APPLICATIONES MATHEMATICAE
26,4 (1999), pp. 437–455

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REFINED RATES OF BIAS CONVERGENCE FOR GENERALIZED L-STATISTICS IN THE I.I.D. CASE

Abstract. Using tools of approximation theory, we evaluate rates of bias convergence for sequences of generalized L-statistics based on i.i.d. samples under mild smoothness conditions on the weight function and simple moment conditions on the score function. Apart from standard methods of weighting, we introduce and analyze L-statistics with possibly random coefficients defined by means of positive linear functionals acting on the weight function.

1. Introduction. We consider a sequence X_n, $n \in \mathbb{N}$, of independent identically distributed (i.i.d.) random variables with a common distribution function F. For a sample of size n, the sequence of order statistics is denoted by $X_{1:n}, \ldots, X_{n:n}$. A sequence L_n, $n \in \mathbb{N}$, of (generalized) L-statistics is defined by

$$L_n = \sum_{i=1}^{n} c_{i,n} g(X_{i:n}), \qquad n \in \mathbb{N}, \tag{1.1}$$

where real coefficients $c_{i,n}$, $1 \le i \le n < \infty$, and a measurable score function g are given. L-statistics have numerous applications in statistical inference (see, e.g., Balakrishnan and Cohen (1991), and David (1981)). The expectation of (1.1) is given by

$$\mathrm{E}_F L_n = \sum_{i=1}^{n} n c_{i,n} \int_0^1 g(F^{-1}(y)) p_{n-1,i-1}(y)\, dy, \tag{1.2}$$

1991 *Mathematics Subject Classification*: 41A35, 62G20, 62G30.
Key words and phrases: generalized L-statistic, random weighting, bias, rate of convergence, modulus of smoothness, K-functional, positive linear operator, Bernstein operator, Kantorovich operator, Bernstein–Durrmeyer operator.

where

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad i = 0, 1, \ldots, n, \tag{1.3}$$

form the Bernstein basis of the linear space of polynomials on $[0,1]$ of degree at most n for $n \in \mathbb{N} \cup \{0\}$. When asymptotic properties of L-statistics are studied, it is usually assumed that the coefficients are defined by means of some weight function $f : [0,1] \to \mathbb{R}$ in two ways, resulting in the following definitions of L-estimate sequences:

$$\begin{aligned} L_n(f,g) &= \sum_{i=1}^{n} l_{i,n} g(X_{i:n}) \\ &= n^{-1} \sum_{i=1}^{n} f\left(\frac{i-1}{n-1}\right) g(X_{i:n}), \qquad n \in \mathbb{N} \setminus \{1\}, \end{aligned} \tag{1.4}$$

(cf., e.g., Chernoff, Gastwirth and Johns (1967), Shorack (1969, 1972), Mason (1981)), and

$$\begin{aligned} K_n(f,g) &= \sum_{i=1}^{n} k_{i,n} g(X_{i:n}) \\ &= \sum_{i=1}^{n} \Big[\int_{(i-1)/n}^{i/n} f(t)\, dt \Big] g(X_{i:n}), \qquad n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

(cf., e.g., Boos (1979), van Zwet (1980), Mason and Shorack (1992)). Under some regularity conditions on the weight, score and distribution functions f, g and F, respectively, both (1.4) and (1.5) tend to

$$\mu(f,g,F) = \mathrm{E}_F[g(X_1) f(F(X_1))] = \int_0^1 g(F^{-1}(y)) f(y)\, dy \tag{1.6}$$

in various modes of convergence. Briefly mentioning some results, we do not pretend to present a comprehensive list of relevant references. Strong laws of large numbers for (1.4) and (1.5) were obtained in Wellner (1977a), Sen (1978), van Zwet (1980), Mason (1982), and Norvaiša (1994). Asymptotic normality was studied by Chernoff, Gastwirth and Johns (1967), Shorack (1969, 1972), Stigler (1974), Sen (1978), Boos (1979), Mason (1981), Helmers and Ruymgaarts (1988), and Mason and Shorack (1992). Laws of the iterated logarithms were established in Wellner (1977b), Boos (1979), Lea and Puri (1988), and Norvaiša and Zitikis (1991), and Berry–Essen bounds were obtained in Helmers, Janssen and Serfling (1990), and Bogdan (1994).

The objective of this paper is to analyze rates of convergence to (1.6) for sequences of expected L-statistics (1.2) defined by (1.4), (1.5) and other formulae under mild conditions on f, g and F. The problem was investigated by Stigler (1974), Mason (1981) and Xiang (1995). Xiang (1995) proposed a method of bias reduction for L-statistics with weight functions having high order derivatives. We impose no smoothness conditions on the score and distribution functions, assuming merely the finiteness of the moments $\mathrm{E}_F|g(X_1)|^q$ for some $q \geq 1$. Another natural requirement that will be tacitly assumed throughout the paper and makes possible defining (1.6) is

$$\mathrm{E}_F|g(X_1)f(F(X_1))| = \int_0^1 |g(F^{-1}(y))f(y)|\,dy < \infty.$$

The only regularity conditions concern the weight function f. In contrast to known results (see, e.g., the above cited articles), where smoothness of functions is mainly expressed by existence and properties of their derivatives, we shall use tools of approximation theory, including moduli of smoothness and K-functionals. In many approximation problems, measuring the smoothness by differentiability is too crude, and the moduli enable us to do that more subtly for larger classes of possibly nondifferentiable functions.

For $f \in C[0,1]$, we define the first (order) modulus of continuity and the second (order) modulus of continuity (smoothness) as

$$\begin{aligned}(1.7)\qquad \omega_1(f,h) &= \sup\{|f(u)-f(v)| :\ u,v\in[0,1],\ |u-v|\le h\},\\ \omega_2(f,h) &= \sup\biggl\{\biggl|f(u)-2f\biggl(\frac{u+v}{2}\biggr)+f(v)\biggr| :\\ &\qquad\qquad u,v\in[0,1],\ |u-v|\le 2h\biggr\},\end{aligned}$$

$h > 0$, respectively. In general, for $f \in L_p[a,b]$, $1 \le p \le \infty$, we define the rth forward difference

$$\Delta_h^r(f,x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh),$$

and $A_{rh} = [a, b-rh]$. The rth modulus of smoothness for $f \in L_p[a,b]$, $1 \le p < \infty$, and $f \in C[a,b]$, $p = \infty$, is defined by

$$\omega_r(f,t)_p = \sup_{0<h\le t} \|\Delta_h^r(f,\cdot)\|_{p,A_{rh}}, \qquad t \ge 0, \tag{1.8}$$

(see DeVore–Lorentz (1993, pp. 40–46) and Schumaker (1981, pp. 53–55)). The latter index of the norm describes the domain of each element of L_p-space and it will be dropped if it coincides with the unit interval. One can check that $\omega_r(f,t)_p$ is a finite, continuous and increasing function of t, with $\omega_r(f,0)_p = 0$ and $\omega_r(f,t)_p \to 0$ as $t \to 0$. There are also some modifications of definitions (1.7)–(1.8) that will be introduced in the sequel. However, these modified moduli of smoothness share the nice properties of the above ones.

Loosely speaking, K-functionals enable us to describe parametrically the accuracy of approximating a function of a space by elements of a subspace. Below we define the Ditzian–Totik version of the K-functional, and others will be presented later. Define the second order symmetric difference as

$$\widetilde{\Delta}_s^2(f,x) = \begin{cases} f(x+s) - 2f(x) + f(x-s) & \text{if } [x-s, x+s] \subset [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\widetilde{\Delta}^2_{h\varphi(\cdot)}(f,\cdot)$ for $h > 0$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, and

$$W^2_\infty(\varphi) = \{g \in C[0,1] : g' \text{ is absolutely continuous on } [0,1] \text{ and } \|\varphi^2 g''\|_\infty < \infty\}.$$

We define the *Ditzian–Totik K-functional* by

$$K(f,t) = K(f,t;C,W^2_\infty(\varphi)) = \inf_{g\in W^2_\infty(\varphi)} \{\|f-g\|_\infty + t\|\varphi^2 g''\|_\infty\} \tag{1.9}$$

and the *Ditzian–Totik modulus of smoothness* by

$$\omega_2^\varphi(f,t) = \sup_{0\le h\le t} \|\widetilde{\Delta}_{h\varphi(\cdot)}(f,\cdot)\|_\infty \tag{1.10}$$

(see DeVore and Lorentz (1993, p. 322)). The latter is a representative of weighted moduli of smoothness, with less emphasis laid on the smoothness at the borders of the domain.

In Section 2 we describe rates of convergence to (1.6) for expectations of (1.4) and (1.5) in terms of moduli of smoothness and K-functionals, making use of their mutual relations to the Bernstein and Kantorovich operators, respectively. In Section 3, we also present nonstandard methods of constructing L-statistics with possibly randomized coefficients determined by a given weight function. Using a general notion of positive linear functionals, we show that the nonstandard L-statistics tend to (1.6) and evaluate the rates of their bias decrease. The rates will be specified more precisely for L-statistics related to Bernstein–Durrmeyer, Mache and Stancu operators. It is worth pointing out that the rates presented here are optimal for the wide classes we study and best constants of approximation are also given in some cases. In Section 4 we refer to saturation theorems that indicate classes of weight functions generating L-statistics with faster vanishing bias.

2. L-Statistics with standard weights. Let

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{2.1}$$

stand for the nth Bernstein operator of a function $f \in C[0,1]$. In Theorems 1–3 we apply approximation properties of the operator to estimate rates of bias convergence of (1.4). In Theorem 1, we describe a relation between the operator and the L-statistic, and evaluate the bias in terms of the modulus of continuity of the derivative of the weight function and the first absolute moment of the score function. Theorem 2 contains analogous results when the derivative of f is replaced by the function itself. In Theorem 3, we apply the notions of K-functional and weighted modulus of smoothness.

THEOREM 1. *For an i.i.d. sequence X_i, $i \in \mathbb{N}$, with a common distribution function F and* $\mathrm{E}|g(X_1)| < \infty$, *and the L-statistic defined by* (1.4), *we have*

$$\mathrm{E}_F L_n(f,g) = \int_0^1 g(F^{-1}(y))(B_{n-1}f)(y)\,dy. \tag{2.2}$$

If $f \in C^1[0,1]$, then for $n \in \mathbb{N}\setminus\{1\}$,

$$|\mathrm{E}_F L_n(f,g) - \mu(f,g,F)| \le \frac{25}{32}(n-1)^{-1/2}\omega_1\Big(f', \frac{1}{4}(n-1)^{-1/2}\Big)\mathrm{E}_F|g(X_1)|.$$

Proof. Combining (1.2), (1.4) and (2.1), we obtain

$$\begin{aligned}\mathrm{E}_F L_n(f,g) &= \mathrm{E}_F n^{-1}\sum_{i=1}^n f\Big(\frac{i-1}{n-1}\Big)g(X_{i:n})\\ &= \int_0^1 g(F^{-1}(y))\sum_{i=1}^n f\Big(\frac{i-1}{n-1}\Big)p_{n-1,i-1}(y)\,dy\\ &= \int_0^1 g(F^{-1}(y))(B_{n-1}f)(y)\,dy,\end{aligned}$$

which proves (2.2). Therefore, applying Anastassiou (1993, Corollary 7.3.4, p. 230), we have, for $n \in \mathbb{N}\setminus\{1\}$,

$$\begin{aligned}|\mathrm{E}_F L_n(f,g) - \mu(f,g,F)| &= \Big|\int_0^1 g(F^{-1}(y))[(B_{n-1}f)(y) - f(y)]\,dy\Big|\\ &\le \|B_{n-1}f - f\|_\infty \int_0^1 |g(F^{-1}(y))|\,dy\\ &\le \frac{25}{32}(n-1)^{-1/2}\omega_1\Big(f', \frac{1}{4}(n-1)^{-1/2}\Big)\mathrm{E}_F|g(X_1)|. \quad\blacksquare\end{aligned}$$

THEOREM 2. *Under the assumptions and notations of Theorem* 1, *with $f \in C[0,1]$, for all $n \in \mathbb{N}\setminus\{1\}$ we have*

$$
\begin{aligned}
(2.3)\qquad |\mathrm{E}_F L_n(f,g)-\mu(f,g,F)| &\le \|B_{n-1}f-f\|_\infty \mathrm{E}_F|g(X_1)| \\
&\le \left\{\begin{aligned} &\frac{4306+837\sqrt{6}}{5832}\omega_1\Big(f,\frac{1}{\sqrt{n-1}}\Big) \\ &\frac{35}{32}\omega_2\Big(f,\frac{1}{\sqrt{n-1}}\Big)\end{aligned}\right\}\mathrm{E}_F|g(X_1)|.
\end{aligned}
$$

Moreover, the first of the two estimates is optimal.

Proof. See Sikkema (1961) for the best constant

$$\frac{4306+837\sqrt{6}}{5832} < 1.09$$

and Paltanea (1995) for the constant $35/32 < 1.094$, which is not optimal. ∎

Remark 1. We have the following approximations of $f \in C[0,1]$ by its Bernstein operators, expressed in terms of the Ditzian–Totik K-functional and modulus of smoothness (cf. (1.9) and (1.10)):

$$
\begin{aligned}
&\|B_n f - f\|_\infty \le 2K(f, n^{-1}),\\
&\|B_n f - f\|_\infty \le C\omega_2^\varphi(f, n^{-1/2}),
\end{aligned}
$$

respectively, where $n \in \mathbb{N}$, and $C > 0$ is an absolute constant (see DeVore and Lorentz (1993, pp. 323–325)).

Direct application of this remark yields

Theorem 3. *With notations and assumptions as in Theorem* 2, *for all* $n \in \mathbb{N}\setminus\{1\}$ *we have*

$$|\mathrm{E}_F L_n(f,g)-\mu(f,g,F)| \le \left\{\begin{matrix} 2K(f,(n-1)^{-1}) \\ C\omega_2^\varphi(f,(n-1)^{-1/2})\end{matrix}\right\}\mathrm{E}_F|g(X_1)|.$$

Remark 2. Applying (1.3), we introduce the Kantorovich operators

$$(2.4)\qquad (K_n f)(x) = (n+1)\sum_{k=0}^{n}\Big(\int\limits_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt\Big)p_{n,k}(x), \qquad n \in \mathbb{N}\cup\{0\},$$

for either $f \in L_p[0,1]$ or $f \in C[0,1]$ in the cases $1 \le p < \infty$ and $p = \infty$, respectively. We also define the Gonska–Zhou version of the K-functional:

$$(2.5)\qquad K^*(f,t)_p = \inf\{\|f-g\|_p + t^2\|(\varphi^2 g')'\|_p : g \in C^2[0,1]\},$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. We shall need the following three theorems:

THEOREM 4 (Gonska and X.-L. Zhou (1995)). *There exists* $C > 0$ *such that*

$$C^{-1}K^*(f, n^{-1/2})_p \le \|K_n f - f\|_p \le CK^*(f, n^{-1/2})_p, \quad 1 \le p \le \infty.$$

THEOREM 5 (Gonska and X.-L. Zhou (1995)). *We have*

$$K^*(f,t)_p \sim \omega_2^{\varphi}(f,t)_p + \omega_1(f,t^2)_p, \quad 1 \le p \le \infty.$$

THEOREM 6 (Gonska and D.-X. Zhou (1995)). *Let* $f \in C[0,1]$ *and* $n \in \mathbb{N}$. *Then*

$$\|K_n f - f\|_\infty \le \frac{M}{n}\Big[\int_{n^{-1/2}}^{1/2} \omega_2^{\varphi}(f,t)_\infty t^{-3}\,dt + E_0(f)_\infty\Big],$$

where $M > 0$ *is independent of* f *and* n, *and*

$$E_0(f)_\infty = \inf_{c\in\mathbb{R}} \|f - c\|_\infty. \tag{2.6}$$

We are now in a position to analyze L-statistics (1.5) with coefficients defined by integrals.

THEOREM 7. *Let* X_i, $i \in \mathbb{N}$, *be an i.i.d. sequence of random variables. Under the assumptions and notation of Remark* 2, *assume also that* $\mathrm{E}|g(X_1)|^q < \infty$ *for* $q = p/(p-1)$ *and* $p > 1$. *Then*

$$\mathrm{E}_F K_n(f,g) = \int_0^1 g(F^{-1}(y))(K_{n-1}f)(y)\,dy$$

and

$$|\mathrm{E}_F K_n(f,g) - \mu(f,g,F)| \le \|K_{n-1}f - f\|_p (\mathrm{E}_F|g(X_1)|^q)^{1/q}, \quad n \in \mathbb{N}\setminus\{1\}.$$

Proof. By (1.2), (1.6) and (2.4)

$$\begin{aligned}\mathrm{E}_F K_n(f,g) &= \mathrm{E}_F \sum_{i=1}^n k_{i,n} g(X_{i:n})\\ &= \int_0^1 g(F^{-1}(y))n\Big[\sum_{i=1}^n \int_{(i-1)/n}^{i/n} f(t)\,dt\Big]p_{n-1,i-1}(y)\,dy\\ &= \int_0^1 g(F^{-1}(y))(K_{n-1}f)(y)\,dy.\end{aligned}$$

Accordingly,

$$\mathrm{E}_F K_n(f,g) - \mu(f,g,F) = \int_0^1 g(F^{-1}(y))[(K_{n-1}f)(y) - f(y)]\,dy,$$

and, by the Hölder inequality,

$$
\begin{aligned}
&|\mathrm{E}_F K_n(f,g) - \mu(f,g,F)| \\
&\quad \le \Big(\int_0^1 |(K_{n-1}f)(y) - f(y)|^p\,dy\Big)^{1/p} \Big(\int_0^1 |g(F^{-1}(y))|^q\,dy\Big)^{1/q} \\
&\quad = \|K_{n-1}f - f\|_p (\mathrm{E}_F |g(X_1)|^q)^{1/q}. \ \blacksquare
\end{aligned}
$$

By Theorems 4–7, we obtain

Theorem 8. *Under the assumptions and notations of Theorem* 7, *we have*

$$
\begin{aligned}
&|\mathrm{E}_F K_n(f,g) - \mu(f,g,F)| \\
&\quad \le \left\{ \begin{matrix} CK^*(f,(n-1)^{-1/2})_p \\ C^*[\omega_2^\varphi(f,(n-1)^{1/2})_p + \omega_1(f,(n-1)^{-1})_p] \end{matrix} \right\} (\mathrm{E}_F |g(X_1)|^q)^{1/q},
\end{aligned}
$$

for $1 < p < \infty$ *and universal constants* C, $C^* > 0$. *Moreover*, *if* $p = \infty$, *then*

$$
\begin{aligned}
&|\mathrm{E}_F K_n(f,g) - \mu(f,g,F)| \\
&\quad \le \|K_{n-1}f - f\|_\infty \mathrm{E}_F |g(X_1)| \\
&\quad \le \left\{ \begin{matrix} CK^*(f,(n-1)^{-1/2})_\infty \\[2mm] C^*[\omega_2^\varphi(f,(n-1)^{1/2})_\infty + \omega_1(f,(n-1)^{-1})_\infty] \\[2mm] \dfrac{M}{n-1}\Big[\displaystyle\int_{(n-1)^{-1/2}}^{1/2} \omega_2^\varphi(f,t)_\infty t^{-3}\,dt + E_0(f)_\infty\Big] \end{matrix} \right\} \mathrm{E}_F |g(X_1)|,
\end{aligned}
$$

where $M > 0$ *and* $E_0(f)_\infty$ *is defined in* (2.6).

3. L-Statistics with nonstandard weights

Remark 3. For $f \in L_1[0,1]$ we define the *Bernstein–Durrmeyer operators* as

$$
(D_n f)(x) = (n+1)\sum_{k=0}^{n} \Big(\int_0^1 f(t) p_{n,k}(t)\,dt\Big) p_{n,k}(x) \tag{3.1}
$$

for $x \in [0,1]$ and $n \in \mathbb{N} \cup \{0\}$. Then there exists a universal $C > 0$ such that

$$
C^{-1} K^*(f, n^{-1/2})_p \le \|D_n f - f\|_p \le CK^*(f, n^{-1/2})_p \tag{3.2}
$$

for any $1 \le p \le \infty$ (see Gonska and D.-X. Zhou (1995)).

Define a sequence of L-statistics by

$$M_n(f,g)=\sum_{i=1}^{n} m_{i,n} g(X_{i:n})=\sum_{i=1}^{n}\Big(\int_0^1 p_{n-1,i-1}(t)f(t)\,dt\Big)g(X_{i:n}). \tag{3.3}$$

This is a modification of (1.5) that consists in replacing the step weight function $\mathbf{1}_{((i-1)/n,i/n]}$ in the integral representation of the coefficients $k_{i,n}$, $1\le i\le n<\infty$, by smooth ones $p_{i-1,n-1}$. Under the above assumptions and notation, we have

THEOREM 9. *For an i.i.d. sequence of random variables X_i, $i\in\mathbb{N}$, with a common distribution function F and a weight function $f\in L_p[0,1]$, $1<p\le\infty$, assume that $\mathrm{E}_F|g(X_1)|^q<\infty$ for $q=p/(p-1)$ and $p<\infty$, and $\mathrm{E}_F|g(X_1)|<\infty$ for $p=\infty$. Then*

$$\mathrm{E}_F M_n(f,g)=\int_0^1 g(F^{-1}(y))(D_{n-1}f)(y)\,dy,\qquad n\in\mathbb{N}. \tag{3.4}$$

If $1<p<\infty$, *then*

$$\begin{aligned}|\mathrm{E}_F M_n(f,g)-\mu(f,g,F)| &\le \|D_{n-1}f-f\|_p(\mathrm{E}_F|g(X_1)|^q)^{1/q}\\ &\le CK^*(f,(n-1)^{-1/2})_p(\mathrm{E}_F|g(X_1)|^q)^{1/q}.\end{aligned} \tag{3.5}$$

If $p=\infty$, *then*

$$|\mathrm{E}_F M_n(f,g)-\mu(f,g,F)| \le \left\{\begin{matrix} CK^*(f,(n-1)^{-1/2})_\infty \\ \dfrac{M}{n-1}\Big[\displaystyle\int_{(n-1)^{-1/2}}^{1/2}\omega_2^\varphi(f,t)_\infty t^{-3}\,dt+E_0(f)_\infty\Big]\end{matrix}\right\}\mathrm{E}_F|g(X_1)|. \tag{3.6}$$

Proof. Formula (3.4) can be immediately deduced from (1.2), (3.1), (3.3). Thus

$$\mathrm{E}_F M_n(f,g)-\mu(f,g,F)=\int_0^1 g(F^{-1}(y))[(D_{n-1}f)(y)-f(y)]\,dy.$$

Applying the Hölder inequality, and then using (3.2), we obtain (3.5). In a similar way, we can conclude the former relation in (3.6). The latter is an implication of Theorem 3 in Gonska and X.-D. Zhou (1995) that asserts

$$\|D_nf-f\|_\infty\le\frac{M}{n}\Big[\int_{n^{-1/2}}^{1/2}\omega_2^\varphi(f,t)_\infty t^{-3}\,dt+E_0(f)_\infty\Big]. \ \blacksquare$$

REMARK 4. One can consider $f\in L_p[0,1]$, $1\le p\le\infty$, such that either

$$\omega_1(f,h)_p=\mathcal{O}(h^\alpha),\qquad 0<\alpha\le1,$$

or

$$\omega_2^{\varphi}(f,h)_p = \mathcal{O}(h^{\alpha}), \qquad 0<\alpha\le 2,$$

(i.e. $K(f,h)_p = \mathcal{O}(h^{\beta})$, $0<\beta\le 1$, cf. (1.9) and (1.10)). We could also have

$$K^*(f,h)_p = \mathcal{O}(h^{\alpha}), \qquad 0<\alpha\le 2,\ 1<p\le\infty,$$

(see (2.5)). All the above are various forms of Lipschitz type conditions for f, and can simplify previous results when applicable. From DeVore and Lorentz (1993, p. 327) we have

$$\|D_n f - f\|_\infty \le 3\omega_1(f,(3/n)^{1/2}), \qquad n\in\mathbb{N},\ f\in C[0,1],$$

where ω_1 is the first (ordinary) modulus of continuity (see (1.7)). Therefore, following the assumptions and notations of Theorem 9, we obtain

$$\begin{aligned}|\mathrm{E}_F M_n(f,g) - \mu(f,g,F)| &\le \|D_{n-1}f - f\|_\infty \mathrm{E}_F|g(X_1)| \\ &\le 3\omega_1(f,(3/n)^{1/2})\mathrm{E}_F|g(X_1)|.\end{aligned} \tag{3.7}$$

Observe that the coefficients of the L-statistic in (3.3) can be expressed as

$$m_{i,n} = \mathrm{E}f(U_{i:n}), \qquad 1\le i\le n<\infty$$

(cf. (1.2)), where $U_{i:n}$, $1\le i\le n$, are the order statistics from a standard uniform i.i.d. sample of size n, which can be easily generated. Therefore, replacing the L-statistics by their randomized modifications

$$\widetilde{M}_n(f,g) = \sum_{i=1}^{n} f(U_{i:n})g(X_{i:n}), \qquad n\in\mathbb{N}, \tag{3.8}$$

we preserve all the conclusions of Theorem 9 and formula (3.7). Also, (1.5) may be substituted by randomized counterparts

$$\widetilde{K}_n(f,g) = \sum_{i=1}^{n} f(V_{i,n})g(X_{i:n}), \qquad n\in\mathbb{N}, \tag{3.9}$$

with the same expectations, if $V_{i,n}$, $1\le i\le n<\infty$, are uniformly distributed on $[(i-1)/n, i/n]$. In fact, it simply suffices to put $V_{i,n} = (i-1+U)/n$ for a single random variable uniformly distributed on $[0,1]$. Formulae (3.8) and (3.9) reveal numerous possibilities for nonstandard choice of randomized coefficients with desired expectations. Below we present even more general constructions, based on positive linear operators which generalize the notion of the expectation operator.

Remark 5. Here we refer to Gavrea and Mache (1995). Let $T_{n,k} : C[0,1]\to\mathbb{R}$, $n\in\mathbb{N}$, $k=0,1,\ldots,n$, be positive linear functionals such that $T_{n,k}1 = 1$. Then

$$(A_n f)(x) = \sum_{k=0}^{n} T_{n,k} f\cdot p_{n,k}(x)$$

is a positive linear operator acting on $f \in C[0,1]$. Set

$$\varDelta_n(x) = \sum_{k=0}^{n} T_{n,k}(\cdot - k/n)^2\, p_{n,k}(x).$$

Notice that $\varDelta_n(x) \ge 0$. Write

$$\widetilde{\varDelta}_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), \qquad x \in [h, 1-h],\ 0 < h < 1,$$

and define

$$\omega_2^*(f,t) = \sup_{0<h\le t}\ \sup_{x\in[h,1-h]} |\widetilde{\varDelta}_h^2 f(x)|.$$

We need

THEOREM 10 (Gavrea and Mache (1995)). *Assume that $\varDelta_n(x) \le C/n^{2\beta}$ for some $C > 0$, $1 < \beta < 2$ and all $x \in [0,1]$ (i.e. $\varDelta_n(x) = \mathcal{O}(n^{-2\beta})$). Then*

$$|(A_n f)(x) - f(x)| \le C\{\varDelta_n^{1/2}(x) + [\varDelta_n(x) + n^{-1}x(1-x)]^{\beta/2}\}$$

for all $x \in [0,1]$ iff $\omega_2^*(f,t) = \mathcal{O}(t^\beta)$.

COROLLARY 1. *If $\varDelta_n(x) \le C/n^{2\beta}$ for some $C > 0$, $1 < \beta < 2$ and all $x \in [0,1]$, and $\omega_2^*(f,t) = \mathcal{O}(t^\beta)$ for $f \in C[0,1]$, then*

$$\|A_n f - f\|_\infty \le C\left[\frac{C^{1/2}}{n^\beta} + \left(\frac{C}{n^{2\beta}} + \frac{1}{4n}\right)^{\beta/2}\right], \qquad n \in \mathbb{N}.$$

Since

$$\mathrm{E}_F g(X_{i:n}) = n\int_0^1 g(F^{-1}(y))p_{n-1,i-1}(y)\,dy,$$

$$\mathrm{E}_F \sum_{i=1}^{n} (T_{n-1,i-1}f) g(X_{i:n}) = n\int_0^1 g(F^{-1}(y))(A_{n-1}f)(y)\,dy, \qquad f \in C[0,1],$$

we can generally define L-statistics $T_n(f,g)$, $n \in \mathbb{N} \setminus \{1\}$, with coefficients $t_{i,n} = n^{-1}T_{n-1,i-1}f$, $i = 1, \ldots, n$, which satisfy

$$\mathrm{E}_F T_n(f,g) = \int_0^1 g(F^{-1}(y))(A_{n-1}f)(y)\,dy,$$

and

$$\mathrm{E}_F T_n(f,g) - \mu(f,g,F) = \int_0^1 g(F^{-1}(y))[(A_{n-1}f)(y) - f(y)]\,dy. \tag{3.10}$$

Therefore we have

THEOREM 11. *Let X_i, $i \in \mathbb{N}$, be i.i.d. random variables with a common distribution function F, and $f \in C[0,1]$. Assume that $\mathrm{E}_F|g(X_1)| < \infty$, $\varDelta_n(x) \le Cn^{-2\beta}$ for some $C > 0$, $1 < \beta < 2$, and all $x \in [0,1]$,* *and* $\omega_2^*(f,t) = \mathcal{O}(t^\beta)$, *using the notions and notations of Remark* 5. *Then*

$$\begin{aligned}|\mathrm{E}_F T_n(f,g) - \mu(f,g,F)| &\\ &\le \|A_{n-1}f - f\|_\infty \mathrm{E}_F|g(X_1)| \\ &\le C\left[\frac{C^{1/2}}{(n-1)^\beta} + \left(\frac{C}{(n-1)^{2\beta}} + \frac{1}{4(n-1)}\right)^{\beta/2}\right]\mathrm{E}_F|g(X_1)|.\end{aligned}$$

Proof. By (3.10), we get

$$\begin{aligned}|\mathrm{E}_F T_n(f,g) - \mu(f,g,F)| &\le \int_0^1 |g(F^{-1}(y))| \cdot |(A_{n-1}f)(y) - f(y)|\,dy \\ &\le \|A_{n-1}f - f\|_\infty \Big(\int_0^1 |g(F^{-1}(y))|\,dy\Big) \\ &= \|A_{n-1}f - f\|_\infty \mathrm{E}_F|g(X_1)|.\end{aligned}$$

Applying Corollary 1, we complete the proof. ∎

Remark 6. Here we follow Mache (1995). We recall the notion of Beta function:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \qquad p,q>0,$$

Let $a,b > -1$, $\alpha \ge 0$ and $c = c_n = [n^\alpha]$ for $n \in \mathbb{N}$. Define positive linear functionals $T_{\alpha,k,n} : C[0,1] \to \mathbb{R}$, $k = 0,\ldots,n$, as follows:

$$T_{\alpha,k,n}f = \frac{\int_0^1 f(t)t^{ck+a}(1-t)^{c(n-k)+b}\,dt}{B(ck+a+1, c(n-k)+b+1)}. \tag{3.11}$$

We also define positive linear operators

$$(M_n^\alpha f)(x) = \sum_{k=0}^n T_{\alpha,k,n}f \cdot p_{n,k}(x), \qquad n \in \mathbb{N},\ \alpha \ge 0. \tag{3.12}$$

When $a = b = 0$, we obtain the so-called Durrmeyer operators with Legendre weights. If, moreover, $\alpha = 0$, we have the standard Bernstein–Durrmeyer operators. In (3.11) and (3.12), we ignore in notation the dependence of the defined notions on a and b, because these do not affect the rates of approximation presented below.

Theorem 12 (Mache (1995)). (i) (Durrmeyer operators with Jacobi weights) *For* $\alpha = 0$, *we get*

$$\|M_n^\alpha f - f\|_\infty \le Cn^{-1}\Big[\int_{n^{-1/2}}^{1/2} \omega_2^\varphi(f,t)_\infty t^{-3}\,dt + \|f\|_\infty\Big].$$

(ii) *For* $0 < \alpha < 1$, *we obtain*

$$\|M_n^\alpha f - f\|_\infty \le C\Big\{n^{-1-\alpha}\Big[\int\limits_{n^{-1/2}}^{1/2} \omega_2^\varphi(f,t)_\infty t^{-3}\,dt + \|f\|_\infty\Big] + \omega_2^\varphi(f,n^{-1/2})_\infty\Big\}.$$

(iii) *For* $\alpha \ge 1$, *we have*

$$\|M_n^\alpha f - f\|_\infty \le C[n^{-1-\alpha}\|f\|_\infty + \omega_2^\varphi(f,n^{-1/2})_\infty].$$

(iv) (Bernstein operators) *For* $\alpha \to \infty$,

$$\|M_n^\alpha f - f\|_\infty \le C\omega_2^\varphi(f,n^{-1/2})_\infty,$$

where $C > 0$ *is independent of* n, α *and* f.

We also use

THEOREM 13 (Mache (1995)). *Let* $\alpha \ge 1$ *and* $0 < \beta < 1$. *Then* $\|M_n^\alpha f - f\|_\infty = \mathcal{O}(n^{-\beta})$ *iff* $\omega_2^\varphi(f,t)_\infty = \mathcal{O}(t^{2\beta})$.

Formula (3.12) describes a large parametric class of generalized L-statistics

$$M_n^\alpha(f,g) = n^{-1}\sum_{i=1}^n T_{\alpha,i-1,n-1}f \cdot g(X_{i:n}), \qquad n \in \mathbb{N}\setminus\{1\},$$

with coefficients defined by specific positive linear functionals (3.11). The L-statistics can be determined randomly by means of a probabilistic model of generalized uniform order statistics introduced by Kamps (1995). Setting $\alpha = 0$, we obtain the so-called fractional order statistics with nonintegral sample sizes, studied in Stigler (1977), and Rohatgi and Saleh (1988). For some choices of parameters, they have practical interpretations as sequential order statistics and certain records (see Kamps (1995)).

Observe that

$$\mathrm{E}_F M_n^\alpha(f,g) = \int\limits_0^1 g(F^{-1}(y))(M_{n-1}^\alpha f)(y)\,dy$$

so that

$$\mathrm{E}_F M_n^\alpha(f,g) - \mu(f,g,F) = \int\limits_0^1 g(F^{-1}(y))[(M_{n-1}^\alpha f)(y) - f(y)]\,dy. \tag{3.13}$$

Now we are ready to present

THEOREM 14. *Let* X_i, $i \in \mathbb{N}$, *be i.i.d. random variables with a common distribution function* F *and* $f \in C[0,1]$. *Assume that* $\mathrm{E}_F|g(X_1)| < \infty$, *and* *use the notions and notations of Remark* 6. *Then*

$$
\begin{aligned}
(3.14)\qquad & |\mathrm{E}_F M_n^\alpha(f,g)-\mu(f,g,F)| \\
&\le \|M_{n-1}^\alpha f-f\|_\infty \int_0^1 |g(F^{-1}(y))|\,dy \\
&\le C\begin{Bmatrix}
(n-1)^{-1}\Big[\int\limits_{(n-1)^{-1/2}}^{1/2}\omega_2^\varphi(f,t)_\infty t^{-3}\,dt+\|f\|_\infty\Big] & (\alpha=0)\\
\Big\{(n-1)^{-1-\alpha}\Big[\int\limits_{(n-1)^{-1/2}}^{1/2}\omega_2^\varphi(f,t)_\infty t^{-3}dt+\|f\|_\infty\Big] & \\
\quad +\,\omega_2^\varphi(f,(n-1)^{-1/2})_\infty\Big\} & (0<\alpha<1)\\
[(n-1)^{-1-\alpha}\|f\|_\infty+\omega_2^\varphi(f,(n-1)^{-1/2})_\infty] & (\alpha\ge 1)\\
\omega_2^\varphi(f,(n-1)^{-1/2})_\infty & (\alpha\to\infty)
\end{Bmatrix}\\
&\quad\times \mathrm{E}_F|g(X_1)|.
\end{aligned}
$$

Proof. From (3.13) we obtain

$$
\begin{aligned}
|\mathrm{E}_F M_n^\alpha(f,g)-\mu(f,g,F)| &\le \int_0^1 |g(F^{-1}(y))|\,|(M_{n-1}^\alpha f)(y)-f(y)|\,dy\\
&\le \|M_{n-1}^\alpha f-f\|_\infty \int_0^1 |g(F^{-1}(y))|\,dy.
\end{aligned}
$$

Then we apply Theorem 12. ■

THEOREM 15. *Under the assumptions of Theorem* 14 *with* $\omega_2^\varphi(f,t)_\infty = \mathcal{O}(t^{2\beta})$ *for some* $0<\beta<1$, *we have*

$$|\mathrm{E}_F M_n^\alpha(f,g)-\mu(f,g,F)| = \mathcal{O}((n-1)^{-\beta}).$$

Proof. Use (3.14) and Theorem 13. ■

REMARK 7. Here we refer to Gonska and Meier (1984). For $f\in C[0,1]$, $m\in\mathbb{N}$, and $0\le\beta\le\gamma$, we define the Stancu-type positive linear operators

$$(L_{m0}^{\langle 0\beta\gamma\rangle} f)(x)=\sum_{k=0}^m f\bigg(\frac{k+\beta}{m+\gamma}\bigg)p_{m,k}(x),\qquad x\in[0,1].$$

We shall apply the following theorem:

THEOREM 16 (Gonska and Meier (1984)). *For* $f \in C[0,1]$, $h > 0$, $0 \le \beta \le \gamma$, $m \in \mathbb{N}$ *and* $x \in [0,1]$ *we have*

$$
\begin{aligned}
(3.15)\quad & |(L_{m0}^{\langle 0\beta\gamma\rangle} f)(x) - f(x)| \\
&\le \left[3 + \max\left\{\frac{1}{h^2}, 1\right\} \frac{(\gamma^2 - m)x^2 + mx + \beta^2}{(m+\gamma)^2}\right] \omega_2(f,h) \\
&\quad + \frac{2|\beta - \gamma x|}{m+\gamma} \max\left\{\frac{1}{h}, 1\right\} \omega_1(f,h).
\end{aligned}
$$

Maximizing the right-hand side of (3.15), we obtain

COROLLARY 2. *For sufficiently large* $m \in \mathbb{N}$,

$$
\begin{aligned}
&\|L_{m0}^{\langle 0\beta\gamma\rangle} f - f\|_\infty \\
&\le \left[3 + \frac{m^3 + 4m^2\beta(\beta - \gamma)}{4(m - \gamma^2)(m+\gamma)^2}\right] \omega_2(f, m^{-1/2}) + \frac{2(\beta+\gamma)m^{1/2}}{m+\gamma} \omega_1(f, m^{-1/2}).
\end{aligned}
$$

Finally, we consider some generalizations of (1.4). Many authors studied modifications of the L-statistics that consist in replacing arguments of the weight function in the coefficients $l_{i,n} = n^{-1} f((i-1)/(n-1))$. The most popular choices were i/n and $i/(n+1)$. These two and many other cases can be examined simultaneously if we define

$$
L_n^{\beta\gamma}(f,g) = n^{-1} \sum_{i=1}^{n} f\left(\frac{i-1+\beta}{n-1+\gamma}\right) g(X_{i:n}),
$$

and apply the statements of Remark 7. Observe that

$$
\mathrm{E}_F L_n^{\beta\gamma}(f,g) = \int_0^1 g(F^{-1}(y)) (L_{n-1,0}^{\langle 0\beta\gamma\rangle} f)(y)\, dy
$$

and

$$
(3.16)\quad \mathrm{E}_F L_n^{\beta\gamma}(f,g) - \mu(f,g,F) = \int_0^1 g(F^{-1}(y)) [(L_{n-1,0}^{\langle 0\beta\gamma\rangle} f)(y) - f(y)]\, dy.
$$

Therefore

THEOREM 17. *Suppose that* X_i, $i \in \mathbb{N}$, *are i.i.d. random variables with a common distribution function* F. *Let* $f \in C[0,1]$ *and* $\mathrm{E}_F|g(X_1)| < \infty$. *Then for sufficiently large* n, *we have*

$$
\begin{aligned}
&|\mathrm{E}_F L_n^{\beta\gamma}(f,g) - \mu(f,g,F)| \\
&\qquad \le \|L_{n-1,0}^{\langle 0\beta\gamma\rangle} f - f\|_\infty \int_0^1 |g(F^{-1}(y))|\,dy \\
&\qquad \le \biggl\{\biggl[3 + \frac{(n-1)^3 + 4(n-1)^2\beta(\beta-\gamma)}{4(n-1-\gamma^2)(n-1+\gamma)^2}\biggr]\omega_2(f,(n-1)^{-1/2}) \\
&\qquad\qquad + \frac{2(\beta+\gamma)(n-1)^{1/2}}{n-1+\gamma}\omega_1(f,(n-1)^{-1/2})\biggr\}\mathrm{E}_F|g(X_1)|.
\end{aligned}
$$

Proof. Use (3.16) and Corollary 2. ∎

4. Concluding remarks. We conclude the paper with discussing rates of convergence to (1.6) of the expectations of the L-statistics (1.4), (1.5), and (3.3) for various classes of weight functions. These rates coincide with the rates of convergence of the Bernstein, Kantorovich and Bernstein–Durrmeyer operators, respectively, to the identity in L_p-norms, $1 \le p \le \infty$. In general, $n^{-1/2}$ is the best rate for these operators (see Knoop and Zhou (1992), DeVore and Lorentz (1993, formula (7.3)), Gonska and X.-L. Zhou (1995), and Gonska and D.-X. Zhou (1995), respectively).

Remark 8. For sufficiently large n, (1.4) has a better constant of approximation than the ones presented in Theorem 1. Namely, the right-hand side of (2.3) can be replaced by $\omega_2(f,(n-1)^{-1/2})\mathrm{E}|g(X_1)|$, which is also optimal (see Paltanea (1998)).

Remark 9. Since B_n reproduces linear functions, $B_n f - f = 0$ for linear f. This means that (1.4) provides unbiased estimates for linear weight functions. The sample mean and Gini mean difference are classical examples here.

Remark 10. If $f \in C^2[0,1]$, then

$$\|B_n f - f\|_\infty = \mathcal{O}(n^{-1})$$

(see Gonska and Meier (1984)), and

$$
\begin{aligned}
\|K_n f - f\|_\infty &= \mathcal{O}(n^{-1}),\\
\|D_n f - f\|_\infty &= \mathcal{O}(n^{-1})
\end{aligned}
$$

(see Cao and Gonska (1989) and Gonska and Kovacheva (1994)).

Remark 11. Since defining the Bernstein operators for discontinuous functions does not make sense, we have L_p-norm estimates, $p < \infty$, for the Kantorovich and Bernstein–Durrmeyer operators only. Writing $\mathcal{L}_n$ for either K_n or D_n, we have the following statements:

THEOREM 18 (see Totik (1984), Ditzian and Ivanov (1989)). *Let $1 \le p < \infty$. Then for $0 < \alpha < 2$, we have*

$$\|\mathcal{L}_n f - f\|_p = \mathcal{O}(n^{-\alpha/2}) \quad \textit{iff} \quad \omega_2^\varphi(f,t)_p = \mathcal{O}(t^\alpha).$$

Here the Ditzian–Totik modulus of smoothness

$$\omega_2^\varphi(f,t)_p = \sup_{0 \le h \le t} \|\widetilde{\Delta}^k_{h\varphi(\cdot)}(f,\cdot)\|_p, \qquad f \in L_p[0,1],$$

is defined analogously to (1.10). For the saturation case, we have

THEOREM 19 (see Maier (1978a,b), Riemenschneider (1978), Totik (1983), Heilmann (1988)). *We have*

$$\|\mathcal{L}_n f - f\|_p = \mathcal{O}(n^{-1}), \qquad 1 \le p < \infty,$$

iff either

$$\omega_2^\varphi(f,t)_p = \mathcal{O}(t^2)$$

for $1 < p < \infty$, or

$$f(x) = K + \int_y^x \frac{h(t)}{t(1-t)}\,dt \qquad \textit{a.e. on } [0,1]$$

for $p = 1$ with $y \in (0,1)$, $h \in BV[0,1]$ and $h(0) = h(1) = 0$.

Acknowledgements. The authors would like to thank deeply Professor H. Gonska of University of Duisburg, Germany, for being an unlimited source of information and advice during the course of writing this article. The work of the second author was done under the support of the Polish State Committee for Scientific Research (KBN) Grant 2 P03A 014 13 while he was visiting Department of Mathematical Sciences, The University of Memphis.

References

G. A. Anastassiou (1993), *Moments in Probability and Approximation Theory*, Pitman Res. Notes Math. Ser. 287, Longman Sci. & Tech., Harlow.

N. Balakrishnan and A. C. Cohen (1991), *Order Statistics and Inference*, Academic Press, Boston.

M. Bogdan (1994), *Asymptotic distributions of linear combinations of order statistics*, Appl. Math. (Warsaw) 24, 201–225.

D. Boos (1979), *A differential for L-statistics*, Ann. Statist. 7, 955–959.

J. D. Cao and H. H. Gonska (1989), *Pointwise estimates for modified positive linear operators*, Portugal. Math. 46, 402–430.

H. Chernoff, J. L. Gastwirth and M. V. Johns (1967), *Asymptotic distribution of linear combinations of order statistics, with applications to estimation*, Ann. Math. Statist. 38, 52–72.

H. A. David (1981), *Order Statistics*, 2nd ed., Wiley, New York.

R. A. DeVore and G. G. Lorentz (1993), *Constructive Approximation*, Grundlehren Math. Wiss. 303, Springer, Berlin.

Z. Ditzian and K. Ivanov (1989), *Bernstein-type operators and their derivatives*, J. Approx. Theory 56, 72–90.

I. Gavrea and D. H. Mache (1995), *Generalization of Bernstein-type approximation methods*, in: *Approximation Theory*, *Proc. IDoMAT95*, M. W. Müller, M. Felten and D. H. Mache (eds.), Math. Res. 86, Akademie-Verlag, Berlin, 115–126.

H. H. Gonska and R. K. Kovacheva (1994), *The second order modulus revisited*: *remarks*, *applications*, *problems*, Confer. Sem. Mat. Univ. Bari 257, 1–32.

H. H. Gonska and I. Meier (1984), *Quantitative theorems on approximation by Bernstein–Stancu operators*, Calcolo 21, 317–335.

H. H. Gonska and D.-X. Zhou (1995), *Local smoothness of functions and Bernstein–Durrmeyer operators*, Comput. Math. Appl. 30, No. 3–6 (special issue *Concrete Analysis*, G. A. Anastassiou (ed.)), 83–101.

H. H. Gonska and X.-L. Zhou (1995), *The strong converse inequality for the Bernstein–Kantorovich operators*, ibid., 103–128.

M. Heilmann (1988), *L_p-saturation of some modified Bernstein operators*, J. Approx. Theory 54, 260–273.

R. Helmers, P. Janssen and R. Serfling (1990), *Berry–Essen and bootstrap results for generalized L-statistics*, Scand. J. Statist. 17, 65–77.

R. Helmers and H. Ruymgaart (1988), *Asymptotic normality of generalized L-statistics with unbounded scores*, J. Statist. Plann. Inference 19, 43–53.

U. Kamps (1995), *A Concept of Generalized Order Statistics*, Teubner Skr. Math. Stochastik, B. G. Teubner, Stuttgart.

H.-B. Knoop and X.-L. Zhou (1992), *The lower estimate for linear positive operators*, part 1: Constr. Approx. 11 (1995), 53–66, part 2: Results Math. 25 (1994), 300–315.

C.-D. Lea and M. L. Puri (1988), *Asymptotic properties of linear functions of order statistics*, J. Statist. Plann. Inference 18, 203–223.

D. H. Mache (1995), *A link between Bernstein polynomials and Durrmeyer polynomials with Jacobi weights*, in: *Approximation Theory VIII*, Vol. 1: *Approximation and Interpolation*, C. K. Chui and L. L. Schumaker (eds.), World Scientific, Singapore, 403–410.

V. Maier (1978a), *L_p approximation by Kantorovich operators*, Anal. Math. 4, 289–295.

V. Maier (1978b), *The L_1 saturation class of the Kantorovich operator*, J. Approx. Theory 22, 223–232.

D. M. Mason (1981), *Asymptotic normality of linear combinations of order statistics with a smooth score function*, Ann. Statist. 9, 899–908.

D. M. Mason (1982), *Some characterizations of strong laws for linear functions of order statistics*, Ann. Probab. 10, 1051–1057.

D. M. Mason and G. R. Shorack (1992), *Necessary and sufficient conditions for asymptotic normality of L-statistics*, ibid. 20, 1779–1804.

R. Norvaiša (1994), *Laws of large numbers for L-statistics*, J. Appl. Math. Stochastic Anal. 7, 125–143.

R. Norvaiša and R. Zitikis (1991), *Asymptotic behavior of linear combinations of functions of order statistics*, J. Statist. Plann. Inference 28, 305–317.

R. Paltanea (1995), *Best constants in estimates with second order moduli of continuity*, in: *Approximation Theory*, *Proc. IDoMAT95*, M. W. Müller, M. Felten and D. H. Mache (eds.), Math. Res. 86, Akademie-Verlag, Berlin, 251–275.

R. Paltanea (1998), *On an optimal constant in approximation by Bernstein operators*, Rend. Circ. Mat. Palermo, to appear.

S. D. Riemenschneider (1978), *The L_p saturation of the Bernstein–Kantorovich polynomials*, J. Approx. Theory 23, 158–162.

V. K. Rohatgi and A. K. M. D. E. Saleh (1988), *A class of distributions connected to order statistics with nonintegral sample size*, Comm. Statist. Theory Methods 17, 2005–2012.
P. K. Sen (1978), *An invariance principle for linear combinations of order statistics*, Z. Wahrsch. Verw. Gebiete 42, 327–340.
G. R. Shorack (1969), *Asymptotic normality of linear combinations of functions of order statistics*, Ann. Math. Statist. 40, 2041–2050.
G. R. Shorack (1972), *Functions of order statistics*, ibid. 43, 412–427.
L. Schumaker (1981), *Spline Functions*, *Basic Theory*, Wiley-Interscience, New York.
P. C. Sikkema (1961), *Der Wert einiger Konstanten in der Theorie der Approximation mit Bernstein-Polynomen*, Numer. Math. 3, 107–116.
S. M. Stigler (1974), *Linear functions of order statistics with smooth weight functions*, Ann. Statist. 2, 676–693.
S. M. Stigler (1977), *Fractional order statistics*, *with applications*, J. Amer. Statist. Assoc. 72, 544–550.
V. Totik (1983), $L_p(p > 1)$*-approximation by Kantorovich polynomials*, Analysis 3, 79–100.
V. Totik (1984), *An interpolation theorem and its application to positive operators*, Pacific J. Math. 111, 447–481.
J. A. Wellner (1977a), *A Glivenko–Cantelli theorem and strong laws of large numbers for functions of order statistics*, Ann. Statist. 5, 473–480.
J. A. Wellner (1977b), *A law of the iterated logarithm for functions of order statistics*, ibid. 5, 481–494.
X. Xiang (1995), *A note on the bias of L-estimators and a bias reduction procedure*, Statist. Probab. Lett. 23, 123–127.
W. R. van Zwet (1980), *A strong law for linear functions of order statistics*, Ann. Probab. 8, 986–990.

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Received on 18.2.1999;
revised version on 16.6.1999