APPLICATIONES MATHEMATICAE
26,3 (1999), pp. 357–362

M. SABATINI (Trento)

QUADRATIC ISOCHRONOUS CENTERS COMMUTE

Abstract. We prove that every quadratic plane differential system having an isochronous center commutes with a polynomial differential system.

1. Introduction. Consider an autonomous differential system in the plane

(S) $$\begin{cases} \dot{x} = F(x, y), \\ \dot{y} = G(x, y), \end{cases}$$

with $(x, y) \in U$, an open connected subset of $\mathbb{R}^2$, and $F, G \in C^2(U, \mathbb{R})$. An isolated critical point O of (S) is said to be a *center* if every orbit in a punctured neighbourhood of O is a nontrivial cycle. It is said to be an *isochronous center* if every cycle in a neighbourhood of O has the same period.

The problem of determining whether a critical point is a center or not has been studied by several authors (see [NS, SC, C]). The related problem of determining whether a center is isochronous or not has attracted less attention (see [NS, SC, MRT]), but a significant number of papers appeared also on this subject. The most studied cases are systems equivalent to second order scalar differential equations, and some classes of polynomial systems. Even in these cases, a complete solution of the isochronicity problem is avalaible only for special subclasses. For instance, this is the case of Liénard differential equations

(L) $$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

1991 *Mathematics Subject Classification*: Primary 34C05.

Key words and phrases: isochronous center, quadratic polynomial system, commuting vector field.

In this paper we collect results contained in the preprint [S2], presented at the Equadiff 95 meeting, Lisboa [S3]. This research has been partially supported by the M.U.R.S.T. group "Equazioni differenziali ordinarie ed applicazioni".

with f and g odd functions [AFG, CDL, S1]. The equations of type (L) contain, as a special subclass ($f(x) \equiv 0$), odd conservative second order equations, studied in [O, U]. Another class of second order conservative equations for which the isochronicity problem has been solved is that of polynomial equations [CJ]. Also, a classification of isochronous centers exists for quadratic polynomial systems [L] and for odd cubic polynomial systems [P].

Methods applied in the study of isochronous centers range from linearizations [MRT] to reduction to second order differential equations [L], computation of isochronicity constants [GGM1], and the commutators method [V, S4]. In this paper we focus on the so-called commutators method, which relies on the following theorem. We recall that two plane differential systems *commute* if their Lie brackets vanish. In this case the systems are said to be *commutators* of each other.

Theorem [V, S4]. *Let O be a center of* (S). *Then O is isochronous if and only if there exists a second differential system* (S^{T}), *defined in a neighbourhood of O, such that* (S^{T}) *is transversal to* (S) *at nonsingular points and commutes with* (S).

In general, finding commutators is not trivial. Given the differential system (S), looking for a commutator (S^{T}) is equivalent to solving a couple of partial differential equations in two unknowns, the components of (S^{T}). So far, no results have been obtained by actually solving a system of partial differential equations. On the other hand, several classes of commutators have been found, and new classes of isochronous systems have been discovered, by applying computer algebra methods to polynomial systems [S2, CGG1, CGG2, CGG3]. Here we describe the first application of this method to the study of plane systems: we show that every quadratic plane system having an isochronous center has a polynomial commutator. In this way, we give a different proof of the isochronicity of Loud's centers. On the other hand, even if the existence of a commutator is a necessary and sufficient condition for a center to be isochronous, the method applied here does not seem to be useful to prove that Loud's centers are the unique isochronous quadratic centers. This is due to the fact that polynomial isochronous centers do not necessarily have polynomial commutators, as shown by Devlin's example [D]. This raises the problem of determining under which conditions a polynomial system admits a polynomial commutator. However, we shall not consider this problem here.

2. Commuting systems. In this section, for every class of quadratic isochronous centers, we give a family of commuting systems. Computations were performed using Maple, version 4.2.1, on an Apple Macintosh IIci. The method used is as follows. Let (S_{Q}) be a quadratic system with an isochronous center at the origin O. After a linear change of variables, it appears in the following form:

$$(\mathrm{S_Q}) \qquad \begin{cases} \dot{x} = -y + p_2(x,y), \\ \dot{y} = \ \ x + q_2(x,y), \end{cases}$$

with $p_2(x,y)$ and $q_2(x,y)$ homogeneous polynomials of degree two [L]. We consider a second polynomial system with indeterminate coefficients, a candidate to be a commuting system. We impose the commutativity condition that gives a system of linear equations in its coefficients. Then we try to solve such equations.

We look for commutators of minimal degree, so that we start with a quadratic candidate commutator:

$$(\mathrm{S_2^T}) \qquad \begin{cases} \dot{x} = x + r_2(x,y), \\ \dot{y} = y + s_2(x,y). \end{cases}$$

Here $r_2(x,y)$ and $s_2(x,y)$ are homogeneous polynomials of degree two. The linear part of $(\mathrm{S_2^T})$ is chosen in order to have transversality of $(\mathrm{S_Q})$ and $(\mathrm{S_2^T})$ at least in a neighbourhood of the origin, which is what we need in order to apply the theorem quoted in the introduction. This ensures the transversality in the whole central region [S5]. Then we compute the commutator of $(\mathrm{S_Q})$ and $(\mathrm{S_2^T})$. It is a vector with polynomial components of degree at most 3. We equate to zero the coefficients of such polynomials. This yields a system of linear equations having the coefficients of $(\mathrm{S_2^T})$ as unknowns. They are usually very simple, and can be even further simplified by successive substitutions. Sometimes this yields incompatible equations. In such cases we look for commuting systems of higher degree, cubic or quartic. In every case we find lowest degree systems of the type

$$\begin{cases} \dot{x} = x + r_2(x,y) + r_3(x,y) + r_4(x,y), \\ \dot{y} = y + s_2(x,y) + s_3(x,y) + s_4(x,y), \end{cases}$$

commuting with the given ones.

The same procedure has been applied to the study of odd cubic systems, providing polynomial commutators of degree up to five [GGM2, MS]. In general, it is not known how to give an *a priori* estimate of the degree of a polynomial commutator.

We refer to the classification of quadratic isochronous centers given in [L]. Loud proved that a (nonlinear) quadratic system has an isochronous center at O if and only if there exists a linear transformation taking the system to a system of the following form:

$$\begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases}$$

with $B \neq 0$, and the ratios D/B and F/B assuming one of the following pairs of values: $(0,1)$, $(0,1/4)$, $(-1/2,2)$, $(-1/2,1/2)$. All such systems are reversible.

For every quadratic system, the commutator we give is the infinitesimal generator of a Lie symmetry. In some cases, this allows us to find invariant curves of the quadratic system. In all cases but the last one, we give an invariant curve.

Case $(0,1)$. The system can be written as follows:

$$(\mathrm{S}_{(0,1)}) \qquad \begin{cases} \dot{x} = -y + Bxy = -y(1-Bx), \\ \dot{y} = x + By^2 = x + By^2. \end{cases}$$

It commutes with the following system:

$$(\mathrm{S}^{\mathrm{T}}_{(0,1)}) \qquad \begin{cases} \dot{x} = x - Bx^2 = x(1-Bx), \\ \dot{y} = y - Bxy = y(1-Bx). \end{cases}$$

$(\mathrm{S}_{(0,1)})$ has only one critical point, $(0,0)$, while $(\mathrm{S}^{\mathrm{T}}_{(0,1)})$ has a critical point, $(0,0)$, and a critical line of equation $Bx-1=0$. This line is invariant for $(\mathrm{S}_{(0,1)})$.

Case $(0,1/4)$. The system can be written as follows:

$$(\mathrm{S}_{(0,1/4)}) \qquad \begin{cases} \dot{x} = -y + Bxy = -y(1-Bx), \\ \dot{y} = x + By^2/4 = x + By^2/4. \end{cases}$$

It commutes with the following system:

$$(\mathrm{S}^{\mathrm{T}}_{(0,1/4)}) \qquad \begin{cases} \dot{x} = x - Bx^2/2 + By^2/4 + B^3y^4/32, \\ \dot{y} = y - Bxy/2 + B^2y^3/8. \end{cases}$$

Computations show that it is not possible to find polynomial systems of degree lower than 4, commuting with $(\mathrm{S}_{(0,1/4)})$. The origin is the unique critical point of $(\mathrm{S}_{(0,1/4)})$. $(\mathrm{S}^{\mathrm{T}}_{(0,1/4)})$ has the same critical point, and a critical parabola of equation $B^2y^2 - 4Bx + 8 = 0$. This parabola is invariant for $(\mathrm{S}_{(0,1/4)})$.

Case $(-1/2,2)$. The system can be written as follows:

$$(\mathrm{S}_{(-1/2,2)}) \qquad \begin{cases} \dot{x} = -y + Bxy = -y(1-Bx), \\ \dot{y} = x - Bx^2/2 + 2By^2 = x(1-Bx/2) + 2By^2. \end{cases}$$

It commutes with the following system:

$$(\mathrm{S}^{\mathrm{T}}_{(-1/2,2)}) \qquad \begin{cases} \dot{x} = x - 3Bx^2/2 + B^2x^3/2 = x(1-Bx)(1-Bx/2), \\ \dot{y} = y - 2Bxy + B^2x^2y = y(1-Bx)^2. \end{cases}$$

Computations show that it is not possible to find polynomial systems of degree lower than 3, commuting with $(\mathrm{S}_{(-1/2,2)})$. $(\mathrm{S}_{(-1/2,2)})$ has two critical points, $(0,0)$, and $(2/B,0)$, both centers. $(\mathrm{S}^{\mathrm{T}}_{(-1/2,2)})$ has the same critical points and a critical line of equation $Bx-1=0$. This line is invariant for

$(S_{(-1/2,2)})$. The vector field of $(S_{(-1/2,2)})$ is symmetric with respect to that line, hence the centers have the same period.

CASE $(-1/2, 1/2)$. The system can be written as follows:

$$(S_{(-1/2,1/2)}) \qquad \begin{cases} \dot{x} = -y + Bxy = -y(1 - Bx), \\ \dot{y} = x - Bx^2/2 + By^2/2 = x(1 - Bx/2) + By^2/2. \end{cases}$$

Its components are a couple of conjugate harmonic functions, so that it commutes with its orthogonal system [V]:

$$(S^{T}_{(-1/2,1/2)}) \qquad \begin{cases} \dot{x} = x - Bx^2/2 + By^2/2 = x(1 - Bx/2) + By^2/2, \\ \dot{y} = y - Bxy = y(1 - Bx). \end{cases}$$

$(S_{(-1/2,1/2)})$ and $(S^{T}_{(-1/2,1/2)})$ have the same two critical points, $(0, 0)$ and $(2/B, 0)$. They are both centers for $(S_{(-1/2,1/2)})$.

References

[AFG] A. Algaba, E. Freire and E. Gamero, *Isochronicity via normal form*, preprint.

[C] R. Conti, *Centers of polynomial systems in R^2*, preprint, Firenze, 1990.

[CDL] C. J. Christopher, J. Devlin and N. G. Lloyd, *On the classification of Liénard systems with amplitude-independent periods*, preprint.

[CGG1] J. Chavarriga, J. Giné and I. García, *Isochronous centers of cubic systems with degenerate infinity*, Differential Equations Dynam. Systems 7 (1999), to appear.

[CGG2] —, —, —, *Isochronous centers of a linear center perturbed by fourth degree homogeneous polynomials*, Bull. Sci. Math. 123 (1999), 77–96.

[CGG3] —, —, —, *Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomials*, preprint, Univ. de Lleida.

[CJ] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields*, Trans. Amer. Math. Soc. 312 (1989), 433–486.

[D] J. Devlin, *Coexisting isochronous and nonisochronous centers*, Bull. Lond. Math. Soc. 28 (1996), 495–500.

[GGM1] A. Gasull, A. Guillamon and V. Mañosa, *An explicit expression of the first Lyapunov and period constants with applications*, J. Math. Anal. Appl. 211 (1997), 190–212.

[GGM2] —, —, —, *Centre and isochronicity conditions for systems with homogeneous nonlinearities*, in: Proc. 2nd Catalan Days on Appl. Math., Collect. Études, Presses Univ. Perpignan, Perpignan, 1995, 105–116.

[L] W. S. Loud, *Behavior of the period of solutions of certain plane autonomous systems near centers*, Contrib. Differential Equations 3 (1964), 21–36.

[MRT] P. Mardešić, C. Rousseau and B. Toni, *Linearization of isochronous centers*, J. Differential Equations 121 (1995), 67–108.

[MS] L. Mazzi and M. Sabatini, *Commutators and linearizations of isochronous centers*, preprint UTM 482, Univ. of Trento, 1996.

[NS] V. V. Nemytskiĭ and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton Univ. Press, Princeton, NJ, 1960.

[O] Z. Opial, *Sur les périodes des solutions de l'équation différentielle* $x'' + g(x) = 0$, Ann. Polon. Math. 10 (1961), 49–72.
[P] I. I. Pleshkan, *A new method of investigating the isochronicity of a system of two differential equations*, Differential Equations 5 (1969), 796–802.
[S1] M. Sabatini, *On the period function of Liénard systems*, J. Differential Equations 152 (1999), 467–487.
[S2] —, *Quadratic isochronous centers commute*, preprint UTM 461, Univ. of Trento, 1995.
[S3] —, *Qualitative analysis of commuting flows on two-dimensional manifolds*, in: EQUADIFF 95—International Conf. on Differential Equations (Lisboã, 1995), L. Magalhaes, C. Rocha and L. Sanchez (eds.), World Sci., Singapore, 1998, 494–497.
[S4] —, *Characterizing isochronous centers by Lie brackets*, Differential Equations Dynam. Systems 5 (1997), 91–99.
[S5] —, *Dynamics of commuting systems on two-dimensional manifolds*, Ann. Mat. Pura Appl. (4) 173 (1997), 213–232.
[SC] G. Sansone e R. Conti, *Equazioni differenziali non lineari*, Cremonese, Roma, 1956.
[U] M. Urabe, *Potential forces which yield periodic motions of a fixed period*, J. Math. Mech. 10 (1961), 569–578.
[V] M. Villarini, *Regularity properties of the period function near a center of a planar vector field*, Nonlinear Anal. 19 (1992), 787–803.

M. Sabatini
Dipartimento di Matematica
Università di Trento
I-38050 Povo (TN), Italy
E-mail: sabatini@science.unitn.it

Received on 7.4.1999