k-m)^n} + \frac{1}{k^n}\Gamma(n)\right] \\ &= \frac{1}{m^2}\left[\left(\frac{k}{k-2m}\right)^n - 2\left(\frac{k}{k-m}\right)^n + 1\right], \end{aligned} \tag{3.3}$$

$$\left(\frac{k}{k-m}\right)^n EY_n^{(k-m)} = \frac{k^n}{m}\left(\frac{1}{(k-2m)^n} - \frac{1}{(k-m)^n}\right), \tag{3.4}$$

which establishes (3.1).

Conversely, assuming that (3.1) is satisfied we see that

$$\int_0^1 \left[F^{-1}(t) - \frac{(1-t)^{-m}-1}{m}\right]^2 [-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt = 0.$$

Since the sequence $\{(-\log(1-t))^n,\ n \ge 1\}$, is complete in $L(0,1)$ (cf. [7]) we conclude that $F(x)$ is of the form (2.1).

Theorem 2′. *Under the assumptions of Theorem* 2 *the distribution function* $F(x)$ *is given by* (2.1) *for* $k > 2m$ *iff the following relations hold*:

$$EY_n^{(s)} = \frac{1}{m}\left[\left(\frac{s}{s-m}\right)^n - 1\right], \qquad s = k, k-m,$$
$$E(Y_n^{(k)})^2 = \frac{1}{m^2}\left[\left(\frac{k}{k-2m}\right)^n - 2\left(\frac{k}{k-m}\right)^n + 1\right]$$

for $n = 1, 2, \ldots$

Putting $m = -1$ we obtain the characterization results given in [6]. For $n = 1$, $m = -1$ we obtain the result of Too and Lin [8].

Similar considerations lead to the analogous characterizations for the distribution (2.6). Namely, we have the following results.

Theorem 3. *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of i.i.d. random variables with a common distribution function* F *such that* $E|\min(X_1, \ldots, X_k)|^{2p} < \infty$ *for a fixed* $k \ge 1$ *and some* $p > 1$. *Then* $F(x)$ *has the form* (2.6) *iff for* $k - 2m > 0$, *where* m *is a positive integer*,

$$E(Y_n^{(k)})^2 - \frac{2}{m}\left[\left(\frac{k}{k-m}\right)^n EY_n^{(k-m)} - EY_n^{(k-m-1)}\right] + \frac{1}{m^2}\left[1 - 2\left(\frac{k}{k-m}\right)^n + \left(\frac{k}{k-2m}\right)^n\right] = 0$$

for $n = 1, 2, \ldots$

Theorem 3′. *Under the assumptions of Theorem* 3, *the distribution function* F *is given by* (2.6) *iff for* $k - 2m > 0$,

$$EY_n^{(s)} = \frac{1}{m}\left[\left(\frac{s}{s-m}\right)^n - 1\right], \qquad s = k, k-m,$$
$$E(Y_n^{(k)})^2 = \frac{1}{m^2}\left[\left(\frac{k}{k-2m}\right)^n - 2\left(\frac{k}{k-m}\right)^n + 1\right]$$

for $n = 1, 2, \ldots$

Letting $m = 1$ we obtain the following characterization result.

COROLLARY 2. $F(x) = 1 - (1+x)^{-1}$, $x > 0$, *iff*

$$E(Y_n^{(k)})^2 - 2\left[\left(\frac{k}{k-1}\right)^n EY_n^{(k-1)} - EY_n^{(k)}\right] + 1 - 2\left(\frac{k}{k-1}\right)^n + \left(\frac{k}{k-2}\right)^n = 0.$$

4. Characterizations by moments of randomly indexed order statistics. Let $X_{k:N}$ be the kth smallest order statistics of a random sample $(X_1, \ldots, X_N)$ with common distribution function F, where N is a random variable independent of $\{X_n,\ n \ge 1\}$ with a probability function $p(k) = P[N = k]$, $k = 1, 2, \ldots$ We write $P_k = P[N \ge k]$.

In this section we give a characterization for the distribution (2.1) in terms of moments of order statistics with a random index.

THEOREM 4. *With the above notation, suppose that* $E(X_{k:N}^2 \mid N \ge k) < \infty$ *for some* k *and a given probability function* $p(\cdot)$ *of* N. *Then*

$$\begin{aligned}(4.1)\qquad & E(X_{k:N}^2 \mid N \ge k) \\ & - \frac{2}{m}\left[E\left(\frac{N_{[k]}}{(N-m)_{[k]}} X_{k:N-m} \,\middle|\, N \ge k\right) - E(X_{k:N} \mid N \ge k)\right] \\ & + \frac{1}{m^2}\left[E\left(\frac{N_{[k]}}{(N-2m)_{[k]}} \,\middle|\, N \ge k\right) - 2E\left(\frac{N_{[k]}}{(N-m)_{[k]}} \,\middle|\, N \ge k\right) + 1\right] = 0,\end{aligned}$$

where $N_{[k]} = N(N-1)\ldots(N-k+1)$, *iff* F *is given by* (2.1).

Proof. Let $F^{-1}(t) = \inf\{x : F(x) \ge t,\ t \in (0,1)\}$. We have

$$(4.2)\qquad EX_{k:n}^l = \frac{n!}{(k-1)!(n-k)!}\int_0^1 (F^{-1}(t))^l t^{k-1}(1-t)^{n-k}\,dt, \qquad l \ge 1.$$

Suppose that F is given by (2.1). Since N is independent of $\{X_n,\ n \ge 1\}$, from (2.3) we have

$$\begin{aligned}(4.3)\qquad E(X_{k:N} \mid N \ge k) &= \frac{1}{mP_k}\sum_{n=k}^{\infty}\frac{n!}{(k-1)!(n-k)!} \\ &\quad \times \int_0^1 [(1-t)^{-m} - 1]t^{k-1}(1-t)^{n-k}\,dt\, P[N = n] \\ &= \frac{1}{m}E\left(\left(\frac{N_{[k]}}{(N-m)_{[k]}} \,\middle|\, N \ge k\right) - 1\right),\end{aligned}$$

$$
\begin{aligned}
(4.4)\qquad & E\bigg(\frac{N_{[k]}}{(N-m)_{[k]}}X_{k:N-m}\,\bigg|\, N\ge k\bigg)\\
&= \frac{1}{mP_k}\sum_{n=k}^{\infty}\frac{(n-m)!}{(k-1)!(n-k-m)!}\\
&\quad\times\frac{n(n-1)\dots(n-k+1)}{(n-m)(n-m-1)\dots(n-m-k+1)}\\
&\quad\times\int_0^1((1-t)^{-m}-1)t^{k-1}(1-t)^{n-m-k}\,dt\,P[N=n]\\
&= \frac{1}{m}\bigg[E\bigg(\frac{N_{[k]}}{(N-2m)_{[k]}}\,\bigg|\,N\ge k\bigg)-E\bigg(\frac{N_{[k]}}{(N-m)_{[k]}}\,\bigg|\,N\ge k\bigg)\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.5)\qquad & E(X_{k:N}^2\,|\,N\ge k)\\
&= \frac{1}{m^2}\bigg[E\bigg(\frac{N_{[k]}}{(N-2m)_{[k]}}\,\bigg|\,N\ge k\bigg)-2E\bigg(\frac{N_{[k]}}{(N-m)_{[k]}}\,\bigg|\,N\ge k\bigg)+1\bigg].
\end{aligned}
$$

We see that (4.1) holds true.

Conversely, assume that (4.1) holds. It can be written as

$$
\sum_{n=k}^{\infty}\int_0^1\bigg(F^{-1}(t)-\frac{(1-t)^{-m}-1}{m}\bigg)^2 t^{k-1}(1-t)^{n-k}\,dt\,P[N=n]=0,
$$

which implies that F is given by (2.1).

Using (4.3)–(4.5) we have the following characterization conditions in terms of conditional moments of order statistics.

THEOREM 4′. *Under the assumptions of Theorem* 4 *the distribution function* F *is given by* (2.1) *iff*

$$
E(X_{k:N}\,|\,N\ge k)=\frac{1}{m}\bigg[E\bigg(\frac{N_{[k]}}{(N-m)_{[k]}}\,\bigg|\,N\ge k\bigg)-1\bigg],
$$

$$
\begin{aligned}
&E\bigg(\frac{N_{[k]}}{(N-m)_{[k]}}X_{k:N-m}\,\bigg|\,N\ge k\bigg)\\
&\qquad=\frac{1}{m}\bigg[E\bigg(\frac{N_{[k]}}{(N-2m)_{[k]}}\,\bigg|\,N\ge k\bigg)-E\bigg(\frac{N_{[k]}}{(N-m)_{[k]}}\bigg)\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
E(X_{k:N}^2\,|\,N\ge k)=\frac{1}{m^2}\bigg[&E\bigg(\frac{N_{[k]}}{(N-2m)_{[k]}}\,\bigg|\,N\ge k\bigg)\\
&-2E\bigg(\frac{N_{[k]}}{(N-m)_{[k]}}\,\bigg|\,N\ge k\bigg)+1\bigg].
\end{aligned}
$$

COROLLARY 3. *Let N be a random variable with probability function*

$$P[N=n]=\frac{\alpha\theta^n}{n},\qquad n=1,2,\ldots;\ \alpha=-\frac{1}{\ln(1-\theta)},\ \theta\in(0,1). \tag{4.7}$$

Then X has the distribution (2.1) *iff*

$$EX_{1:N}=\frac{1}{m}\left[\theta^m\left(1-\alpha\sum_{n=1}^{-m}\frac{\theta^n}{n}\right)-1\right],$$

$$E\frac{N}{N-m}X_{1:N-m}=\frac{1}{m}\left[\theta^{2m}-\theta^m-\theta^{2m}\alpha\sum_{n=1}^{-2m}\frac{\theta^n}{n}+\theta^m\alpha\sum_{n=1}^{-m}\frac{\theta^n}{n}\right],$$

$$EX_{1:N}^2=\frac{1}{m^2}\left[\theta^{2m}-2\theta^m-\alpha\theta^{2m}\sum_{n=1}^{-2m}\frac{\theta^n}{n}+2\theta^m\alpha\sum_{n=1}^{-m}\frac{\theta^n}{n}+1\right].$$

REMARK. Putting $m=-1$ we obtain a characterization of the uniform distribution in terms of $X_{1:N}$, which after using the equality

$$EX_{1:N}^2-E\frac{N}{N+1}X_{1:N}=\alpha EX_1+\left(1-\frac{1}{\theta}\right)EX_{1:N}$$

leads to the result of [9], i.e.

$$EX_{1:N}^2-2\left[\alpha EX+\left(1-\frac{1}{\theta}\right)EX_{1:N}\right]=-\alpha\left[\frac{3}{2}-\frac{1}{\theta}-\left(1-\frac{1}{\theta}\right)^2\ln(1-\theta)\right].$$

5. Characterizations via moments of randomly indexed record statistics

THEOREM 5. *Let $Y_N^{(k)}$ be the kth record value, where N is a positive integer-valued random variable independent of $\{X_n,\ n\ge 1\}$, and suppose that $E(Y_N^{(k)})^2<\infty$. Then F is given by* (2.1) *iff*

$$E(Y_N^{(k)})^2-\frac{2}{m}\left[E\left(\frac{k}{k-m}\right)^N Y_N^{(k-m)}-EY_N^{(k)}\right]+\frac{1}{m^2}\left[1-2E\left(\frac{k}{k-m}\right)^N+E\left(\frac{k}{k-2m}\right)^N\right]=0. \tag{5.1}$$

P r o o f. Suppose that F is given by (2.1). Since

$$E(Y_n^{(k)})^l=\frac{k^n}{(n-1)!}\int_0^1(F^{-1}(t))^l[-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt$$

and N and $\{X_n,\ n\ge 1\}$ are independent, it follows that

$$EY_N^{(k)}=\frac{1}{m}E\left[\left(\frac{k}{k-m}\right)^N-1\right],$$

$$E\left(\frac{k}{k-m}\right)^N Y_N^{(k-m)}=\frac{1}{m}\left[E\left(\frac{k}{k-2m}\right)^N-E\left(\frac{k}{k-m}\right)^N\right],$$

$$E(Y_N^{(k)})^2 = \frac{1}{m^2}\left[E\left(\frac{k}{k-2m}\right)^N - 2E\left(\frac{k}{k-m}\right)^N + 1\right],$$

which establishes (5.1).

Assuming now that (2.1) is satisfied we see that

$$\sum_{n=1}^{\infty} \frac{k^n}{(n-1)!} \int_0^1 \left(F^{-1}(t) - \frac{(1-t)^{-m}-1}{m}\right)^2 \times [-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt\,P[N=n] = 0.$$

Since the sequence $\{(-\log(1-t))^n,\ n \ge 1\}$ is complete in $L(0,1)$ (cf. [7]) it follows that $F(x)$ has the form (2.1).

Putting $m = -1$ we have the following characterization.

Corollary 4. $F(x) = x$, $x \in (0,1)$, *iff*

$$E(Y_N^{(k)})^2 + 2\left(E\left(\frac{k}{k+1}\right)^N Y_N^{(k+1)} - EY_N^{(k)}\right) + E\left(\frac{k}{k+2}\right)^N - 2E\left(\frac{k}{k+1}\right)^N + 1 = 0$$

(cf. [5] with $m = 1$).

Corollary 5. *Let N be a random variable with the probability function* (4.7). *Then X has the distribution* (2.1) *iff*

$$E(Y_N^{(k)})^2 - \frac{2}{m}\left[E\left(\frac{k}{k-m}\right)^N Y_N^{(k-m)} - EY_N^{(k)}\right] + \frac{1}{m^2}\left[1 + 2\alpha\log\frac{k(1-\theta)-m}{k-m} - \alpha\log\frac{k(1-\theta)-2m}{k-2m}\right].$$

Remark. $F(x) = x$, $x \in (0,1)$, iff

$$E(Y_N^{(k)})^2 + 2\left(E\left(\frac{k}{k+1}\right)^N Y_N^{(k+1)} - EY_N^{(k)}\right) + 1 + 2\alpha\log\frac{k(1-\theta)+1}{k+1} - \alpha\log\frac{k(1-\theta)+2}{k+2} = 0.$$

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