APPLICATIONES MATHEMATICAE
26,3 (1999), pp. 293–298

W. DZIUBDZIELA (Kielce)
A. TOMICKA-STISZ (Częstochowa)

STOCHASTIC ORDERING OF RANDOM kTH RECORD VALUES

Abstract. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with continuous distribution function $F(x)$. Denote by $X(1,k), X(2,k), \ldots$ the kth record values corresponding to $X_1, X_2, \ldots$ We obtain some stochastic comparison results involving the random kth record values $X(N,k)$, where N is a positive integer-valued random variable which is independent of the X_i.

1. Introduction. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with continuous distribution function $F(x) = P(X_1 \le x)$. Denote by $X_{1,n} \le \ldots \le X_{n,n}$ the order statistics of $X_1, \ldots, X_n$. For a fixed integer $k \ge 1$, we define the corresponding kth *record times*, $\{L(n,k), n \ge 1\}$, and kth *record values*, $\{X(n,k), n \ge 1\}$, by setting

$$L(1,k) = k, \quad L(n+1,k) = \min\{j > L(n,k) : X_j > X_{j-k,j-1}\} \quad \text{for } n \ge 1,$$

and

$$X(n,k) = X_{L(n,k)-k+1, L(n,k)} \quad \text{for} \quad n \ge 1.$$

The study of kth record values and kth record times was initiated by Dziubdziela and Kopociński [2] (see e.g. Deheuvels [1] and Nevzorov [11] and the references therein). In the literature there are numerous results concerning partial ordering in the case of order statistics and record values. Interesting results for the latter are stated in Gupta and Kirmani [5], Kochar [10] and Kamps [6].

Let N be a positive integer-valued random variable which is independent of the X_i. The random variables $X(N,k)$ are called the *random kth record*

1991 *Mathematics Subject Classification*: Primary 60G70; Secondary 60E15.

Key words and phrases: extreme value theory, k-record values, random k-record values, random sums, hazard rate order, likelihood ratio order, stochastic comparison.

values. The exact and limit distributions of $X(N,k)$ have been studied in the literature (see Freudenberg and Szynal [3], Grudzień [4]). The purpose of the present note is to obtain stochastic comparison results involving $X(N_1,k)$ and $X(N_2,k)$, where N_1 and N_2 are two positive integer-valued random variables which are independent of the X_i. The case when $k=1$ was considered in Kirmani and Gupta [9]. We are concerned with the following stochastic relations: the likelihood ratio ordering and the hazard rate ordering. Convenient references for further properties and uses of these stochastic orders are Shaked and Shanthikumar [12], Shanthikumar and Yao [13], and the references therein.

2. Preliminaries. Let U and W be continuous random variables with densities $f_U(\cdot)$ and $f_W(\cdot)$, respectively. Let $F_U(\cdot)$ and $F_W(\cdot)$ be their respective distribution functions, and let $\overline{F}_U(\cdot)=1-F_U(\cdot)$ and $\overline{F}_W(\cdot)=1-F_W(\cdot)$ be their respective survival functions. Throughout, $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution; "increasing" and "decreasing" are used in the non-strict sense.

First recall the following definition and known results concerning the likelihood ratio ordering ($\geq_{\mathrm{lr}}$) and the hazard rate ordering ($\geq_{\mathrm{hr}}$).

We say that

$$U \geq_{\mathrm{lr}} W \quad \text{if} \quad \frac{f_U(x)}{f_W(x)} \text{ is increasing for all } x,$$
$$U \geq_{\mathrm{hr}} W \quad \text{if} \quad \frac{\overline{F}_U(x)}{\overline{F}_W(x)} \text{ is increasing for all } x.$$

It is well known (see for instance Shaked and Shanthikumar [12]) that

$$U \geq_{\mathrm{lr}} W \quad \text{implies that} \quad U \geq_{\mathrm{hr}} W.$$

The following results regarding the preservation of the order $\geq_{\mathrm{lr}}$ under monotone transformations is proved in Keilson and Sumita [8].

THEOREM 1. *If $U \geq_{\mathrm{lr}} W$ and ψ is any increasing* [*resp. decreasing*] *function, then $\psi(U) \geq_{\mathrm{lr}} [\leq_{\mathrm{lr}}]\, \psi(W)$.*

Recall that a random variable U has the PF_2 (Pólya frequency of order 2) property, denoted as $U\in \mathrm{PF}_2$, if $f_U(x)/f_U(x+a)$ is increasing in x for any given $a\geq 0$ (Karlin [7]).

The following version of the closure properties of random sums will be useful for our needs.

THEOREM 2. *Let $U_1,U_2,\ldots$ and $W_1,W_2,\ldots$ be two sequences of independent random variables such that for all i: $U_i\geq 0$ (a.s.), $U_i \geq_{\mathrm{lr}} W_i$, and either $U_i\in \mathrm{PF}_2$ or $W_i \in \mathrm{PF}_2$ (or both). For $r=1,2,\ldots$, let S_r and T_r be two independent random variables such that $S_r \stackrel{\mathrm{d}}{=} \sum_{i=1}^r U_i$ and $T_r \stackrel{\mathrm{d}}{=} \sum_{i=1}^r W_i$;*

and let f_{S_r} *and* f_{T_r} *denote their respective density functions. If for all* $u \geq v$ *and* $m \geq n$,

$$f_{S_m}(u)f_{T_n}(v) - f_{S_m}(v)f_{T_n}(u) \geq f_{S_n}(v)f_{T_m}(u) - f_{S_n}(u)f_{T_m}(v), \tag{1}$$

then

$$\sum_{i=1}^{M} U_i \geq_{\mathrm{lr}} \sum_{i=1}^{N} W_i,$$

where M *and* N *are any two positive integer-valued random variables which are independent of the* $\{U_i\}$ *and* $\{W_i\}$, *respectively*, *and* $M \geq_{\mathrm{lr}} N$.

Proof. See Shanthikumar and Yao [13].

3. Distribution of *k*th record values. Let $X_1, X_2, \ldots$ be i.i.d. random variables with common continuous distribution function $F(x) = P(X_1 \leq x)$. The function $R(x)$ defined by

$$R(x) = -\log(1 - F(x)), \quad -\infty \leq x \leq \infty,$$

is called the *hazard function* corresponding to the distribution function F.

Let $E_1, E_2, \ldots$ be i.i.d. exponential with mean one. Denote by $X_E(n,k)$ the kth record values for $\{E_n, n \geq 1\}$. The random variables $X(n,k)$ and $R^{\leftarrow}(X_E(n,k))$, where $R^{\leftarrow}(y) = \inf\{s : R(s) \geq y\}$ is the inverse of R, have the same distribution which we write as

$$X(n,k) \stackrel{\mathrm{d}}{=} R^{\leftarrow}(X_E(n,k)).$$

It is known that $\{X_E(n,k), n \geq 1\}$ are the points of a homogeneous Poisson process on $(0,\infty)$ with rate k (see, for example, Deheuvels [1]). Thus

$$P(X(n,k) \leq x) = P(R^{\leftarrow}(X_E(n,k)) \leq x) = P(X_E(n,k) \leq R(x)),$$

so that

$$P(X(n,k) \leq x) = P\Big(\sum_{j=1}^{n} Y_j \leq R(x)\Big),$$

where $Y_1, Y_2, \ldots$ are i.i.d. exponential with mean k^{-1}.

Note that (Dziubdziela and Kopociński [2])

$$P(X(n,k) \leq x) = \int_0^{kR(x)} \frac{1}{(n-1)!} u^{n-1} e^{-u}\, du.$$

Remark 1. It is easy to see that for $n \geq 1$ the kth record value $X(n,k)$ from an i.i.d. sequence with continuous distribution function $F(x)$ and the first record value $\widetilde{X}(n,1)$ from an i.i.d. sequence with distribution function $1 - [1 - F(x)]^k$ have the same distribution.

Let N be a positive integer-valued random variable which is independent of the X_n and let p_N denote the discrete density function of N. Then the distribution function of $X(N,k)$ is given by

$$P(X(N,k) \le x) = \sum_{n=1}^{\infty} P(X(n,k) \le x) p_N(n) = \sum_{n=1}^{\infty} P\Big(\sum_{j=1}^{n} Y_j \le R(x)\Big) p_N(n),$$

where $Y_1, Y_2, \ldots$ are i.i.d. exponential with mean k^{-1} which are independent of the N. Therefore

$$P(X(N,k) \le x) = P\Big(\sum_{j=1}^{N} Y_j \le R(x)\Big),$$

and

$$X(N,k) \stackrel{\mathrm{d}}{=} R^{\leftarrow}\Big(\sum_{j=1}^{N} Y_j\Big). \tag{2}$$

4. Stochastic comparisons of random kth record values. In the following theorem we obtain some comparison results involving the random kth record values.

THEOREM 3. *Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Let N_1 and N_2 be two positive integer-valued random variables which are independent of the X_i. If $k_1 \le k_2$ and $N_1 \ge_{\mathrm{lr}} N_2$, then*

$$X(N_1, k_1) \ge_{\mathrm{lr}} X(N_2, k_2). \tag{3}$$

Proof. From (2) we have

$$X(N_i, k_i) \stackrel{\mathrm{d}}{=} R^{\leftarrow}\Big(\sum_{j=1}^{N_i} Y_j^{(i)}\Big), \quad i = 1, 2, \tag{4}$$

where $Y_1^{(i)}, Y_2^{(i)}, \ldots$ are i.i.d. exponential with mean k_i^{-1} and are independent of the N_i. Note that $Y_j^{(i)} \in \mathrm{PF}_2$ for $i = 1, 2$. From $k_1 \le k_2$, it follows that $Y_j^{(1)} \ge_{\mathrm{lr}} Y_j^{(2)}$. Writing $S_r \stackrel{\mathrm{d}}{=} \sum_{i=1}^{r} Y_i^{(1)}$ and $T_r \stackrel{\mathrm{d}}{=} \sum_{i=1}^{r} Y_i^{(2)}$, it is easy to see that S_r and T_r have the Erlang distributions with parameters (r, k_1^{-1}) and (r, k_2^{-1}), respectively, and that (1) holds. Thus the conditions of Theorem 2 are satisfied, so that $\sum_{i=1}^{N_1} Y_i^{(1)} \ge_{\mathrm{lr}} \sum_{i=1}^{N_2} Y_i^{(2)}$. Finally, by (4) and Theorem 1, we obtain (3), as desired.

Remark 2. Kochar [10] and Kamps [6] have shown that record values and kth record values are partially ordered in the sense of likelihood ratio. These results are contained in Theorem 3.

The following result is proven in Kirmani and Gupta [9].

Theorem 4. *Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Let N_1 and N_2 be two positive integer-valued random variables which are independent of the X_i. If $N_1 \geq_{\mathrm{hr}} N_2$, then*

$$X(N_1, 1) \geq_{\mathrm{hr}} X(N_2, 1). \tag{5}$$

From Theorem 4 and Remark 2, we obtain the following particular variant of Theorem 3 for the hazard rate ordering.

Corollary 1. *Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Let N_1 and N_2 be two positive integer-valued random variables which are independent of the X_i. If $N_1 \geq_{\mathrm{hr}} N_2$, then*

$$X(N_1, k) \geq_{\mathrm{hr}} X(N_2, k). \tag{6}$$

References

[1] P. Deheuvels, *Strong approximations of kth records and kth record times by Wiener processes*, Probab. Theory Related Fields 77 (1988), 195–209.

[2] W. Dziubdziela and B. Kopoci/nski, *Limiting properties of the k-th record values*, Zastos. Mat. 15 (1976), 187–190.

[3] W. Freudenberg and D. Szynal, *Limit laws for a random number of record values*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), 193–199.

[4] Z. Grudzie/n, *On distributions and moments of k-th record statistics with random index*, Ann. Univ. Mariae Curie-Sk/lodowska Sect. A 33 (1979), 99–102.

[5] R. C. Gupta and S. N. U. A. Kirmani, *Closure and monotonicity properties of nonhomogeneous Poisson processes and record values*, Probab. Engrg. Inform. Sci. 2 (1988), 475–484.

[6] U. Kamps, *Reliability properties of record values from non-identically distributed random variables*, Comm. Statist. Theory Methods 23 (1994), 2101–2112.

[7] S. Karlin, *Pólya-type distributions, II*, Ann. Math. Statist. 28 (1957), 281–308.

[8] J. Keilson and U. Sumita, *Uniform stochastic ordering and related inequalities*, Canad. J. Statist. 10 (1982), 181–198.

[9] S. N. U. A. Kirmani and R. C. Gupta, *Some results on randomly stopped minimal repair processes*, Comm. Statist. Stochastic Models 11 (1995), 631–644.

[10] S. C. Kochar, *Some partial ordering results on record values*, Comm. Statist. Theory Methods 19 (1990), 299–306.

[11] V. B. Nevzorov, *Records*, Theory Probab. Appl. 32 (1987), 201–228.

[12] M. Shaked and J. G. Shanthikumar, *Stochastic Orders and Their Applications*, Academic Press, New York, 1994.

[13] J. G. Shanthikumar and D. D. Yao, *Bivariate characterization of some stochastic order relations*, Adv. Appl. Probab. 23 (1991), 642-659.

Wiesław Dziubdziela
University of Commerce
Peryferyjna 15
25-562 Kielce, Poland
E-mail: wieslaw@wsh-kielce.edu.pl

Agata Tomicka-Stisz
Institute of Mathematics and Computer Science
Częstochowa Technical University
Dąbrowskiego 73
42-201 Częstochowa, Poland

Received on December 7, 1998