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W. HAN (Iowa City) and M. SOFONEA (Perpignan)

ANALYSIS AND NUMERICAL APPROXIMATION OF AN ELASTIC FRICTIONAL CONTACT PROBLEM WITH NORMAL COMPLIANCE

Abstract. We consider the problem of frictional contact between an elastic body and an obstacle. The elastic constitutive law is assumed to be nonlinear. The contact is modeled with normal compliance and the associated version of Coulomb's law of dry friction. We present two alternative yet equivalent weak formulations of the problem, and establish existence and uniqueness results for both formulations using arguments of elliptic variational inequalities and fixed point theory. Moreover, we show the continuous dependence of the solution on the contact conditions. We also study the finite element approximations of the problem and derive error estimates. Finally, we introduce an iterative method to solve the resulting finite element system.

1. Introduction. Processes of frictional contact between deformable bodies are very common in industry and everyday life. Contact without lubrication can be found for example in the process of metal forming, in car's braking systems, in engines, motors and transmissions. Despite the difficulties that the process of frictional contact presents because of the complicated surface phenomena involved, considerable progress has been made in the modeling and analysis of contact problems. An early attempt to study frictional contact problems for elastic and viscoelastic materials within the framework of variational inequalities was made in [6]. Steady-state as well as time-dependent frictional contact problems for linearly and nonlinearly elastic materials may be found in [13]. An excellent reference to the field of contact problems with or without friction is [9].

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Dynamic problems with normal compliance contact condition were first considered in [11] where the existence of a weak solution was proven. This condition allows the interpenetration of the body's surface into the obstacle. It was justified in [11] and [10] by considering the interpenetration and deformation of surface asperities. Moreover, the normal compliance condition has been employed as a mathematical regularization of Signorini's nonpenetration condition and used in numerical solution algorithms. Contact problems with normal compliance have been discussed in numerous papers, e.g. [1, 7, 9, 10, 14] and the references therein.

In this work we consider the problem of frictional contact between an elastic body and a foundation. We assume that the forces and tractions acting upon the body change slowly in time so that the acceleration in the system is negligible. Neglecting the inertial term in the equation of motion leads to a quasistatic approximation for the process. The material's constitutive law is assumed to be nonlinear. The contact is modeled with a normal compliance condition and the static version of Coulomb's law of dry friction. We establish the existence of a unique solution to the problem, using fixed point arguments. Then we prove the stability of the solution with respect to perturbation of the normal compliance functions, which is important from the point of view of applications. We also discuss the numerical treatment of the problem and derive some error estimates.

The paper is organized as follows. In Section 2 we introduce the function spaces for various quantities, and state assumptions on given data. The mechanical problem is stated in Section 3, where two alternative yet equivalent variational forms of the problem are formulated. The unknown for one of the variational problems is the displacement, while for the other, the unknown is the stress. For both variational problems, we show the unique solvability in Section 4. The proofs are based on arguments from elliptic variational inequalities and fixed point properties of certain maps. We also study the link between the solutions of the variational problems and we prove that the displacement and the stress field are related by the elastic constitutive law. Section 5 is devoted to a result on the continuous dependence of the solution on contact conditions, which indicates that a small perturbation in the contact condition leads to a small change in the solution. Finally, in Section 6 we study the finite element approximation of the displacement variational formulation. We prove Céa's type inequalities, from which we can conclude the convergence of the finite element method and derive order error estimates under appropriate regularity assumptions on the exact solution. We introduce an iterative method to solve the resulting finite element system, which converges under certain assumptions.

2. Preliminaries. Let us introduce various notations and spaces which will be used in the formulation and analysis of the mechanical problem. For further details on this preliminary material we refer the reader to [6, 8, 13].

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N = 2,3$ in applications) with a Lipschitz boundary Γ and let Γ_1 be a measurable part of Γ such that $\operatorname{meas}(\Gamma_1) > 0$. Since the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on Γ.

Let S_N represent the space of second order symmetric tensors on $\mathbb{R}^N$, or equivalently, the space of symmetric matrices of order N. We define the inner products and the corresponding norms on $\mathbb{R}^N$ and S_N by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \qquad |\boldsymbol{v}| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}, \quad \forall \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^N,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad |\boldsymbol{\tau}| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}, \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau}\in S_N.$$

Here and below, $i,j = 1,\dots,N$, and the summation convention over repeated indices is adopted. Moreover, in the sequel, the index that follows a comma indicates a partial derivative, e.g., $u_{i,j} = \partial u_i/\partial x_j$.

We now introduce several function spaces. Let

$$H = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \qquad H_1 = \{\boldsymbol{u} = (u_i) \mid u_i \in H^1(\Omega)\},$$
$$\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma}\in\mathcal{H} \mid \sigma_{ij,j}\in H\}.$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega u_i v_i\,dx, \qquad (\boldsymbol{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau})_H.$$

Here $\boldsymbol{\varepsilon} : H_1 \to \mathcal{H}$ and $\operatorname{Div} : \mathcal{H}_1 \to H$ are the *deformation* and the *divergence* operators, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \qquad \varepsilon_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad \operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$.

Let $H_\Gamma = H^{1/2}(\Gamma)^N$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\boldsymbol{v}\in H_1$, we also use the notation $\boldsymbol{v}$ to denote the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the *normal* and *tangential* components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}. \tag{2.1}$$

Let H'_Γ be the dual of H_Γ and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between H'_Γ and H_Γ. For every $\boldsymbol{\sigma}\in\mathcal{H}_1$, $\boldsymbol{\sigma\nu}$ can be defined as the element in H'_Γ which satisfies

$$\langle\boldsymbol{\sigma\nu},\gamma\boldsymbol{v}\rangle = (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma},\boldsymbol{v})_H \quad \forall\boldsymbol{v}\in H_1. \tag{2.2}$$

Denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the *normal* and *tangential* traces of $\boldsymbol{\sigma}$, respectively. If $\boldsymbol{\sigma}$ is continuously differentiable on $\overline{\Omega}$, then

$$\sigma_\nu = (\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}, \tag{2.3}$$

$$\langle\boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}\rangle = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,da \tag{2.4}$$

for all $\boldsymbol{v}\in H_1$, where da is the surface measure element.

In the sequel we use V to denote the closed subspace of H_1 defined by

$$V = \{\boldsymbol{v}\in H_1 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\}.$$

Since $\mathrm{meas}(\Gamma_1) > 0$, the following Korn inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \ge c_K\|\boldsymbol{v}\|_{H_1} \quad \forall\boldsymbol{v}\in V \tag{2.5}$$

(see e.g. [12]). Here c_K denotes a strictly positive constant which depends only on Ω and Γ_1.

On V we consider the inner product given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{2.6}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (2.5) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space.

3. The mechanical problem and weak formulations. In this section we introduce the physical setting, list the assumptions on the problem data and present the variational formulations of the model.

We consider an elastic body occupying a bounded domain $\Omega\subset\mathbb{R}^N$ with a Lipschitz boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\mathrm{meas}(\Gamma_1) > 0$. A volume force of density $\boldsymbol{\varphi}_1$ acts in Ω and a surface traction of density $\boldsymbol{\varphi}_2$ acts on Γ_2. The body is clamped on Γ_1 and thus the displacement field vanishes there. A gap g exists between the potential contact surface Γ_3 and a foundation, and it is measured along the direction of the outward normal $\boldsymbol{\nu}$.

We denote by $\boldsymbol{u}$ the displacement vector, $\boldsymbol{\sigma}$ the stress field and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the small strain tensor. The elastic constitutive law of the material is assumed to be

$$\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u})) \tag{3.1}$$

with a given (nonlinear) function F. Here and below, in order to simplify the notation, we usually do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}\in\Omega\cup\Gamma$.

Next, we describe the condition on the potential contact surface Γ_3. We assume that the normal stress σ_ν satisfies the normal compliance condition

$$-\sigma_\nu = p_\nu(u_\nu - g) \tag{3.2}$$

where u_ν represents the normal displacement, p_ν is a prescribed nonnegative function with $p_\nu(t) = 0$ for $t \le 0$, and $u_\nu - g$, when it is positive, represents the penetration of the body in the foundation. Such a contact condition was proposed in [11] and used in a number of publications (see e.g. [1, 7, 9, 10, 14] and references therein). In this condition the interpenetration is allowed but penalized. In [7, 10, 11] the following function was employed:

$$p_\nu(t) = c_\nu (t)_+^{m_\nu} \tag{3.3}$$

where c_ν is a positive constant, m_ν is a positive exponent and $t_+ = \max\{0, t\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$. Here we allow for a more general expression, similar to the one used in [1, 14].

The associated friction law on Γ_3 is chosen as

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \le p_\tau(u_\nu - g), \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu - g) \Rightarrow \boldsymbol{u}_\tau = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu - g) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda \boldsymbol{u}_\tau,\ \lambda \ge 0. \end{cases} \tag{3.4}$$

Here p_τ is a nonnegative function, the so-called *friction bound*, which satisfies $p_\tau(t) = 0$ for $t \le 0$, $\boldsymbol{u}_\tau$ denotes the tangential displacement and $\boldsymbol{\sigma}_\tau$ represents the tangential force on the contact boundary. This is a static version of Coulomb's law of dry friction and should be seen either as a mechanical model suitable for the proportional loadings or as a first approximation of a more realistic model, based on a friction law involving the time derivative of $\boldsymbol{u}_\tau$ (see for instance [14, 15]). It states that the tangential shear cannot exceed the maximal frictional resistance p_τ. When strict inequality holds the surface adheres to the foundation and is in the so-called *stick* state, and when equality holds there is relative sliding, the so-called *slip* state. Therefore, the contact surface Γ_3 is divided into three zones: stick, slip and the separation zone in which $u_\nu < g$, i.e. there is no contact. The boundaries of these zones are free boundaries since they are unknown a priori, and are part of the problem.

In the references [3, 4, 5, 6, 9], the friction law (3.4) was used with

$$p_\tau = \mu p_\nu \tag{3.5}$$

where $\mu > 0$ is a coefficient of friction. In [7] the friction law (3.4) was used with

$$p_\tau(t) = c_\tau \tag{3.6}$$

where $c_\tau > 0$. Recently, a new version for Coulomb's law of friction was derived in [16, 17] from thermodynamic consideration. It consists in using in (3.4) the friction bound function

$$p_\tau = \mu p_\nu (1 - \alpha p_\nu)_+ \tag{3.7}$$

where α is a small positive coefficient related to the wear and hardness of the surface.

With (3.1) as the constitutive relation, (3.2) and (3.4) for the contact condition, the mechanical problem of frictional contact of the elastic body may be formulated classically as follows.

Problem P. Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^N$ and a stress field $\boldsymbol{\sigma} : \Omega \to S_N$ such that

$$\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega, \tag{3.8}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{\varphi}_1 = \mathbf{0} \quad \text{in } \Omega, \tag{3.9}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{3.10}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{\varphi}_2 \quad \text{on } \Gamma_2, \tag{3.11}$$

and on Γ_3,

$$\begin{cases} -\sigma_\nu = p_\nu(u_\nu - g), \\ |\boldsymbol{\sigma}_\tau| \le p_\tau(u_\nu - g), \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu - g) \Rightarrow \boldsymbol{u}_\tau = \mathbf{0}, \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu - g) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda \boldsymbol{u}_\tau \text{ for some } \lambda \ge 0. \end{cases} \tag{3.12}$$

In the study of the mechanical problem (3.8)–(3.12) we assume that the *elasticity operator*

$$F : \Omega \times S_N \to S_N$$

satisfies

(3.13) (a) There exists an $M > 0$ such that $|F(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - F(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \le M |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_N$, a.e. in Ω.

(b) There exists an $m > 0$ such that $(F(\boldsymbol{x}, \boldsymbol{\varepsilon}_1)) - F(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_N$, a.e. in Ω.

(c) The mapping $\boldsymbol{x} \mapsto F(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is Lebesgue measurable on Ω for any $\boldsymbol{\varepsilon} \in S_N$.

(d) $F(\boldsymbol{x}, \mathbf{0}) \in \mathcal{H}$ for all $\boldsymbol{x} \in \Omega$.

A family of elasticity operators satisfying the condition (3.13) is provided by nonlinear Hencky materials (for details, cf. e.g. [18]). For a Hencky material, the stress-strain relation is

$$\boldsymbol{\sigma} = K_0 \operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}) \boldsymbol{I} + \psi(|\boldsymbol{\varepsilon}^D(\boldsymbol{u})|^2) \boldsymbol{\varepsilon}^D(\boldsymbol{u}),$$

so that the elasticity operator is

$$F(\boldsymbol{\varepsilon}) = K_0 \operatorname{tr} \boldsymbol{\varepsilon} \boldsymbol{I} + \psi(|\boldsymbol{\varepsilon}^D|^2) \boldsymbol{\varepsilon}^D. \tag{3.14}$$

Here, $K_0 > 0$ is a material coefficient, $\boldsymbol{I}$ is the identity tensor of the second order, $\operatorname{tr} \boldsymbol{\varepsilon} = \varepsilon_{ii}$ is the trace of $\boldsymbol{\varepsilon}$, and $\boldsymbol{\varepsilon}^D$ denotes the deviatoric part of $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - \frac{1}{N} \operatorname{tr} \boldsymbol{\varepsilon} \boldsymbol{I}.$$

The function ψ is assumed to be piecewise continuously differentiable, and there exist positive constants c_1, c_2, d_1 and d_2 such that for $\xi \geq 0$,

$$\psi(\xi) \leq d_1, \quad -c_1 \leq \psi'(\xi) \leq 0, \quad c_2 \leq \psi(\xi) + 2\psi'(\xi)\xi \leq d_2.$$

Let us show that the condition (3.13) is satisfied for the elasticity operator defined in (3.14). We have

$$\begin{aligned}
&F(\boldsymbol{\varepsilon}_1) - F(\boldsymbol{\varepsilon}_2)\\
&\quad = K_0 \operatorname{tr}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)\boldsymbol{I} + \psi(|\boldsymbol{\varepsilon}_1^D|^2)\boldsymbol{\varepsilon}_1^D - \psi(|\boldsymbol{\varepsilon}_2^D|^2)\boldsymbol{\varepsilon}_2^D\\
&\quad = K_0 \operatorname{tr}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)\boldsymbol{I} + \int_0^1 \frac{d}{dt}[\psi(|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2)(\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D))]\,dt\\
&\quad = K_0 \operatorname{tr}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)\boldsymbol{I}\\
&\qquad + \int_0^1 [2\psi'(|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2)\\
&\qquad \times (\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)(\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D))\\
&\qquad + \psi(|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2)(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)]\,dt.
\end{aligned}$$

Then the condition (3.13)(a) is satisfied for some constant M depending on K_0, d_1, d_2 and c_1. Now

$$\begin{aligned}
&(F(\boldsymbol{\varepsilon}_1) - F(\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)\\
&\quad = K_0|\operatorname{tr}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|^2\\
&\qquad + \int_0^1 [2\psi'(|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2)|(\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|^2\\
&\qquad + \psi(|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2)|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2]\,dt\\
&\quad \geq K_0|\operatorname{tr}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|^2\\
&\qquad + \int_0^1 [2\psi'(|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2)|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2\\
&\qquad + \psi(|\boldsymbol{\varepsilon}_2^D + t(\boldsymbol{\varepsilon}_1^D - \boldsymbol{\varepsilon}_2^D)|^2)]|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2\,dt\\
&\quad \geq K_0|\operatorname{tr}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|^2 + c_2|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2.
\end{aligned}$$

Hence, the condition (3.13)(b) is satisfied with m depending on K_0 and c_2. Conditions (3.13)(c), (d) are obviously valid.

Using the condition (3.13), we see that for all $\boldsymbol{\tau} \in \mathcal{H}$ the function $\boldsymbol{x} \mapsto F(\boldsymbol{x}, \boldsymbol{\tau}(\boldsymbol{x}))$ belongs to $\mathcal{H}$ and hence we may consider F as an operator defined on $\mathcal{H}$ with range in $\mathcal{H}$. Moreover, $F : \boldsymbol{\tau} \in \mathcal{H} \mapsto F(\cdot, \boldsymbol{\tau}) \in \mathcal{H}$ is a strongly monotone Lipschitz continuous operator and therefore F is invertible and its inverse $F^{-1} : \mathcal{H} \to \mathcal{H}$ is also a strongly monotone Lipschitz continuous operator.

We assume the *normal compliance* functions

$$p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \qquad (r = \nu, \tau)$$

satisfy the following hypothesis for $r = \nu, \tau$.

(3.15) (a) There exists an $L_r > 0$ such that $|p_r(\boldsymbol{x}, t_1) - p_r(\boldsymbol{x}, t_2)| \le L_r |t_1 - t_2|$ for all $t_1, t_2 \in \mathbb{R}$, a.e. on Γ_3.

(b) The mapping $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, t)$ is Lebesgue measurable on Γ_3 for any $t \in \mathbb{R}$.

(c) $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, t) = 0$ for $t \le 0$.

We observe that the assumptions (3.15) on the functions p_ν and p_τ are fairly general. The most severe restriction comes from the condition (a), which, roughly speaking, requires the functions to grow at most linearly. Certainly the function defined in (3.6) satisfies (3.15)(a), whereas that defined in (3.3) satisfies it if and only if $m_\nu = 1$. We also observe that if the functions p_ν and p_τ are related by (3.5) and p_ν satisfies (3.15)(a), then so does p_τ with $L_\tau = \mu L_\nu$. It can be verified that this statement is still valid if the functions p_ν and p_τ are related by (3.7).

The condition (3.15)(c) shows that when there is no penetration (i.e. $u_\nu \le g$) then the tractions vanish ($\sigma_\nu = 0$, $\boldsymbol{\sigma}_\tau = \boldsymbol{0}$). This condition is satisfied for the function (3.3) if $m_\nu > 0$. We also observe that if the functions p_ν and p_τ are related by (3.5) or (3.7) and p_ν satisfies (3.15)(c), then so does p_τ.

We also suppose that the forces and the tractions have the regularity

$$\boldsymbol{\varphi}_1 \in H, \qquad \boldsymbol{\varphi}_2 \in L^2(\Gamma_2)^N \tag{3.16}$$

while the gap function g is such that

$$g \in L^\infty(\Gamma_3), \qquad g(\boldsymbol{x}) \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{3.17}$$

Next, using Riesz's representation theorem, we define $\boldsymbol{f} \in V$ by

$$(\boldsymbol{f}, \boldsymbol{v})_V = (\boldsymbol{\varphi}_1, \boldsymbol{v})_H + (\boldsymbol{\varphi}_2, \gamma \boldsymbol{v})_{L^2(\Gamma_2)^N} \qquad \forall \boldsymbol{v} \in V \tag{3.18}$$

and let $j : V \times V \to \mathbb{R}$ be the functional

$$j(\boldsymbol{\eta}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu(\eta_\nu - g) v_\nu \, da + \int_{\Gamma_3} p_\tau(\eta_\nu - g) |\boldsymbol{v}_\tau| \, da. \tag{3.19}$$

For all $\boldsymbol{\eta} \in V$, set

$$\Sigma(\boldsymbol{\eta}) = \{\boldsymbol{\sigma} \in \mathcal{H} \mid (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{\eta}, \boldsymbol{v}) \ge (\boldsymbol{f}, \boldsymbol{v})_V \ \forall \boldsymbol{v} \in V\} \tag{3.20}$$

and

$$D(T) = \{\boldsymbol{z} \in \mathcal{H} \mid \exists \boldsymbol{v} \in V \text{ such that } F(\boldsymbol{\varepsilon}(\boldsymbol{v})) = \boldsymbol{z}\}. \tag{3.21}$$

From (3.13)(b) and Korn's inequality (2.5) it follows that the operator $F \circ \boldsymbol{\varepsilon} : V \to D(T)$ is invertible; we let $T : D(T) \to V$ denote its inverse. We have

$$\boldsymbol{v} = T(\boldsymbol{z}) \Leftrightarrow F(\boldsymbol{\varepsilon}(\boldsymbol{v})) = \boldsymbol{z}. \tag{3.22}$$

Moreover we obtain the following result.

Lemma 3.1. *If* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *are sufficiently smooth functions satisfying* (3.8)–(3.12), *then*

$$\begin{aligned} &\boldsymbol{u} \in V, \\ &(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V, \end{aligned} \tag{3.23}$$

$$\boldsymbol{\sigma} \in D(T) \cap \Sigma(\boldsymbol{u}), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\boldsymbol{u}). \tag{3.24}$$

Proof. The regularity $\boldsymbol{u} \in V$ follows from (3.10). Let $\boldsymbol{v} \in V$. Using (2.2), (2.4), (3.9)–(3.11) we have

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{\varphi}_1, \boldsymbol{v})_V + (\boldsymbol{\varphi}_2, \gamma \boldsymbol{v})_{L^2(\Gamma_2)^N} + \int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot \boldsymbol{v}\, da, \tag{3.25}$$

and using (2.1), (2.3), (3.12) and (3.19) gives

$$\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot \boldsymbol{v}\, da \geq -j(\boldsymbol{u}, \boldsymbol{v}). \tag{3.26}$$

Therefore, by (3.25), (3.26) and (3.18) we deduce

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_V. \tag{3.27}$$

The regularity $\boldsymbol{\sigma} \in D(T) \cap \Sigma(\boldsymbol{u})$ now follows from (3.8), (3.21), (3.20) and (3.27).

Moreover, from (3.12) and (3.19) we have

$$\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot \boldsymbol{u}\, da = -j(\boldsymbol{u}, \boldsymbol{u})$$

and therefore, taking $\boldsymbol{v} = \boldsymbol{u}$ in (3.25) and using again (3.18), we deduce

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \tag{3.28}$$

The inequalities (3.23) and (3.24) are now a consequence of (3.27), (3.28) and (3.20). ■

Lemma 3.1, (3.8) and (3.22) lead us to consider the following two variational problems.

Problem P_1. Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^N$ such that

$$\boldsymbol{u} \in V, \quad (F(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V. \tag{3.29}$$

PROBLEM P_2. Find a stress field $\boldsymbol{\sigma}: \Omega \to S_N$ such that

$$\boldsymbol{\sigma} \in D(T) \cap \Sigma(T(\boldsymbol{\sigma})), \quad (F^{-1}(\boldsymbol{\sigma}), \boldsymbol{\tau} - \boldsymbol{\sigma})_{\mathcal{H}} \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(T(\boldsymbol{\sigma})). \tag{3.30}$$

Problems P_1 and P_2 are formally equivalent to the mechanical problem P. Indeed, if $\boldsymbol{u}$ represents a sufficiently regular solution of P_1 and $\boldsymbol{\sigma}$ is defined by $\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u}))$, then, using the arguments of [6], it follows that $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ is a solution of problem P. Similarly, if $\boldsymbol{\sigma}$ represents a regular solution of P_2 and $\boldsymbol{u} \in V$ is given by $\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ then, using the same arguments, it follows that $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ is a solution of P. For this reason we may consider problems P_1 and P_2 as *variational formulations* of the mechanical problem P.

4. Existence and uniqueness. The main results of this section are on the existence and uniqueness for the two weak formulations P_1 and P_2. We have:

THEOREM 4.1. *Let the conditions* (3.13), (3.15)–(3.17) *hold. Then there exists* $L_0 > 0$ *depending only on* Ω, Γ_1 *and* F *such that if* $L_\nu + L_\tau < L_0$ *then there exists a unique solution* $\boldsymbol{u}$ *to problem* P_1.

THEOREM 4.2. *Let the conditions* (3.13), (3.15)–(3.17) *hold. Let* $L_0 > 0$ *be defined as in Theorem* 4.1. *If* $L_\nu + L_\tau < L_0$ *then there exists a unique solution* $\boldsymbol{\sigma}$ *to problem* P_2. *Moreover*, $\boldsymbol{\sigma} \in \mathcal{H}_1$.

The proof of Theorem 4.1 will be carried out in several steps. It is based on fixed point arguments similar to those used in [14, 15]. We suppose in the sequel that the assumptions of Theorem 4.1 are fulfilled and let $\boldsymbol{\eta} \in V$. We consider the following variational problem.

PROBLEM P_1^η. Find $\boldsymbol{u}_\eta \in V$ such that

$$\begin{aligned}(F(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta))_{\mathcal{H}} + j(\boldsymbol{\eta}, \boldsymbol{v}) - j(\boldsymbol{\eta}, \boldsymbol{u}_\eta) \\ \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\eta)_V \quad \forall \boldsymbol{v} \in V.\end{aligned} \tag{4.1}$$

We have the following result.

LEMMA 4.3. *For any* $\boldsymbol{\eta} \in V$, *problem* P_1^η *has a unique solution* $\boldsymbol{u}_\eta \in V$.

Proof. Using Riesz's representation theorem we may define an operator $A : V \to V$ by

$$(A(\boldsymbol{w}), \boldsymbol{v})_V = (F(\boldsymbol{\varepsilon}(\boldsymbol{w})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall \boldsymbol{w}, \boldsymbol{v} \in V. \tag{4.2}$$

Keeping in mind (3.13)(a) we see that A is a Lipschitz continuous operator. Using (3.13)(b) and Korn's inequality (2.5) we conclude that A is a strongly monotone operator. Moreover, by (3.19) it follows that $j(\boldsymbol{\eta}, \cdot)$ is a continuous seminorm on V. Lemma 4.3 now results from (4.2) and standard arguments of elliptic variational inequalities. ∎

Lemma 4.3 allows us to consider an operator $\Lambda : V \to V$ defined by

$$\Lambda(\boldsymbol{\eta}) = \boldsymbol{u}_\eta \quad \forall \boldsymbol{\eta} \in V. \tag{4.3}$$

LEMMA 4.4. *There exists $L_0 > 0$ depending only on Ω, Γ_1 and F such that if $L_\nu + L_\tau < L_0$ then the operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in V$.*

Proof. We use the Banach fixed point theorem. Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in V$ be given and denote the corresponding solutions of the problem (4.1) by $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Then we have $\boldsymbol{u}_1 \in V$, $\boldsymbol{u}_2 \in V$, and

$$(F(\boldsymbol{\varepsilon}(\boldsymbol{u}_1)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1))_{\mathcal{H}} + j(\boldsymbol{\eta}_1, \boldsymbol{v}) - j(\boldsymbol{\eta}_1, \boldsymbol{u}_1) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_1)_V \quad \forall \boldsymbol{v} \in V,$$
$$(F(\boldsymbol{\varepsilon}(\boldsymbol{u}_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} + j(\boldsymbol{\eta}_2, \boldsymbol{v}) - j(\boldsymbol{\eta}_2, \boldsymbol{u}_2) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_2)_V \quad \forall \boldsymbol{v} \in V.$$

We take $\boldsymbol{v} = \boldsymbol{u}_2$ in the first inequality, $\boldsymbol{v} = \boldsymbol{u}_1$ in the second, and add the two inequalities to obtain

$$\begin{aligned}
(F(\boldsymbol{\varepsilon}(\boldsymbol{u}_1)) - F(\boldsymbol{\varepsilon}(\boldsymbol{u}_2)), \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} &\\
\le j(\boldsymbol{\eta}_1, \boldsymbol{u}_2) - j(\boldsymbol{\eta}_1, \boldsymbol{u}_1) + j(\boldsymbol{\eta}_2, \boldsymbol{u}_1) - j(\boldsymbol{\eta}_2, \boldsymbol{u}_2) &\\
= \int_{\Gamma_3} (p_\nu(\eta_{1\nu} - g) - p_\nu(\eta_{2\nu} - g))(u_{2\nu} - u_{1\nu})\, da &\\
+ \int_{\Gamma_3} (p_\tau(\eta_{1\nu} - g) - p_\tau(\eta_{2\nu} - g))(|\boldsymbol{u}_{2\tau}| - |\boldsymbol{u}_{1\tau}|)\, da. &
\end{aligned}$$

Thus, using (3.13) and (3.15) we deduce

$$\begin{aligned}
m\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{\mathcal{H}}^2 &\le \int_{\Gamma_3} |\eta_{1\nu} - \eta_{2\nu}|(L_\nu|u_{1\nu} - u_{2\nu}| + L_\tau|\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}|)\, da\\
&\le (L_\nu + L_\tau) \int_{\Gamma_3} |\eta_{1\nu} - \eta_{2\nu}|\, |\boldsymbol{u}_1 - \boldsymbol{u}_2|\, da\\
&\le (L_\nu + L_\tau)\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^2(\Gamma_3)^N}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^2(\Gamma_3)^N}.
\end{aligned}$$

By the Sobolev trace theorem and the fact that $\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}$ is an equivalent norm on V, we have a constant c_0 depending only on the domain Ω and Γ_1 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^N} \le c_0\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \quad \forall \boldsymbol{v} \in V. \tag{4.4}$$

Hence,

$$m\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{\mathcal{H}}^2 \le (L_\nu + L_\tau)c_0^2\|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)\|_{\mathcal{H}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{\mathcal{H}},$$

i.e.,

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{\mathcal{H}} \le \frac{c_0^2}{m}(L_\nu + L_\tau)\|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)\|_{\mathcal{H}}.$$

Let

$$L_0 = m/c_0^2. \tag{4.5}$$

Then if $L_\nu + L_\tau < L_0$, the mapping Λ is a contraction of V. By the Banach fixed point theorem, the mapping Λ has a unique fixed point on V. ■

Proof of Theorem 4.1. Let $L_\nu + L_\tau < L_0$ and let $\boldsymbol{\eta}^*$ be the fixed point of the operator Λ. We denote by $\boldsymbol{u}^*$ the solution of the variational problem P_1^η for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. Using (4.1) and (4.3) it is straightforward to see that $\boldsymbol{u}^*$ is a solution of (3.29). This proves the existence part of Theorem 4.1. The uniqueness part follows from the uniqueness of the fixed point of the operator Λ given by (4.3). ■

Next, we present the proof of Theorem 4.2. For this we suppose in the sequel that conditions (3.13), (3.15)–(3.17) hold and let again $\boldsymbol{\eta} \in V$. We consider the following variational problem.

PROBLEM P_2^η. Find $\boldsymbol{\sigma}_\eta : \Omega \to S_N$ such that

$$\boldsymbol{\sigma}_\eta \in D(T) \cap \Sigma(\boldsymbol{\eta}), \qquad (F^{-1}(\boldsymbol{\sigma}_\eta), \boldsymbol{\tau} - \boldsymbol{\sigma}_\eta)_{\mathcal{H}} \ge 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\boldsymbol{\eta}). \tag{4.6}$$

We have the following result.

LEMMA 4.5. *For any* $\boldsymbol{\eta} \in V$, *problem* P_2^η *has a unique solution* $\boldsymbol{\sigma}_\eta$. *Moreover*, $\boldsymbol{\sigma}_\eta \in \mathcal{H}_1$.

Proof. Let $\boldsymbol{\sigma}_\eta \in D(T)$ be given by

$$\boldsymbol{\sigma}_\eta = F(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta)) \tag{4.7}$$

where $\boldsymbol{u}_\eta$ is the solution of problem P_1^η. Taking $\boldsymbol{v} = 2\boldsymbol{u}_\eta$ and $\boldsymbol{v} = \boldsymbol{0}$ in (4.1) we get

$$(\boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta))_{\mathcal{H}} + j(\boldsymbol{\eta}, \boldsymbol{u}_\eta) = (\boldsymbol{f}, \boldsymbol{u}_\eta)_V. \tag{4.8}$$

Moreover, from (4.1), (4.7) and (4.8) we deduce that

$$(\boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{\eta}, \boldsymbol{v}) \ge (\boldsymbol{f}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V,$$

which implies $\boldsymbol{\sigma}_\eta \in \Sigma(\boldsymbol{\eta})$. Using again (4.7), (3.20) and (4.8) we see that

$$(F^{-1}(\boldsymbol{\sigma}_\eta), \boldsymbol{\tau} - \boldsymbol{\sigma}_\eta)_{\mathcal{H}} = (\boldsymbol{\tau} - \boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta))_{\mathcal{H}} \ge 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\boldsymbol{\eta}),$$

which proves the existence part in Lemma 4.5. The uniqueness part follows from standard arguments since $F^{-1} : \mathcal{H} \to \mathcal{H}$ is a strongly monotone operator.

The regularity $\boldsymbol{\sigma}_\eta \in \mathcal{H}_1$ follows from $\boldsymbol{\sigma}_\eta \in \Sigma(\boldsymbol{\eta})$. Indeed, taking $\boldsymbol{v} = \pm\boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^N$ in the constraint inequality defining the set (3.20) and using (3.18), (3.19) we deduce

$$\operatorname{Div} \boldsymbol{\sigma}_\eta + \boldsymbol{\varphi}_1 = \boldsymbol{0} \quad \text{a.e. in } \Omega. \tag{4.9}$$

Then with the condition (3.16), it follows that $\boldsymbol{\sigma}_\eta \in \mathcal{H}_1$.

Proof of Theorem 4.2. Let $L_\nu + L_\tau < L_0$ and let $\boldsymbol{\eta}^*$ be the fixed point of the operator Λ defined by (4.3). We denote by $\boldsymbol{\sigma}^*$ the solution of the variational problem P_2^η for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. From (4.7) and (3.22) it follows that

$\boldsymbol{u}^* = T(\boldsymbol{\sigma}^*)$ where $\boldsymbol{u}^*$ is the solution of P_1^{η} for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. Moreover, by (4.3) we have

$$\boldsymbol{\eta}^* = T(\boldsymbol{\sigma}^*). \tag{4.10}$$

From (4.6) and (4.10) it now follows that $\boldsymbol{\sigma}^*$ satisfies (3.30), which proves the existence part of Theorem 4.2. The regularity $\boldsymbol{\sigma}^* \in \mathcal{H}_1$ follows from Lemma 4.5.

In order to prove the uniqueness part let $\boldsymbol{\sigma}^*$ be the solution of problem P_2 obtained above and let $\boldsymbol{\sigma}$ be another solution of P_2. We denote by $\boldsymbol{\eta}$ the element of V given by

$$\boldsymbol{\eta} = T(\boldsymbol{\sigma}). \tag{4.11}$$

From (3.30) and (4.11) it follows that $\boldsymbol{\sigma}$ is a solution of problem P_2^{η} and, since by Lemma 4.5 this problem has the unique solution $\boldsymbol{\sigma}_\eta$ given by (4.7), we have

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_\eta. \tag{4.12}$$

Using again (4.7) and (3.22) we get

$$\boldsymbol{u}_\eta = T(\boldsymbol{\sigma}_\eta) \tag{4.13}$$

and therefore, by (4.11)–(4.13), it follows that $\boldsymbol{u}_\eta = \boldsymbol{\eta}$. We conclude that $\boldsymbol{\eta}$ is the fixed point of the operator Λ given by (4.3), and by Lemma 4.4 it follows that

$$\boldsymbol{\eta} = \boldsymbol{\eta}^*. \tag{4.14}$$

The uniqueness part in Theorem 4.1 is now a consequence of (4.12) and (4.14). ■

As was pointed out in Section 3, problems P_1 and P_2 represent two variational formulations of the mechanical problem P. Problem P_1 is formulated in terms of displacements while problem P_2 is formulated in terms of stresses. We are interested in the relation between the solutions $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ obtained in Theorems 4.1 and 4.2, which is stated in the next result.

THEOREM 4.6. *Let the conditions* (3.13), (3.15)–(3.17) *hold*, *and assume* $L_\nu + L_\tau < L_0$.

1) *Let* $\boldsymbol{u}$ *be the solution of problem* P_1 *and* $\boldsymbol{\sigma}$ *be given by*

$$\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u})). \tag{4.15}$$

Then $\boldsymbol{\sigma}$ *belongs to* $\mathcal{H}_1$ *and* $\boldsymbol{\sigma}$ *is the solution of problem* P_2.

2) *Conversely*, *let* $\boldsymbol{\sigma}$ *be the solution of problem* P_2. *Then there exists a unique* $\boldsymbol{u} \in V$ *such that* (4.15) *holds and* $\boldsymbol{u}$ *is the solution of problem* P_1.

Proof. 1) Let $\boldsymbol{u}$ be the solution of problem P_1. It follows from the proof of Theorem 4.1 that $\boldsymbol{u}$ is the solution of problem $P_1^{\eta^*}$ where η^* is the fixed point of the operator Λ given by (4.3). Therefore, from the proof of Lemma 4.5 we deduce that $\boldsymbol{\sigma}$ given by (4.15) is a solution of $P_2^{\eta^*}$, which concludes the proof.

2) Conversely, let $\boldsymbol{\sigma}$ be the solution of P_2 and let $\boldsymbol{u} = T(\boldsymbol{\sigma}) \in V$. Using (3.22) we deduce that (4.15) holds and moreover

$$(4.16) \qquad (\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} \geq \mathbf{0} \quad \forall \boldsymbol{\tau} \in \Sigma(\boldsymbol{u}).$$

Using now the subdifferentiability of the seminorm $j(\boldsymbol{u}, \cdot)$ on V and (2.6) we deduce that there exists $\widetilde{\boldsymbol{\tau}} \in \mathcal{H}$ such that

$$(4.17) \qquad (\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V.$$

Taking $\boldsymbol{v} = 2\boldsymbol{u}$ and $\boldsymbol{v} = \mathbf{0}$ in (4.17) we obtain

$$(4.18) \qquad (\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V.$$

From (4.17), (4.18) and (3.20), it now follows that $\widetilde{\boldsymbol{\tau}} \in \Sigma(\boldsymbol{u})$. Therefore, taking $\boldsymbol{\tau} = \widetilde{\boldsymbol{\tau}}$ in (4.16) and using again (4.18) we deduce

$$(\boldsymbol{f}, \boldsymbol{u})_V \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{u}).$$

The converse inequality follows from (3.20) since $\boldsymbol{\sigma} \in \Sigma(T(\boldsymbol{\sigma}))$ and $T(\boldsymbol{\sigma}) = \boldsymbol{u}$. Therefore, we conclude that

$$(4.19) \qquad (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V.$$

Using again (3.20) we have

$$(4.20) \qquad (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_V$$

and from (4.20), (4.19) and (4.15) it results that $\boldsymbol{u}$ is a solution of P_1. ■

The mechanical interpretation of the results in Theorem 4.6 is the following.

1) If the displacement field $\boldsymbol{u}$ is the solution of the variational problem P_1, then the associated stress field $\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ is the solution of the variational problem P_2.

2) If the stress field $\boldsymbol{\sigma}$ is the solution of P_2 then there exists a displacement field $\boldsymbol{u} \in V$ associated with $\boldsymbol{\sigma}$ by the elastic constitutive law $\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ and $\boldsymbol{u}$ is the solution of P_1.

Under the assumptions of Theorems 4.1 and 4.2, we also see that if the displacement field $\boldsymbol{u}$ is the solution of P_1 and the stress field $\boldsymbol{\sigma}$ is the solution of P_2, then $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are connected by the elastic constitutive law $\boldsymbol{\sigma} = F(\boldsymbol{\varepsilon}(\boldsymbol{u}))$. For this reason we shall consider in the sequel the couple $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ given by Theorems 4.1 and 4.2 as a *weak solution* for the problem (3.8)–(3.12) and we conclude that this mechanical problem has a unique weak solution provided $L_\nu + L_\tau < L_0$.

5. Continuous dependence on contact conditions. Next, we investigate the behavior of the weak solution to the problem (3.8)–(3.12) with respect to perturbations of the normal compliance functions p_ν and p_τ. To this end, suppose that the conditions (3.13), (3.15)–(3.17) hold. For every $\alpha \geq 0$, let p_r^α be a perturbation of p_r which satisfies (3.15) with Lipschitz constant L_r^α $(r = \nu, \tau)$. Let us also introduce the functionals j^α which are obtained by replacing p_ν and p_τ by p_ν^α and p_τ^α in j, and let $\Sigma^\alpha(\boldsymbol{\eta})$ be given by (3.20) with j replaced by j^α. We now consider the following problems.

PROBLEM P_1^α. For $\alpha \geq 0$, find a displacement field $\boldsymbol{u}^\alpha \in V$ such that

$$(5.1)\qquad (F(\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha))_{\mathcal{H}} + j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{v}) - j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^\alpha)_V \quad \forall \boldsymbol{v} \in V.$$

PROBLEM P_2^α. For $\alpha \geq 0$, find a stress field $\boldsymbol{\sigma}^\alpha \in D(T) \cap \Sigma^\alpha(T(\boldsymbol{\sigma}^\alpha))$ such that

$$(5.2)\qquad (F^{-1}(\boldsymbol{\sigma}^\alpha), \boldsymbol{\tau} - \boldsymbol{\sigma}^\alpha)_{\mathcal{H}} \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma^\alpha(T(\boldsymbol{\sigma}^\alpha)).$$

We suppose that $L_\nu^\alpha + L_\tau^\alpha < L_0$. Using Theorems 4.1 and 4.2, we deduce that for each $\alpha \geq 0$, problem P_1^α has a unique solution $\boldsymbol{u}^\alpha \in V$ and problem P_2^α has a unique solution $\boldsymbol{\sigma}^\alpha \in \mathcal{H}_1$. Moreover, by Theorem 4.6 it follows that $\boldsymbol{\sigma}^\alpha$ and $\boldsymbol{u}^\alpha$ are connected by the elastic constitutive law, i.e.,

$$(5.3)\qquad \boldsymbol{\sigma}^\alpha = F(\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha)).$$

Suppose now that the normal compliance functions satisfy the following assumption: there exist $\varphi_r : \mathbb{R}_+ \to \mathbb{R}_+$ $(r = \nu, \tau)$ and $\beta > 0$ such that

(5.4) (a) $|p_r^\alpha(x,t) - p_r(x,t)| \leq \varphi_r(\alpha)|t|$ for all $t \in \mathbb{R}$, a.e. on Γ_3.
(b) $\lim_{\alpha\to 0} \varphi_r(\alpha) = 0$.
(c) $L_\nu^\alpha + L_\tau^\alpha + \beta \leq L_0$ for all $\alpha \geq 0$.

Under these assumptions we have the following result.

THEOREM 5.1. *Let* (5.4) *hold. Then*

$$(5.5)\qquad \boldsymbol{u}^\alpha \to \boldsymbol{u} \text{ in } V, \quad \boldsymbol{\sigma}^\alpha \to \boldsymbol{\sigma} \text{ in } \mathcal{H}_1, \quad \text{as } \alpha \to 0.$$

Proof. Let $\alpha \geq 0$. Using (3.29) and (5.1) we obtain

$$\begin{aligned}(F(\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha)) - F(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} &\leq j(\boldsymbol{u}, \boldsymbol{u}^\alpha) - j(\boldsymbol{u}, \boldsymbol{u}) + j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}) - j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) \\ &= \int_{\Gamma_3} (p_\nu(u_\nu - g) - p_\nu^\alpha(u_\nu^\alpha - g))(u_\nu^\alpha - u_\nu)\, da \\ &\quad + \int_{\Gamma_3} (p_\tau(u_\nu - g) - p_\tau^\alpha(u_\nu^\alpha - g))(|\boldsymbol{u}_\tau^\alpha| - |\boldsymbol{u}_\tau|)\, da.\end{aligned}$$

Thus, using (3.13), we deduce

$$m\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|^2_{\mathcal{H}} \le \int_{\Gamma_3} \{|p_\nu(u_\nu - g) - p^\alpha_\nu(u^\alpha_\nu - g)| + |p_\tau(u_\nu - g) - p^\alpha_\tau(u^\alpha_\nu - g)|\}|\boldsymbol{u}^\alpha - \boldsymbol{u}|\, da. \tag{5.6}$$

Let now $r = \nu$ or τ. Then

$$|p_r(u_\nu - g) - p^\alpha_r(u^\alpha_\nu - g)| \le |p_r(u_\nu - g) - p^\alpha_r(u_\nu - g)| + |p^\alpha_r(u_\nu - g) - p^\alpha_r(u^\alpha_\nu - g)|$$

a.e. on Γ_3. Taking into account (5.4)(a) and (3.15), we get

$$|p_r(u_\nu - g) - p^\alpha_r(u^\alpha_\nu - g)| \le \varphi_r(\alpha)|\boldsymbol{u}| + L^\alpha_r|\boldsymbol{u}^\alpha - \boldsymbol{u}| \tag{5.7}$$

a.e. on Γ_3. Combining now (5.5) and (5.6) we deduce

$$\begin{aligned} m\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|^2_{\mathcal{H}} &\le \int_{\Gamma_3} [(\varphi_\nu(\alpha) + \varphi_\tau(\alpha))|\boldsymbol{u}|\,|\boldsymbol{u}^\alpha - \boldsymbol{u}| + (L^\alpha_\nu + L^\alpha_\tau)|\boldsymbol{u}^\alpha - \boldsymbol{u}|^2]\, da \\ &\le (\varphi_\nu(\alpha) + \varphi_\tau(\alpha))\|\boldsymbol{u}\|_{L^2(\Gamma_3)^N}\|\boldsymbol{u}^\alpha - \boldsymbol{u}\|_{L^2(\Gamma_3)^N} \\ &\quad + (L^\alpha_\nu + L^\alpha_\tau)\|\boldsymbol{u}^\alpha - \boldsymbol{u}\|^2_{L^2(\Gamma_3)^N} \end{aligned}$$

and by (4.4) it follows that

$$m\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|^2_{\mathcal{H}} \le c_0^2(\varphi_\nu(\alpha) + \varphi_\tau(\alpha))\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{\mathcal{H}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|_{\mathcal{H}} + c_0^2(L^\alpha_\nu + L^\alpha_\tau)\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|^2_{\mathcal{H}}.$$

Using now (4.5) and (5.4)(c) in the previous inequality, we deduce

$$\beta\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|_{\mathcal{H}} \le (\varphi_\nu(\alpha) + \varphi_\tau(\alpha))\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{\mathcal{H}}. \tag{5.8}$$

Moreover, from (4.15), (5.3) and (3.13) it follows that

$$\|\boldsymbol{\sigma}^\alpha - \boldsymbol{\sigma}\|_{\mathcal{H}} \le M\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|_{\mathcal{H}}$$

and since $\operatorname{Div}\boldsymbol{\sigma}^\alpha = \operatorname{Div}\boldsymbol{\sigma} = -\boldsymbol{\varphi}_1$ (see (4.9)) we obtain

$$\|\boldsymbol{\sigma}^\alpha - \boldsymbol{\sigma}\|_{\mathcal{H}_1} = \|\boldsymbol{\sigma}^\alpha - \boldsymbol{\sigma}\|_{\mathcal{H}} \le M\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha - \boldsymbol{u})\|_{\mathcal{H}}. \tag{5.9}$$

The convergence result (5.5) is now a consequence of (5.7), (5.8) and (5.4)(b). ■

In addition to the mathematical interest in the result of Theorem 5.1, it is of importance in applications, as it indicates that a small perturbation in the contact condition leads to a small change in the solution.

6. Finite element approximation. In this section we study the finite element approximation of the variational problem P_1. Everywhere in the sequel c will denote a strictly positive generic constant which may depend on Ω, Γ_1 and F and whose value may vary from place to place. Let $V_h \subset V$

be a finite element subspace. Then the finite element solution of problem P_1 is $\boldsymbol{u}_h \in V_h$ which satisfies

$$(6.1)\qquad (F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_{\mathcal{H}} + j(\boldsymbol{u}_h, \boldsymbol{v}_h) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h)_V \quad \forall \boldsymbol{v}_h \in V_h.$$

Under the assumptions of Theorem 4.1 with the same value of L_0, the discrete system (6.1) has a unique solution $\boldsymbol{u}_h \in V_h$. Here we focus on error analysis of the numerical solution. We first derive a Céa's type inequality.

THEOREM 6.1. *Under the assumptions of Theorem* 4.1 *with the same value of* L_0, *for some constant* $c > 0$ *we have*

$$(6.2)\qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq c \inf_{\boldsymbol{v}_h \in V_h} \{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Gamma_3)^N} + \|F(\boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathcal{H}}^{1/2} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V^{1/2} + (\|p_\nu(u_\nu - g)\|_{L^2(\Gamma_3)} + \|p_\tau(u_\nu - g)\|_{L^2(\Gamma_3)})^{1/2} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Gamma_3)^N}^{1/2}\}.$$

Proof. First we have

$$m\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{H}}^2 \leq (F(\boldsymbol{\varepsilon}(\boldsymbol{u})) - F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h))_{\mathcal{H}} = (F(\boldsymbol{\varepsilon}(\boldsymbol{u})) - F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}_h))_{\mathcal{H}} + (F(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}))_{\mathcal{H}} + (F(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h))_{\mathcal{H}} - (F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}_h))_{\mathcal{H}}.$$

We then use (3.29) with $\boldsymbol{v} = \boldsymbol{u}_h$ and (6.1) to get

$$m\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{H}}^2 \leq (F(\boldsymbol{\varepsilon}(\boldsymbol{u})) - F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}_h))_{\mathcal{H}} + (F(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{u}_h) - j(\boldsymbol{u}, \boldsymbol{u}) + j(\boldsymbol{u}_h, \boldsymbol{v}_h) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) - (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u})_V,$$

i.e.,

$$(6.3)\qquad m\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{H}}^2 \leq R_1 + R_2 + R_3 + R_4,$$

where

$$R_1 = (F(\boldsymbol{\varepsilon}(\boldsymbol{u})) - F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}_h))_{\mathcal{H}},$$
$$R_2 = (F(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}_h) - j(\boldsymbol{u}, \boldsymbol{u}) - (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u})_V,$$
$$R_3 = j(\boldsymbol{u}, \boldsymbol{u}_h) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) + j(\boldsymbol{u}_h, \boldsymbol{u}) - j(\boldsymbol{u}, \boldsymbol{u}),$$
$$R_4 = j(\boldsymbol{u}_h, \boldsymbol{v}_h) - j(\boldsymbol{u}, \boldsymbol{v}_h) + j(\boldsymbol{u}, \boldsymbol{u}) - j(\boldsymbol{u}_h, \boldsymbol{u}).$$

Let us estimate each of the four terms. For the first term, we have

$$(6.4)\qquad |R_1| \leq M\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{H}} \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}_h)\|_{\mathcal{H}}.$$

The second term R_2 can be viewed as a residual, and by a straightforward estimation, we have

$$(6.5)\quad |R_2| \le \|F(\boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathcal{H}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}_h)\|_{\mathcal{H}} + (\|p_\nu(u_\nu-g)\|_{L^2(\Gamma_3)} + \|p_\tau(u_\nu-g)\|_{L^2(\Gamma_3)})\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(\Gamma_3)^N}.$$

Since

$$R_3 = \int_{\Gamma_3} [(p_\nu(u_\nu-g)-p_\nu(u_{h\nu}-g))(u_{h\nu}-u_\nu) + (p_\tau(u_\nu-g)-p_\tau(u_{h\nu}-g))(|\boldsymbol{u}_{h\tau}|-|\boldsymbol{u}_\tau|)]\,da,$$

we have

$$|R_3| \le \int_{\Gamma_3} [L_\nu|u_\nu-u_{h\nu}|^2 + L_\tau|u_\nu-u_{h\nu}|\,|\boldsymbol{u}_\tau-\boldsymbol{u}_{h\tau}|]\,da$$

and thus

$$(6.6)\qquad |R_3| \le c(L_\nu+L_\tau)\|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{\mathcal{H}}.$$

Similarly,

$$R_4 = \int_{\Gamma_3} [(p_\nu(u_{h\nu}-g)-p_\nu(u_\nu-g))(v_{h\nu}-u_\nu) + (p_\tau(u_{h\nu}-g)-p_\tau(u_\nu-g))(|\boldsymbol{v}_{h\tau}|-|\boldsymbol{u}_\tau|)]\,da$$

and then

$$(6.7)\quad |R_4| \le \int_{\Gamma_3} [L_\nu|u_\nu-u_{h\nu}|\,|u_\nu-v_{h\nu}| + L_\tau|u_\nu-u_{h\nu}|\,|\boldsymbol{u}_\tau-\boldsymbol{v}_{h\tau}|]\,da \le c\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Gamma_3)^N}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(\Gamma_3)^N} \le c\|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mathcal{H}}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(\Gamma_3)^N}.$$

Using the bounds (6.4)–(6.7) in (6.3) and applying the elementary inequality

$$ab \le \delta a^2 + \frac{1}{4\delta}b^2 \qquad \forall\delta>0,$$

we have

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{\mathcal{H}} \le c\{\|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}_h)\|^2_{\mathcal{H}} + \|\boldsymbol{u}-\boldsymbol{v}_h\|^2_{L^2(\Gamma_3)^N} + \|F(\boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathcal{H}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}_h)\|_{\mathcal{H}} + (\|p_\nu(u_\nu-g)\|_{L^2(\Gamma_3)} + \|p_\tau(u_\nu-g)\|_{L^2(\Gamma_3)})\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(\Gamma_3)^N}\},$$

so the inequality (6.2) holds. ■

The inequality (6.2) is the basis for convergence analysis. Indeed, we see immediately that the finite element method converges under the basic solution regularity $\boldsymbol{u} \in V$, as long as the finite element triangulation is regular and the finite element space V_h contains piecewise linear functions.

We can improve the estimate (6.2) under the regularity assumption $\boldsymbol{\sigma}_\tau \in (L^2(\Gamma_3))^N$. In this case, we can perform integration by parts to obtain

$$R_2 = \int_{\Gamma_3} [\boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_{h\tau} - \boldsymbol{u}_\tau) + p_\tau(u_\nu - g)(|\boldsymbol{v}_{h\tau}| - |\boldsymbol{u}_\tau|)]\, da.$$

Therefore we can use

$$|R_2| \le (\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^N} + \|p_\tau(u_\nu - g)\|_{L^2(\Gamma_3)})\|\boldsymbol{u}_\tau - \boldsymbol{v}_{h\tau}\|_{L^2(\Gamma_3)^N}) \tag{6.8}$$

to replace (6.5). As a result we have the following variant of Theorem 6.1.

THEOREM 6.2. *Under the assumptions of Theorem* 4.1 *with the same value of* L_0, *assume additionally* $\boldsymbol{\sigma}_\tau \in (L^2(\Gamma_3))^N$. *Then for some constant* $c > 0$, *we have*

$$\begin{aligned} \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{H}} &\le c \inf_{\boldsymbol{v}_h \in V_h} \{\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}_h)\|_{\mathcal{H}} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Gamma_3)^N} \\ &\quad + (\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^N} + \|p_\tau(u_\nu - g)\|_{L^2(\Gamma_3)})^{1/2}\|\boldsymbol{u} - \boldsymbol{v}_h\|^{1/2}_{L^2(\Gamma_3)^N}\}. \end{aligned} \tag{6.9}$$

To derive an error estimate, we need to make additional assumptions on the solution regularity. We present a sample result.

Assume

$$\boldsymbol{u} \in H^2(\Omega)^N, \quad \boldsymbol{u}|_{\Gamma_3} \in H^2(\Gamma_3)^N. \tag{6.10}$$

We use linear elements for the finite element space V_h. Let $\Pi_h \boldsymbol{u} \in V_h$ be the finite element interpolant of the solution $\boldsymbol{u}$. Then from (6.2), we obtain

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_V &\le c\{\|\boldsymbol{u} - \Pi_h\boldsymbol{u}\|_V + \|\boldsymbol{u} - \Pi_h\boldsymbol{u}\|_{L^2(\Gamma_3)^N} \\ &\quad + (\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^N} + \|p_\tau(u_\nu - g)\|_{L^2(\Gamma_3)})^{1/2}\|\boldsymbol{u} - \Pi_h\boldsymbol{u}\|^{1/2}_{L^2(\Gamma_3)^N}\}. \end{aligned}$$

The standard finite element interpolation theory yields (cf. [2])

$$\|\boldsymbol{u} - \Pi_h\boldsymbol{u}\|_V \le Ch|\boldsymbol{u}|_{H^2(\Omega)^N}, \qquad \|\boldsymbol{u} - \Pi_h\boldsymbol{u}\|_{L^2(\Gamma_3)^N} \le ch^2|\boldsymbol{u}|_{H^2(\Gamma_3)^N}.$$

Therefore, under the regularity assumption (6.10), we have the following error estimate:

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_V &\le ch(|\boldsymbol{u}|_{H^2(\Omega)^N} + |\boldsymbol{u}|_{H^2(\Gamma_3)^N} \\ &\quad + (\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^N} + \|p_\tau(u_\nu - g)\|_{L^2(\Gamma_3)})^{1/2}|\boldsymbol{u}|^{1/2}_{H^2(\Gamma_3)^N}). \end{aligned}$$

The finite element system (6.1) can be approximated by a fixed-point iteration method. This follows from a discrete analogue of the proof of Theorem 4.1. Choosing an initial guess $\boldsymbol{u}_h^0 \in V_h$, we define a sequence $\{\boldsymbol{u}_h^n\} \subset V_h$ recursively by

$$\begin{aligned} (F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1})), \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1}))_{\mathcal{H}} + j(\boldsymbol{u}_h^n, \boldsymbol{v}_h) - j(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}) \\ \ge (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^{n+1})_V \quad \forall \boldsymbol{v}_h \in V_h. \end{aligned} \tag{6.11}$$

We have the following convergence result.

THEOREM 6.3. *Under the assumptions of Theorem* 4.1 *with the same value of* L_0, *the iteration method* (6.11) *converges*:

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \to 0 \quad \text{as } n \to \infty.$$

Furthermore, *for some constant* $0 < \kappa < 1$, *we have the estimate*

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \le c\kappa^n. \tag{6.12}$$

Proof. We take $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}$ in (6.1),

$$(F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_{\mathcal{H}} + j(\boldsymbol{u}_h, \boldsymbol{u}_h^{n+1}) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) \ge (\boldsymbol{f}, \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)_V, \tag{6.13}$$

and take $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (6.11),

$$(F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1})), \boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1}))_{\mathcal{H}} + j(\boldsymbol{u}_h^n, \boldsymbol{u}_h) - j(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}) \ge (\boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{u}_h^{n+1})_V. \tag{6.14}$$

Adding (6.13) and (6.14), we obtain

$$(F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) - F(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1})), \boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1})) \le j(\boldsymbol{u}_h, \boldsymbol{u}_h^{n+1}) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) + j(\boldsymbol{u}_h^n, \boldsymbol{u}_h) - j(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}).$$

Then as in the proof of Lemma 4.4, we can derive the estimate

$$\begin{aligned}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1})\|_{\mathcal{H}} &\le \frac{c_0^2}{m}(L_\nu + L_\tau)\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h - \boldsymbol{u}_h^n)\|_{\mathcal{H}} \\ &= \frac{L_\nu + L_\tau}{L_0}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h - \boldsymbol{u}_h^n)\|_{\mathcal{H}}.\end{aligned}$$

Under the stated assumption, $\kappa \equiv (L_\nu + L_\tau)/L_0 < 1$, and we have the estimate (6.12). ■

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Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242, USA
E-mail: whan@math.uiowa.edu

Mircea Sofonea
Laboratoire de Théorie des Systèmes
Université de Perpignan
52 Avenue de Villeneuve
66 860 Perpignan, France
E-mail: sofonea@univ-perp.fr

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