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# NONSTATIONARY MARANGONI CONVECTION

**Introduction.** In this paper we are concerned with a free boundary problem for the Navier–Stokes system.

Imagine a volume of fluid bounded by a free surface with surface tension. As an example we may think of a drop of a molten substance in outer space. The fluid inside is driven by the Navier–Stokes equations. We also have a temperature distribution inside, described by the heat equation. The only coupling of the fluid with the temperature is via the surface tension, which is assumed to be temperature dependent. Fluctuations in the surface tension cause tangential stress and lead to an onset of motion inside. This is what we call *Marangoni convection*.

The boundary of the drop is also an unknown. Its shape is determined by the stress tensor and the variable surface tension, and it moves with the fluid.

We will be concerned with the following system:

$$\begin{aligned} \partial_t v - \mathrm{Pr}\,\Delta v + v\cdot\nabla v + \nabla p &= f,\\ \nabla\cdot v &= 0,\\ \partial_t\theta - \Delta\theta + v\cdot\nabla\theta &= g \end{aligned}$$

in $\bigcup_{0\le t\le T}\Omega_t\times\{t\}$ together with the boundary conditions

$$\begin{aligned} T(v,p)\nu - \mathrm{Ma}\ \mathrm{Pr}\,\nabla\theta &= 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,H\nu,\\ v\cdot\nu &= \partial_t\eta,\\ \nu\cdot\nabla\theta &= h \end{aligned}$$

in $\bigcup_{0\le t\le T}\partial\Omega_t\times\{t\}$ and initial values $v(\cdot,0)=v_0$, $\theta(\cdot,0)=\theta_0$ and $\eta(\cdot,0)=\eta_0$.

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NOTATION. $v$ denotes the velocity field, $p$ the pressure of the fluid, $\theta$ the temperature, and $\eta$ represents the boundary of the domain, written as a graph over $S^2$: $\partial\Omega_t = \{(\xi, 1+\eta(\xi,t)) : \xi \in S^2\}$; $\nu$ will always denote the outer normal vector on $\partial\Omega_t$. These are the quantities which have to be determined by the above system. Moreover,

$$T(v,p)_{ij} = -p\delta_{ij} + \mathrm{Pr}(\partial_i v^j + \partial_j v^i)$$

denotes the stress tensor, whose divergence is the Stokes operator. $H$ denotes the mean curvature of $\partial\Omega_t$.

The forces are denoted by $f$, $g$ and $h$. They will always have to satisfy the compatibility condition

$$\int_{\Omega_t} f\,dx = 0, \qquad \int_{\Omega_t} g\,dx = \int_{\partial\Omega_t} h\,dS$$

for all time.

The system is written in dimensionless form.

The *Prandtl number* Pr measures the importance of diffusion relative to heat conductivity. In low Prandtl number fluids, heat diffuses significantly faster than vorticity, a typical situation in a liquid or in molten metals.

The *Marangoni number* Ma gives the ratio of surface tension tractions generated by temperature inhomogeneities at the surface to the dissipation and heat conduction. The word "temperature" may be replaced by "chemical concentration". Fixing dissipation and heat conduction implies, in the case of low Ma numbers, that the surface tension changes only a little if the temperature changes.

The meaning of the *Crispation number* Cr can be seen if it tends to zero. This corresponds to the case of a stress free surface. The number always appears as the inverse $\mathrm{Cr}^{-1}$.

The equations in the interior of the unknown domain represent the transport of momentum, of mass and of internal energy. We always assume the density to be constant; as a consequence, the Navier–Stokes equations do not contain any buoyancy terms.

The boundary conditions consist of one vector equation and two scalar equations. The vector equation is a balance equation for the stress tensor $T(v,p)\nu$. In its simplest form ($v = \theta = 0$) it expresses the fact that the mean curvature of an interface is determined by the pressure difference in the two media (Laplace law).

The first scalar equation is a transport equation for the free boundary. To see this assume for a moment that the free boundary is represented by the level set of a function $\Psi$: $\partial\Omega_t = \{(x,t) : \Psi(x,t) = 0\}$. Then the motion of $\Psi$ is given by the transport equation

$$\partial_t\Psi + v\cdot\nabla\Psi = 0.$$

Since $\nabla\Psi$ points in the normal direction, the difference between the two equations is the normalisation: $\nu = \nabla\Psi/|\nabla\Psi|$. However, since we will only work locally in space and time the difference does not affect the result.

The second scalar equation is a Neumann condition for the heat equation.

We wish to use energy methods to prove existence of a solution. However, since we do not have any information about the free boundary a priori, tools like Green's formula or Korn's inequality cannot be applied. To get around this difficulty we have to parametrise all quantities over a known smooth manifold. In particular the free boundary will be a graph over this reference manifold. As a consequence, we will only obtain results that are local in space and time: the free boundary might loose its property of being a graph after short time.

We proceed as follows: We assume that for given forces we have found a solution. We linearise around this solution and obtain estimates for the corresponding stationary problem. Existence and regularity for the stationary problem will be the central point of the paper. With the help of the method of Rothe these estimates carry over to estimates for the nonstationary linear case. The nonlinear problem is then solved with the help of a fixed point argument. While the solution of the linear nonstationary problem exists for all time, we only expect to get short time existence in the nonlinear case: The nonlinearity of the Navier–Stokes operator may cause singularities for the velocity field in finite time and the transformation from the reference manifold to the free boundary may loose its bijectivity after short time. We do not know what happens first.

Our work is closely related to that of V. A. Solonnikov. In a series of papers he considered the incompressible nonstationary Navier–Stokes equations as a free boundary problem with constant surface tension and without surface tension. In [Sol1] the solvability of that problem (without surface tension) on a finite time interval has been established in the class of Hölder spaces. In [Beale1] and [Beale2] a related problem describing the unsteady motion of a fluid over an infinite bottom has been described, both without surface tension in [Beale1] and with surface tension in [Beale2].

In [Sol2] the problem with surface tension was investigated and the existence of a solution on a finite time interval was proved in Sobolev spaces.

In [Sol3] the long time behaviour of the above problem was studied under the condition that there are no external forces and that the initial data are close to the equilibrium data. For the two-dimensional case it can be shown that the limiting domain as $t \to \infty$ is a circle [Sol4]. In [Sol5] forces between particles were added to the system, again with constant surface tension. The unique solvability was proved for a finite time interval.

In later work with A. Tani the compressibility condition was dropped and similar results to those mentioned above were obtained (see e.g. [Sol&Tan]).

Heat conducting fluids with and without surface tension were considered by Zadrzyńska and Zajączkowski in a series of papers [ZZ1]–[ZZ7].

The techniques in most of those papers are based on the transformation to lagrangian coordinates, while we use the eulerian coordinates.

## 1. The model

**1.1.** We consider the nonstationary model for the Marangoni convection

$$\partial_t v - \mathrm{Pr}\,\Delta v + v\cdot\nabla v + \nabla p = f, \tag{1}$$

$$\nabla\cdot v = 0, \tag{2}$$

$$\partial_t\theta - \Delta\theta + v\cdot\nabla\theta = g \tag{3}$$

in $\bigcup_{0\le t\le T}\Omega_t\times\{t\}$, together with the boundary conditions

$$T(v,p)\nu - \mathrm{Ma}\,\mathrm{Pr}\,\nabla\theta = 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,H\nu, \tag{4}$$

$$v\cdot\nu = \partial_t\eta, \tag{5}$$

$$\nu\cdot\nabla\theta = h \tag{6}$$

in $\bigcup_{0\le t\le T}\partial\Omega_t\times\{t\}$.

The surface moves as the fluid starts to move. This is indicated by the subscript in $\Omega_t$ and $\partial\Omega_t$.

As initial data we choose $\eta(0)=0, v(0)=0$ and $\theta(0)=0$ in $\Omega(0)=\Omega_0$.

We look for solutions close to the equilibrium given by zero forces. This is not the most general case. We could ask for solutions close to any known stationary or nonstationary solution given at $t=0$. However, this would require a rather complicated discussion of "compatibility conditions". We wish to avoid this and we will assume that all the forces vanish together with all time derivatives at $t=0$.

**1.2.** We assume we have found a smooth stationary solution $\widehat{v},\widehat{\theta},\widehat{\eta}$. Linearisation around this solution gives

$$\partial_t v - \mathrm{Pr}\,\Delta v + \widehat{v}\cdot\nabla v + v\cdot\nabla\widehat{v} + \nabla p = f, \tag{7}$$

$$\nabla\cdot v = 0, \tag{8}$$

$$\partial_t\theta - \Delta\theta + \widehat{v}\cdot\nabla\theta + v\cdot\nabla\widehat{\theta} = g \tag{9}$$

in $\widehat{\Omega}\times(0,T)$, together with the boundary conditions

$$T(v,p)\widehat{\nu} - \mathrm{Ma}\,\mathrm{Pr}\,\nabla\theta = 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,\widetilde{H}(\eta)\widehat{\nu}, \tag{10}$$

$$v\cdot\widehat{\nu} = \partial_t\eta, \tag{11}$$

$$\widehat{\nu}\cdot\nabla\theta = h \tag{12}$$

in $\partial\widehat{\Omega}\times(0,T)$, with zero initial values. $\widetilde{H}(\eta)$ denotes the linearised mean curvature operator on $\partial\widehat{\Omega}$:

$$\widetilde{H}(\eta) = H(\widehat{\eta}) + \Delta^*\eta - 2\eta$$

where $\Delta^*$ denotes the Laplace–Beltrami operator on $\partial\widehat{\Omega}$, $H(\widehat{\eta})$ is the mean curvature of $\partial\widehat{\Omega} = \{(\xi, 1+\widehat{\eta}(\xi,t)) : \xi \in S^2\}$ and $\widehat{\nu}$ is the outer normal vector on $\partial\widehat{\Omega}$.

We wish to point out again that the linearised system is now defined on a space time cylinder $\widehat{\Omega} \times (0,T)$ where $\widehat{\Omega}$ is a smooth domain.

**1.3.** We introduce the spaces we will work in. Their properties are extensively described in [L&M], thus we only give the most important ones. For $r \geq 0$ we define

$$K^r(\Omega \times (0,T)) = H^{r/2,2}((0,T), H^{0,2}(\Omega)) \cap H^{0,2}((0,T), H^{r,2}(\Omega)).$$

We recall some properties:

$$\begin{aligned} \partial_x^\alpha \partial_t^k u &\in K^{r-|\alpha|-2k}(\Omega \times (0,T)), && |\alpha| + 2k < r, \\ \partial_x^\alpha \partial_t u|_{\partial\Omega} &\in K^{r-|\alpha|-1/2}(\partial\Omega \times (0,T)), && \alpha + 1/2 < r, \\ \partial_t^k u(\cdot,0) &\in H^{r-2k-1}(\Omega), && 2k+1 < r \end{aligned}$$

(see [L&M], Prop. 4.2.3, Theorem 4.2.1). The corresponding extension theorems also hold.

Furthermore we have an imbedding

$$K^r(\Omega \times (0,T)) \hookrightarrow C^k(0,T; H^{r-2k-1,2}(\Omega)), \qquad 2k < r-1.$$

**1.4.** We decompose our system: $v = v_1 + v_2$, where $(v_1, q, \theta)$ and $(v_2, p - q)$ solve two different systems:

$$\partial_t v_1 - \operatorname{Pr}\Delta v_1 + \widehat{v}\cdot\nabla v_1 + v_1\cdot\nabla\widehat{v} + \nabla q = f, \tag{13}$$

$$\nabla\cdot v_1 = 0, \tag{14}$$

$$\partial_t\theta - \Delta\theta + \widehat{v}\cdot\nabla\theta + (v_1+v_2)\cdot\nabla\widehat{\theta} = g, \tag{15}$$

in $\widehat{\Omega} \times (0,T)$, together with the boundary conditions

$$\widehat{\tau}_i \cdot T(v_1,q)\widehat{\nu} - \operatorname{Ma}\operatorname{Pr}\widehat{\tau}_i\cdot\nabla\theta = 0, \qquad i = 1,2, \tag{16}$$

$$v_1\cdot\widehat{\nu} = 0, \tag{17}$$

$$\widehat{\nu}\cdot\nabla\theta = h \tag{18}$$

in $\partial\widehat{\Omega} \times (0,T)$, and

$$\partial_t v_2 - \operatorname{Pr}\Delta v_2 + \widehat{v}\cdot\nabla v_2 + v_2\cdot\nabla\widehat{v} + \nabla(p-q) = 0, \tag{19}$$

$$\nabla\cdot v_2 = 0 \tag{20}$$

in $\widehat{\Omega} \times (0,T)$, together with the boundary conditions

$$\widehat{\tau}_i\cdot T(v_2, p-q)\widehat{\nu} = 0, \qquad i = 1,2, \tag{21}$$

$$v_2 \cdot \widehat{\nu} = \partial_t \eta, \tag{22}$$

$$\widehat{\nu} \cdot T(v_2, p-q)\widehat{\nu} - \mathrm{Ma}\,\mathrm{Pr}\, h = -\widehat{\nu} \cdot T(v_1, q)\widehat{\nu} + 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,\widetilde{H}(\eta) \tag{23}$$

in $\partial\widehat{\Omega} \times (0,T)$.

The first system (13)–(18) of this decomposition simply contains the Navier–Stokes equations and the heat equation in the domain $\widehat{\Omega} \times (0,T)$. They are coupled via the boundary condition (16), where we find the "Marangoni term" $\mathrm{Ma}\,\mathrm{Pr}\,\widehat{\tau}_i \cdot \nabla\theta$. The analysis of the first system will concentrate on the behaviour of this term.

The second system (19)–(20) consists of the Navier–Stokes system in $\widehat{\Omega} \times (0,T)$ with four (!) boundary conditions in $\partial\widehat{\Omega} \times (0,T)$, containing the unknown graph of the free boundary. Thus we may try to use one boundary equation as an equation for the graph.

Both systems satisfy the complementing boundary conditions as formulated in [ADN] while the system (7)–(12) does not.

## 2. Stationary estimates

**2.1.** A prerequisite for estimates for the two systems are estimates for the two corresponding stationary systems:

$$\lambda v_1 - \mathrm{Pr}\,\Delta v_1 + \widehat{v} \cdot \nabla v_1 + v_1 \cdot \nabla \widehat{v} + \nabla q = f, \tag{24}$$

$$\nabla \cdot v_1 = 0, \tag{25}$$

$$\lambda\theta - \Delta\theta + \widehat{v} \cdot \nabla\theta + (v_1 + v_2) \cdot \nabla\widehat{\theta} = g \tag{26}$$

in $\widehat{\Omega}$, with the boundary conditions

$$\widehat{\tau}_i \cdot T(v_1, q)\widehat{\nu} - \mathrm{Ma}\,\mathrm{Pr}\,\widehat{\tau}_i \cdot \nabla\theta = 0, \qquad i = 1,2, \tag{27}$$

$$v_1 \cdot \widehat{\nu} = 0, \tag{28}$$

$$\widehat{\nu} \cdot \nabla\theta = h \tag{29}$$

in $\partial\widehat{\Omega}$, and

$$\lambda v_2 - \mathrm{Pr}\,\Delta v_2 + \widehat{v} \cdot \nabla v_2 + v_2 \cdot \nabla\widehat{v} + \nabla(p-q) = 0, \tag{30}$$

$$\nabla \cdot v_2 = 0 \tag{31}$$

in $\widehat{\Omega}$, with the boundary conditions

$$\widehat{\tau}_i \cdot T(v_2, p-q)\widehat{\nu} = 0, \qquad i = 1,2, \tag{32}$$

$$v_2 \cdot \widehat{\nu} = \lambda(\eta + \widetilde{h}), \tag{33}$$

$$\widehat{\nu} \cdot T(v_2, p-q)\widehat{\nu} - \mathrm{Ma}\,\mathrm{Pr}\, h = -\widehat{\nu} \cdot T(v_1, q)\widehat{\nu} + 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,\widetilde{H}(\eta) \tag{34}$$

in $\partial\widehat{\Omega}$. Here $\widetilde{h}$ is a prescribed function on $\partial\widehat{\Omega}$ with $\int_{\partial\widehat{\Omega}} \widetilde{h}\, d\widehat{S} = 0$ and $\lambda \geq 0$. The introduction of this function will become clear when we consider the nonstationary problem.

Recalling the definition of $\widetilde{H}(\eta)$ we consider equation (34) as a Laplace equation for $\eta$ on the closed surface $\partial\widehat{\Omega}$:

$$-2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,\Delta^*\eta + 2\eta = -\widehat{\nu}\cdot T(v,p)\widehat{\nu} + \mathrm{Ma}\,\mathrm{Pr}\,h.$$

Conditions for the solvability of such an equation can be found in [Aubin]; see in particular Theorem 4.7, p. 104. Observe that the right hand side of the above equation can always be modified to have zero mean value, since the pressure is only determined up to a constant. $\eta$ also has mean value zero as we will see below. Thus there are no extra compatibility conditions coming up.

**2.2.** In what follows, the bilinear form

$$K(v,v) \equiv \int\limits_{\widehat{\Omega}} D(v) : D(v)\,dx = \sum_{i,j}\int\limits_{\widehat{\Omega}} (\partial_i v^j + \partial_j v^i)(\partial_i v^j + \partial_j v^i)\,dx$$

will play a crucial role. It was studied by many authors, e.g. [Sol&Shch], [Bem1], [Olej] and others.

Two facts will be used in the sequel:

1) If $\widehat{\Omega}$ is rotationally symmetric w.r.t. some axis $\beta$ lying in $\widehat{\Omega}$, then the bilinear form has a kernel, consisting of all rotations around this axis. Thus we make the following definition.

Definition 2.1. Let $L(\widehat{\Omega})$ denote the closure of the divergence free $C^\infty$-vector fields with vanishing normal component on the boundary, with respect to the $H^{1,2}$-norm.

If the domain is rotationally symmetric define $L^\perp(\widehat{\Omega}) \equiv L(\widehat{\Omega})/\{u : u(x) = t\beta\wedge x,\ t\in\mathbb{R}\}$.

$\mathcal{L}(\widehat{\Omega})$ then denotes—according to the properties of the domain—the corresponding space.

2) On $\mathcal{L}(\widehat{\Omega})$ we have Korn's inequality, provided the domain is smooth enough:

Lemma 2.1. *Let $\widehat{\Omega}$ be a Lipschitz domain. There exists a constant $c_0$ such that*

$$K(v,v) \ge c_0\|v\|^2_{H^{1,2}(\widehat{\Omega})} \qquad \forall v\in\mathcal{L}(\widehat{\Omega}).$$

For a proof see [Sol&Shch] (Lemma 4 p. 191) or [Bem1] (Corollary to Lemma 2, p. 249). In particular $K(v,v)$ defines a norm on $\mathcal{L}(\Omega)$ which is equivalent to the $H^{1,2}$-norm.

We will treat (24)–(29) and (30)–(34) as two coupled systems. Consequently, we define a weak solution $(v_1,\theta,v_2,\eta)$ of (24)–(34) as an element in

$\mathcal{L}(\widehat{\Omega}) \times H_0^{1,2}(\widehat{\Omega}) \times \mathcal{L}(\widehat{\Omega}) \times H_0^{1,2}(\widehat{\Omega})$ which satisfies the integral equations

$$\begin{aligned}(35)\qquad &\lambda \int_{\widehat{\Omega}} v_1 \cdot \phi_1\, dx + \mathrm{Pr} \int_{\widehat{\Omega}} D(v_1) : D(\phi_1)\, dx \\ &- \int_{\partial\widehat{\Omega}} \mathrm{Ma}\,\mathrm{Pr}\, \phi_1 \cdot \widehat{\tau}_1 \widehat{\tau}_1 \cdot \nabla\theta + \mathrm{Ma}\,\mathrm{Pr}\, \phi_1 \cdot \widehat{\tau}_2 \widehat{\tau}_2 \cdot \nabla\theta\, d\widehat{S} \\ &+ \int_{\widehat{\Omega}} \widehat{v} \cdot \nabla v_1 \phi_1\, dx + \int_{\widehat{\Omega}} v_1 \cdot \nabla \widehat{v} \phi_1\, dx = \int_{\widehat{\Omega}} f \cdot \phi_1\, dx\end{aligned}$$

and

$$\begin{aligned}(36)\qquad &\lambda \int_{\widehat{\Omega}} \theta\psi\, dx + \int_{\widehat{\Omega}} \nabla\theta \cdot \nabla\psi\, dx + \int_{\widehat{\Omega}} \widehat{v} \cdot \nabla\theta\psi\, dx \\ &+ \int_{\widehat{\Omega}} (v_1 + v_2) \cdot \nabla\widehat{\theta}\psi\, dx = \int_{\widehat{\Omega}} g\psi\, dx - \int_{\partial\widehat{\Omega}} h\psi\, dx\end{aligned}$$

for all $\phi_1 \in \mathcal{L}(\widehat{\Omega}), \psi \in H_0^{1,2}(\widehat{\Omega})$, and

$$\begin{aligned}(37)\qquad &\lambda \int_{\widehat{\Omega}} v_2 \cdot \phi_2\, dx + \mathrm{Pr} \int_{\widehat{\Omega}} D(v_2) : D(\phi_2)\, dx \\ &- \mathrm{Ma}\,\mathrm{Pr} \int_{\partial\widehat{\Omega}} \phi_2 \cdot \widehat{\nu} h\, d\widehat{S} + \int_{\partial\widehat{\Omega}} \phi_2 \cdot \widehat{\nu}\widehat{\nu} \cdot T(v_1, q)\widehat{\nu}\, d\widehat{S} \\ &- 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr} \int_{\partial\widehat{\Omega}} \phi_2 \cdot \widehat{\nu} \widetilde{H}(\eta)\, d\widehat{S} + \int_{\widehat{\Omega}} \widehat{v} \cdot \nabla v_2 \phi_2\, dx \\ &+ \int_{\widehat{\Omega}} v_2 \cdot \nabla\widehat{v}\phi_2\, dx = 0\end{aligned}$$

for all $\phi_2 \in \mathcal{L}(\widehat{\Omega})$. Here the space $H_0^{1,2}(\widehat{\Omega})$ denotes the functions in $H^{1,2}(\widehat{\Omega})$ with zero mean value. Note that we have Poincaré's inequality for such functions.

Remark. We wish to point out that $\eta$ has zero mean value as a function over $\partial\widehat{\Omega}$. This is a consequence of the condition that the volume is preserved: Let $\Omega$ be a surface, parametrised over $\widehat{\Omega}$ and enclosing the same volume. A point in $\widehat{\Omega}$ is denoted by $x_0$, and a point in $\Omega$ by $x$. Then

$$\begin{aligned}|\Omega| &= \int_{\Omega} 1\, dx = \frac{1}{n} \int_{\partial\Omega} x \cdot \nu\, dS = \frac{1}{n} \int_{\partial\widehat{\Omega}} (x_0 + \eta\widehat{\nu}) \cdot \widehat{\nu}\, d\widehat{S} \\ &= \frac{1}{n} \int_{\partial\widehat{\Omega}} x_0 \cdot \widehat{\nu}\, d\widehat{S} + \frac{1}{n} \int_{\partial\widehat{\Omega}} \eta\, d\widehat{S} = |\widehat{\Omega}| + \frac{1}{n} \int_{\partial\widehat{\Omega}} \eta\, d\widehat{S}.\end{aligned}$$

This implies $\int_{\partial\widehat{\Omega}} \eta\, d\widehat{S} = 0$.

**2.3.** We derive an a priori bound for $v_1$ and $\theta$ if we set $\phi_1 = v_1$ and $\psi = \theta$ in the first two integral equations.

Since

$$\Big|\int_{\partial\widehat{\Omega}} \mathrm{Ma}\,\mathrm{Pr}\, v_1\cdot\widehat{\tau}_1\widehat{\tau}_1\cdot\nabla\theta + \mathrm{Ma}\,\mathrm{Pr}\, v_1\cdot\widehat{\tau}_2\widehat{\tau}_2\cdot\nabla\theta\, d\widehat{S}\Big|$$
$$\le \mathrm{Ma}\,\mathrm{Pr}\,\|v_1\|_{H^{1/2,2}(\partial\widehat{\Omega})}\|\nabla^*\theta\|_{H^{-1/2}(\partial\widehat{\Omega})}$$
$$\le \mathrm{Ma}\,\mathrm{Pr}\,\|v_1\|_{H^{1/2,2}(\partial\widehat{\Omega})}\|\theta\|_{H^{1/2,2}(\partial\widehat{\Omega})}$$
$$\le \mathrm{Ma}\,\mathrm{Pr}\,c_1^2\|v_1\|_{H^{1,2}(\widehat{\Omega})}\|\theta\|_{H^{1,2}(\widehat{\Omega})}$$

where $\nabla^*$ denotes the tangential gradient on $\partial\widehat{\Omega}$ ($\nabla^*_i = \widehat{\tau}_i\cdot\nabla\theta$), we obtain the following inequality ($c_1$ will always stand for the constants connected with trace theorems):

$$\lambda\int_{\widehat{\Omega}}|v_1|^2\,dx + c_0\,\mathrm{Pr}\,\|v_1\|^2_{H^{1,2}(\widehat{\Omega})}$$
$$\le \mathrm{Ma}\,\mathrm{Pr}\,c_1^2\|v_1\|_{H^{1,2}(\widehat{\Omega})}\|\theta\|_{H^{1,2}(\widehat{\Omega})}$$
$$+ \max_{\widehat{\Omega}}|\nabla\widehat{v}|\cdot\|v_1\|^2_{H^{0,2}(\widehat{\Omega})} + \|f\|_{H^{0,2}(\widehat{\Omega})}\|v_1\|_{H^{0,2}(\widehat{\Omega})}.$$

On the other hand, for the heat equation we have

$$\lambda\int_{\widehat{\Omega}}|\theta|^2\,dx + c_2\|\theta\|^2_{H^{1,2}(\widehat{\Omega})}$$
$$\le (\|g\|_{H^{0,2}(\widehat{\Omega})} + c_1\|h\|_{H^{1/2,2}(\partial\widehat{\Omega})})\|\theta\|_{H^{1,2}(\widehat{\Omega})}$$
$$+ \max_{\widehat{\Omega}}|\nabla\widehat{\theta}|\ (\|v_1\|_{H^{0,2}(\widehat{\Omega})} + \|v_2\|_{H^{0,2}(\widehat{\Omega})})\|\theta\|_{H^{0,2}(\widehat{\Omega})}$$

where $c_2$ is the constant in the inequality $\int_{\widehat{\Omega}}|\nabla\theta|^2\,dx \ge c_2\|\theta\|^2_{H^{1,2}(\widehat{\Omega})}$.

Combining the two estimates gives us

$$\lambda\int_{\widehat{\Omega}}|v_1|^2\,dx + \lambda\int_{\widehat{\Omega}}|\theta|^2\,dx + c_3\|v_1\|^2_{H^{1,2}(\widehat{\Omega})} + \frac{c_2}{2}\|\theta\|^2_{H^{1,2}(\widehat{\Omega})}$$
$$\le c_4\{\|f\|^2_{H^{0,2}(\widehat{\Omega})} + \|g\|^2_{H^{0,2}(\widehat{\Omega})} + \|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})} + \|v_2\|^2_{H^{0,2}(\widehat{\Omega})}\}.$$

The constants $c_3$ and $c_4$ can be computed as

$$c_3 = c_0\,\mathrm{Pr} - \max_{\widehat{\Omega}}|\nabla\widehat{v}| - \mathrm{Ma}^2\,\mathrm{Pr}^2\,c_1^2\frac{1+\max_{\widehat{\Omega}}|\nabla\widehat{\theta}|}{c_2}$$
$$- \frac{c_2}{4(1+\max_{\widehat{\Omega}}|\nabla\widehat{\theta}|)} - \frac{\max_{\widehat{\Omega}}|\nabla\widehat{\theta}|(1+\max_{\widehat{\Omega}}|\nabla\widehat{\theta}|)}{c_2},$$
$$c_4 = \frac{1+\max_{\widehat{\Omega}}|\nabla\widehat{\theta}|}{c_2}\max(1, c_1^2, \max_{\widehat{\Omega}}|\nabla\widehat{\theta}|).$$

We see that the Pr number has to be chosen large enough, while the Ma number has to satisfy a smallness condition.

By choosing $v_1$ as a test function the pressure term dropped out. We may rediscover it by standard methods (see e.g. [Lad], [Tem]). Adding the pressure estimate to our inequality then gives us the desired a priori esimate.

LEMMA 2.2. *For the first system let* $f, g \in H^{0,2}(\widehat{\Omega})$, $h \in H^{1/2,2}(\partial\widehat{\Omega})$ *and let* $\widehat{\Omega}$ *be of class* $C^{3,\alpha}$. *For sufficiently small Marangoni numbers and sufficiently large Prandtl numbers we have the a priori estimate*

$$\lambda \int_{\widehat{\Omega}} |v_1|^2\,dx + \lambda \int_{\widehat{\Omega}} |\theta|^2\,dx + c_3 \|v_1\|^2_{H^{1,2}(\widehat{\Omega})} + \|q - q_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})} + \frac{c_2}{2}\|\theta\|^2_{H^{1,2}(\widehat{\Omega})} \le c_6\{\|f\|^2_{H^{0,2}(\widehat{\Omega})} + \|g\|^2_{H^{0,2}(\widehat{\Omega})} + \|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})} + \|v_2\|^2_{H^{0,2}(\widehat{\Omega})}\}$$

*with* $c_6 = c_4 + c_5$ *where* $c_5$ *is the constant in the pressure estimate.*

REMARK. 1) In the above estimate we assumed $\widehat{v}, p_0, \widehat{\theta}, \widehat{\Omega}$ to be sufficiently smooth. This is a useful assumption though not necessary. The least we need is $\widehat{v} \in H^{1,2}(\widehat{\Omega})$, $\widehat{p} \in H^{0,2}(\widehat{\Omega})$, $\widehat{\theta} \in H^{1,2}(\widehat{\Omega})$ and $\partial\widehat{\Omega} \in C^{1,1}$.

2) If we had linearised the system around the solution $\widehat{v} = 0$, $p_0 = \text{const}$, $\widehat{\theta} = 0$, $\widehat{\Omega} = B(0)$ the heat equation would have been decoupled from the rest of the system. We could solve this equation in advance. In that case no smallness condition on the Ma number would appear—only the Prandtl number would have to be sufficiently large.

3) In some sense the condition on the Ma number can be considered as a consequence of the free boundary. If we look at the fixed boundary problem

$$\begin{aligned} \partial_t v - \mathrm{Pr}\,\Delta v + v \cdot \nabla v + \nabla p &= f, \\ \nabla \cdot v &= 0, \\ \partial_t \theta - \Delta\theta + v \cdot \nabla\theta &= g \end{aligned}$$

in $\Omega \times (0, T)$, together with the boundary conditions

$$\begin{aligned} \tau_i T(v, p)\nu - \mathrm{Ma}\,\mathrm{Pr}\,\tau_i \nabla\theta &= 0 \qquad i = 1, 2, \\ v \cdot \nu &= 0, \\ \nu \cdot \nabla\theta &= h \end{aligned}$$

in $\partial\Omega \times (0, T)$ (with $\partial\Omega$ sufficiently smooth), we observe that no condition on the Ma or Pr number is required to obtain existence of a weak solution.

**2.4.** For the third integral equation we work similarly by setting $\phi_2 = v_2$ to obtain

$$\lambda \int_{\widehat{\Omega}} |v_2|^2\,dx + \mathrm{Pr} \int_{\widehat{\Omega}} D(v_2) : D(v_2)\,dx - \mathrm{Ma}\,\mathrm{Pr} \int_{\partial\widehat{\Omega}} v_2 \cdot \widehat{\nu} h\,d\widehat{S} + \int_{\partial\widehat{\Omega}} v_2 \cdot \widehat{\nu}\widehat{\nu} \cdot T(v_1, q)\widehat{\nu}\,d\widehat{S} - 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr} \int_{\partial\widehat{\Omega}} v_2 \cdot \widehat{\nu}\widetilde{H}(\eta)\,d\widehat{S} + \int_{\widehat{\Omega}} v_2 \cdot \nabla\widehat{v} v_2\,dx = 0.$$

The only difference in comparison to the first system are the boundary integrals. It is sufficient to consider these terms only and to carry over all the other estimates from the $v_1$-system.

Thus we have to estimate the integrals

$$-\operatorname{Ma}\operatorname{Pr}\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}h\,d\widehat{S}+\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}\widehat{\nu}\cdot T(v_1,q)\widehat{\nu}\,d\widehat{S}-2\operatorname{Cr}^{-1}\operatorname{Pr}\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}\widetilde{H}(\eta)\,d\widehat{S}.$$

The first two integrals can be estimated straightaway:

$$\Big|\operatorname{Ma}\operatorname{Pr}\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}hd\widehat{S}\Big|\le \operatorname{Ma}\operatorname{Pr}c_1\|v_2\|_{H^{1,2}(\widehat{\Omega})}\|h\|_{H^{1/2,2}(\partial\widehat{\Omega})},$$

$$\begin{aligned}\Big|\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}\widehat{\nu}\cdot T(v_1,q)\widehat{\nu}\,d\widehat{S}\Big| &= \Big|\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}\widehat{\nu}\cdot T(v_1,q-q_{\widehat{\Omega}})\widehat{\nu}\,d\widehat{S}\Big|\\ &\le c_1^2\|v_2\|_{H^{1,2}(\widehat{\Omega})}\{\|v_1\|_{H^{1,2}(\widehat{\Omega})}+\|q-q_{\widehat{\Omega}}\|_{H^{0,2}(\widehat{\Omega})}\}.\end{aligned}$$

The integral $-2\operatorname{Cr}^{-1}\operatorname{Pr}\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}\widetilde{H}(\eta)\,d\widehat{S}$ requires more care. We insert the boundary condition for $v_2\cdot\widehat{\nu}$ and the expression for $\widetilde{H}$ to obtain

$$\begin{aligned}-2\operatorname{Cr}^{-1}\operatorname{Pr}\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}\widetilde{H}(\eta)\,d\widehat{S} = &-2\operatorname{Cr}^{-1}\operatorname{Pr}\int_{\partial\widehat{\Omega}} v_2\cdot\widehat{\nu}(H(\widehat{\eta})-2\eta)\,d\widehat{S}\\ &-2\operatorname{Cr}^{-1}\operatorname{Pr}\int_{\partial\widehat{\Omega}}\lambda(\eta+\widetilde{h})\Delta^*\eta\,d\xi.\end{aligned}$$

Thus the curvature term gives us a positive term. We obtain the inequality

$$\begin{aligned}\lambda\int_{\widehat{\Omega}}|v_2|^2\,dx &+ 2\operatorname{Cr}^{-1}\operatorname{Pr}\lambda\int_{\partial\widehat{\Omega}}|\eta|^2\,dx+(c_0\operatorname{Pr}-\max_{\widehat{\Omega}}|\nabla\widehat{v}|-\varepsilon)\|v_2\|^2_{H^{1,2}(\widehat{\Omega})}\\ &+\operatorname{Cr}^{-1}\operatorname{Pr}\lambda\int_{\partial\widehat{\Omega}}|\nabla^*\eta|^2\,d\xi\\ &\le \operatorname{Ma}^2\operatorname{Pr}^2c_1^2\|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})}+2\operatorname{Cr}^{-1}\operatorname{Pr}\lambda\|\widetilde{h}\|^2_{H^{1,2}(\partial\widehat{\Omega})}\\ &\quad+c_1^2\|v_2\|_{H^{1,2}(\widehat{\Omega})}\{\|v_1\|_{H^{1,2}(\widehat{\Omega})}+\|q-q_{\widehat{\Omega}}\|_{H^{0,2}(\widehat{\Omega})}\}\\ &\quad+\operatorname{Cr}^{-1}\operatorname{Pr}c_1^2\|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}.\end{aligned}$$

We insert the estimate for the $v_1$-system and rearrange terms. Including the pressure we summarise our results:

LEMMA 2.3. *For the second system let $\widetilde{h}\in H^{3/2,2}(\partial\widehat{\Omega})$. Then we have the a priori estimate*

$$\lambda \int_{\widehat{\Omega}} |v_2|^2\,dx + \mathrm{Cr}^{-1}\,\mathrm{Pr}\,\lambda\|\eta\|^2_{H^{1,2}(\partial\widehat{\Omega})} + c_8\|v_2\|^2_{H^{1,2}(\widehat{\Omega})}$$
$$+ \|p - q - (p-q)_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})}$$
$$\le c_9\{\|f\|^2_{H^{0,2}(\partial\widehat{\Omega})} + \|g\|^2_{H^{0,2}(\partial\widehat{\Omega})} + \|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})}$$
$$+ \lambda\|\widetilde{h}\|^2_{H^{1,2}(\partial\widehat{\Omega})} + \|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}\}.$$

We can compute $c_8 = c_0\,\mathrm{Pr} - \max_{\widehat{\Omega}} |\nabla\widehat{v}| - \varepsilon - c_7$, where $c_7 = c_1/2 + c_5/\min(1, c_4)$. Thus we have an additional condition on the Prandtl number to obtain a positive $c_8$. $c_9$ can be computed from the other constants. No constant depends on $\lambda$ and no constant vanishes or blows up if $\mathrm{Ma} \to 0, \mathrm{Cr}^{-1} \to 0$ or if one of the $\widehat{\cdot}$ quantities tends to zero. This statement holds true for the rest of the paper.

We add the estimates in Lemmas 2.2 and 2.3 to obtain an existence theorem for the weak solution of the linearised problem.

THEOREM 2.1. *Let $f, g \in H^{0,2}(\widehat{\Omega})$, $h \in H^{1/2,2}(\partial\widehat{\Omega})$, $\widetilde{h} \in H^{1,2}(\partial\widehat{\Omega})$ and let the boundary of the domain be of class $C^{3,\alpha}$. Let $\lambda \ge 0$ and let the compatibility conditions $\int_{\widehat{\Omega}} g\,dx = \int_{\partial\widehat{\Omega}} h\,dS$ and $\int_{\widehat{\Omega}} f\,dx = 0$ hold. For small Marangoni numbers and large Prandtl numbers there exists one and only one weak solution*

$$(v, p - p_{\widehat{\Omega}}, \theta, \eta) \in H^{1,2}(\widehat{\Omega}) \times H^{0,2}(\widehat{\Omega}) \times H^{1,2}(\widehat{\Omega}) \times H^{1,2}(\partial\widehat{\Omega}).$$

*Furthermore the solution satisfies the estimate*

$$\lambda \int_{\widehat{\Omega}} |v|^2\,dx + \lambda \int_{\widehat{\Omega}} |\theta|^2\,dx + \|v\|^2_{H^{1,2}(\widehat{\Omega})}$$
$$+ \|p - p_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})} + \|\theta\|^2_{H^{1,2}(\widehat{\Omega})} + \|\eta\|^2_{H^{1,2}(\partial\widehat{\Omega})}$$
$$\le c\{\|f\|^2_{H^{0,2}(\widehat{\Omega})} + \|g\|^2_{H^{0,2}(\widehat{\Omega})} + \|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})}$$
$$+ \lambda\|\widetilde{h}\|^2_{H^{1,2}(\partial\widehat{\Omega})} + \|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}\}.$$

*The constant does not depend on $\lambda$.*

## 3. Regularity of the stationary solution

**3.1.** We now turn to the question of regularity of the weak solution. We restrict ourselves to the more complicated case of the boundary regularity. For that we will use cut-off functions $\chi$ with support in a small tube around the boundary of $\widehat{\Omega}$. We extend the coordinate system given on the boundary by the normal and tangential vectors smoothly into the interior of the tube. In that sense we may speak about a tangential (resp. normal) vector at an interior point.

The cut-off functions $\chi$ are always assumed to be constant in tangential directions in the tube and furthermore we assume the estimates $|\nabla\chi| < c/\delta$ and $|\Delta\chi| < c/\delta^2$ for $\delta > 0$.

As in Section 2 we use the decomposition of our system. In particular we write (24)–(34) as

$$\lambda \int_{\widehat{\Omega}} v_1\phi_1\,dx + \int_{\widehat{\Omega}} (-\Pr\Delta v_1 + \nabla q)\phi_1\,dx + \int_{\widehat{\Omega}} \widehat{v}\cdot\nabla v_1\phi_1\,dx + \int_{\widehat{\Omega}} v_1\cdot\nabla\widehat{v}\phi_1\,dx = \int_{\widehat{\Omega}} f\phi_1\,dx, \tag{38}$$

$$\lambda \int_{\widehat{\Omega}} \theta\psi\,dx - \int_{\widehat{\Omega}} \Delta\theta\psi\,dx + \int_{\widehat{\Omega}} \widehat{v}\cdot\nabla\theta\psi\,dx + \int_{\widehat{\Omega}} (v_1+v_2)\cdot\nabla\widehat{\theta}\psi\,dx = \int_{\widehat{\Omega}} g\psi\,dx, \tag{39}$$

$$\lambda \int_{\widehat{\Omega}} v_2\phi_2\,dx + \int_{\widehat{\Omega}} (-\Pr\Delta v_2 + \nabla(p-q))\phi_2\,dx + \int_{\widehat{\Omega}} \widehat{v}\cdot\nabla v_2\phi_2\,dx + \int_{\widehat{\Omega}} v_2\cdot\nabla\widehat{v}\phi_2\,dx = 0, \tag{40}$$

for suitable test functions $\phi_1, \phi_2, \psi$.

Our aim is to find an $H^{2,2}$ bound for $v_1, v_2$ and $\theta$ on $\widehat{\Omega}$, an $H^{0,2}$ bound for $\nabla q$ and $\nabla(p-q)$ on $\widehat{\Omega}$, and an $H^{5/2,2}$ bound for $\eta$ on $\partial\widehat{\Omega}$.

We proceed as follows: By choosing a right test function we find a bound for the tangential derivatives of $\nabla v_1$ close to the boundary. By another choice of the test function we find an $H^{0,2}$ estimate for the tangential derivatives of the pressure $q$. The corresponding bounds for the normal derivatives are then given by the system itself. As in Section 2 we concentrate on the estimates where the Marangoni term is involved.

For the $v_2$-system we work similarly. The difference is that we can only bound the $H^{2,2}$ norm of $v_2$ (resp. the $H^{0,2}$ norm of $\nabla(p-q)$) by the $H^{3/2}$ norm of $\eta$ on $\partial\widehat{\Omega}$. Equation (34) then gives a bound of the $H^{5/2,2}$ norm of $\eta$ by the $H^{2,2}$ norm of $v_2$ (resp. the $H^{0,2}$ norm of $\nabla(p-q)$). Arranging the estimates in the right way we obtain the regularity result.

We wish to point out that the arrangement of the arguments cannot be changed. Thus knowing the existence of a weak solution of our system, we do not see any possibility to gain regularity for the fluid variables from the geometrical operators.

**3.2.** We first define the test function $\phi_1$ as

$$\phi_1 \equiv -\Delta^*(v_1\chi^2).$$

Basic calculations give the inequality

$$\begin{aligned}\lambda \int_{\widehat{\Omega}} |\nabla^*(v_1\chi)|^2\,dx + \|\nabla^*(v_1\chi)\|^2_{H^{1,2}(\widehat{\Omega})} \\ \le c\Big\{ \|v_1\|^2_{H^{1,2}(\widehat{\Omega})} + \|q - q_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})} \\ + \operatorname{Ma}\operatorname{Pr}\Big|\sum_{i=1}^{2}\int_{\partial\widehat{\Omega}} \Delta^* v_1 \cdot \widehat{\tau}_i\widehat{\tau}_i \cdot \nabla\theta\, d\widehat{S}\Big| + \|f\|^2_{H^{0,2}(\widehat{\Omega})}\Big\}.\end{aligned}$$

To obtain this inequality we have to keep in mind that the test function is no longer divergence free (this explains the pressure term) and that interchanging euclidian derivatives with $\nabla^*_i$ $(= \widehat{\tau}_i \cdot \nabla)$ causes the appearance of additional terms of lower differentiability order in $v_1$.

We estimate the Marangoni term:

$$\begin{aligned}\operatorname{Ma}\operatorname{Pr}\Big|\sum_{i=1}^{2}\int_{\partial\widehat{\Omega}} \Delta^* v_1 \cdot \tau_i\tau_i \cdot \nabla\theta\, dS\Big| \\ \le \operatorname{Ma}\operatorname{Pr}\|\Delta^* v_1\|_{H^{-1/2}(\partial\widehat{\Omega})}\|\nabla^*\theta\|_{H^{1/2,2}(\partial\widehat{\Omega})} \\ \le c_1 \operatorname{Ma}\operatorname{Pr}\|\nabla^*(v_1\chi)\|_{H^{1/2,2}(\partial\widehat{\Omega})}\|\nabla\theta\|_{H^{1,2}(\widehat{\Omega})} \\ \le c_1^2 \operatorname{Ma}\operatorname{Pr}\|\nabla^*(v_1\chi)\|_{H^{1,2}(\widehat{\Omega})}\|\theta\|_{H^{2,2}(\widehat{\Omega})}.\end{aligned}$$

Thus we end up with a local estimate for the tangential derivatives for $\nabla v_1$.

Intermediate Result 1.

$$\begin{aligned}\lambda\|\nabla^*(v_1\chi)\|^2_{H^{0,2}(\widehat{\Omega})} + \|\nabla^*(v_1\chi)\|^2_{H^{1,2}(\widehat{\Omega})} \\ \le c\{\|v_1\|^2_{H^{1,2}(\widehat{\Omega})} + \|q - q_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})} \\ + \operatorname{Ma}\operatorname{Pr}\|\nabla^* v_1\|_{H^{1,2}(\widehat{\Omega})}\|\theta\|_{H^{2,2}(\widehat{\Omega})} + \|f\|^2_{H^{0,2}(\widehat{\Omega})}\}.\end{aligned}$$

The "Marangoni term" $\operatorname{Ma}\operatorname{Pr}\|\nabla^* v_1\|_{H^{1,2}(\widehat{\Omega})}\|\theta\|_{H^{2,2}(\widehat{\Omega})}$ will be estimated later.

We are now left with the problem of finding a bound for the second derivatives of $v_1$ in the normal direction, and a bound for the pressure. Following [Sol&Shch], p. 197, we briefly sketch the procedure. First we obtain an estimate for the tangential derivatives of the pressure.

As a test function we choose $\Delta^*\Phi\chi^2$, where $\Phi$ is the solution of

$$\nabla \cdot \Phi = q \quad \text{in } \widehat{\Omega}, \qquad \Phi \cdot \widehat{\nu} = 0 \quad \text{in } \partial\widehat{\Omega}.$$

For this $\Phi$ we have the estimates

$$\|\nabla\Phi\|_{H^{0,2}(\widehat{\Omega})} \le c\|q\|_{H^{0,2}(\widehat{\Omega})}, \qquad \|\nabla^*\nabla\Phi\|_{H^{0,2}(\widehat{\Omega})} \le c\|\nabla^* q\|_{H^{0,2}(\widehat{\Omega})}$$

(see (9) and (13) in [Sol&Shch], pp. 188, 189). Inserting this test function into the integral equation (38)–(40) we obtain the integral equation

$$\lambda \int_{\widehat{\Omega}} v_1 \cdot \Delta^* \Phi \chi^2 \, dx - \mathrm{Pr} \int_{\widehat{\Omega}} \Delta v_1 \cdot \Delta^* \Phi \chi^2 \, dx + \int_{\widehat{\Omega}} \nabla q \cdot \Delta^* \Phi \chi^2 \, dx + \int_{\widehat{\Omega}} v_1 \cdot \nabla \widehat{v} \Delta^* \Phi \chi^2 \, dx + \int_{\widehat{\Omega}} \widehat{v} \cdot \nabla v_1 \Delta^* \Phi \chi^2 \, dx = \int_{\widehat{\Omega}} f \cdot \Delta^* \Phi \chi^2 \, dx.$$

The estimates are done as before. Observe that now the Marangoni term is estimated as

$$\begin{aligned} \mathrm{Ma}\,\mathrm{Pr} \sum_{i=1}^{2} \int_{\partial\widehat{\Omega}} \Delta^* \Phi \cdot \widehat{\tau}_0 \widehat{\tau}_0 \cdot \nabla \theta \, d\widehat{S} &\le \mathrm{Ma}\,\mathrm{Pr}\, \|\Delta^* \Phi\|_{H^{-1/2,2}(\partial\widehat{\Omega})} \|\nabla^* \theta\|_{H^{1/2,2}(\partial\widehat{\Omega})} \\ &\le c\, \mathrm{Ma}\,\mathrm{Pr}\, \|\nabla^* \Phi\|_{H^{1/2,2}(\partial\widehat{\Omega})} \|\theta\|_{H^{3/2,2}(\partial\widehat{\Omega})} \\ &\le c\, \mathrm{Ma}\,\mathrm{Pr}\, \|\nabla^* \Phi\|_{H^{1,2}(\widehat{\Omega})} \|\theta\|_{H^{2,2}(\widehat{\Omega})} \\ &\le c\, \mathrm{Ma}\,\mathrm{Pr}\, \|\nabla^* \nabla \Phi\|_{H^{0,2}(\widehat{\Omega})} \|\theta\|_{H^{2,2}(\widehat{\Omega})}. \end{aligned}$$

The last inequality follows from the fact that interchanging $\nabla^*$ and euclidian derivatives produces terms that can be estimated by $\|\nabla^* \nabla \Phi\|_{H^{0,2}(\widehat{\Omega})}$. We obtain an estimate for the tangential derivatives of the pressure:

Intermediate Result 2.

$$\begin{aligned} \|\nabla^*(q\chi)\|_{H^{0,2}(\widehat{\Omega})} \le c\{ &\|\nabla^*(v_1\chi)\|_{H^{1,2}(\widehat{\Omega})} + \lambda \|v_1\|_{H^{0,2}(\widehat{\Omega})} + \|f\|_{H^{0,2}(\widehat{\Omega})} \\ &+ \|\theta\|_{H^{2,2}(\widehat{\Omega})} + \|v_1\|_{H^{1,2}(\widehat{\Omega})} + \|q - q_{\widehat{\Omega}}\|_{H^{0,2}(\widehat{\Omega})}\}. \end{aligned}$$

Remark. In the estimate the term $\lambda \|v_1\|_{H^{0,2}(\widehat{\Omega})}$ appears. It was not estimated before. However, if in the weak formulation (35)–(37) in Section 2 we choose $\phi = \lambda v_1$ resp. $\psi = \lambda \theta$ we obtain an estimate for this term. There will be no additional condition for the Ma or Pr number. We will not give the computations in detail, since no new estimates for the Marangoni term will be required.

Later we will see that the estimate for $\lambda \|v_1\|_{H^{0,2}(\widehat{\Omega})}$ corresponds to the estimate for the first time derivative of $v_1$.

Now we add up the two "tangential estimates" to obtain:

Intermediate Result 3.

$$\begin{aligned} \lambda \|\nabla^*(v_1\chi)\|^2_{H^{0,2}(\widehat{\Omega})} &+ \|\nabla^*(v_1\chi)\|^2_{H^{1,2}(\widehat{\Omega})} + \|\nabla^*(q\chi)\|^2_{H^{0,2}(\widehat{\Omega})} \\ &\le c\{ \|v_1\|^2_{H^{1,2}(\widehat{\Omega})} + \|q - q_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})} + \|\theta\|^2_{H^{2,2}(\widehat{\Omega})} \\ &\quad + \mathrm{Ma}\,\mathrm{Pr}\, \|\nabla^*(v_1\chi)\|_{H^{1,2}(\widehat{\Omega})} \|\theta\|_{H^{2,2}(\widehat{\Omega})} + \|f\|^2_{H^{0,2}(\widehat{\Omega})}\}. \end{aligned}$$

The missing estimates are given by the system itself: We transform the Navier–Stokes system near the boundary (i.e. in the support of $\chi$), where the new coordinate system is generated by $(\widehat{\tau}_1, \widehat{\tau}_2, \widehat{\nu})$.

We write down the new system:

$$\lambda V_1^j - \mathrm{Pr}\, \widehat{g}_0^{ik} \partial_i \partial_k V_1^j + \frac{1}{\det D\widehat{\Phi}} \widehat{g}_0^{jk} \partial_k Q + M^j(V_1, \nabla V_1) = F^j,$$
$$\nabla \cdot V_1 = 0.$$

We wish to point out that we transformed the velocity field as

$$V^i(X) \equiv \frac{\partial \widehat{\Phi}^i(\widehat{\Phi}^{-1}(X))}{\partial x^j} v^j(\widehat{\Phi}^{-1}(X)) \cdot (\det D\widehat{\Phi}(\widehat{\Phi}^{-1}(X)))^{-1}.$$

This transformation has the property that

$$\sum_{i=1}^3 \frac{\partial V^i}{\partial X^i} = 0 \quad \text{if} \quad \sum_{i=1}^3 \frac{\partial v^i}{\partial x^i} = 0.$$

Choose the enumeration of the vector components in such a way that $V_1^1, V_1^2$ give the components of the velocity field in the tangential directions and $V_1^3$ gives the component in the normal direction. $M$ always contains at most first derivatives in $V_1$.

The first and second equations give us a bound for the second normal derivatives of $V_1^1$ and $V_1^2$. The second radial derivatives of $V_1^3$ can be bounded if we differentiate the equation $\sum_{i=1}^3 \partial V^i/\partial X^i = 0$ in the normal direction. Finally, we can rewrite the first equation as an equation for the normal derivative of the pressure. In that way the full second derivatives of $V_1$ and the full gradient of the pressure are bounded in the $H^{0,2}$ sense.

For the above considerations the temperature equation does not play any role, because the coupling is only via the boundary terms, but we never integrated by parts.

We now transform back and obtain inequalities for $v_1$ and $q$. These inequalities are valid locally but with a covering argument they may be extended all over the domain.

The regularity considerations for the heat equation are omitted. The necessary arguments for bounding the second and higher derivatives can be found e.g. in [GT], Chapter 8.

Rearranging terms and using the smallness condition on the Marangoni number, we obtain the final estimate, recorded as a lemma.

Lemma 3.1. *Under the assumptions of Theorem* 2.1 *the solution to* (24)–(34) *is regular*, *i.e.* $v_1 \in H^{2,2}(\widehat{\Omega})$, $\nabla q \in H^{0,2}(\widehat{\Omega})$, $\theta \in H^{2,2}(\widehat{\Omega})$. *The following estimate holds*:

$$\lambda\|v_1\|^2_{H^{1,2}(\widehat{\Omega})} + \lambda\|\theta\|^2_{H^{1,2}(\widehat{\Omega})} + \|v_1\|^2_{H^{2,2}(\widehat{\Omega})} + \|\theta\|^2_{H^{2,2}(\widehat{\Omega})} + \|q - q_{\widehat{\Omega}}\|^2_{H^{1,2}(\widehat{\Omega})}$$
$$\leq c\{\|v_1\|^2_{H^{1,2}(\widehat{\Omega})} + \|v_2\|^2_{H^{1,2}(\widehat{\Omega})} + \|\theta\|^2_{H^{1,2}(\widehat{\Omega})} + \|q - q_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})}$$
$$+ \|f\|^2_{H^{0,2}(\widehat{\Omega})} + \|g\|^2_{H^{0,2}(\widehat{\Omega})} + \|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})}\}.$$

We wish to point out that there is no additional condition on the parameters. Higher regularity now follows from the [ADN] theory.

**3.3.** We work similarly for the $v_2$-system (48)–(49). We choose $\phi_2 = -\Delta^*(v_2\chi^2)$ as a test function. The only difference compared to the $v_1$-system are the different boundary integrals after using Green's formula:

$$\operatorname{Ma}\operatorname{Pr}\int_{\partial\widehat{\Omega}} \Delta^*(v_2\cdot\widehat{\nu})h\,d\widehat{S} - \int_{\partial\widehat{\Omega}} \Delta^*(v_2\cdot\widehat{\nu})\widehat{\nu}\cdot T(v_1,q)\widehat{\nu}\,d\widehat{S}$$
$$+ 2\operatorname{Cr}^{-1}\operatorname{Pr}\int_{\partial\widehat{\Omega}} \Delta^*(v_2\widehat{\nu})\widetilde{H}(\eta)\,d\widehat{S}.$$

As in the $v_1$-system, the first two integrals can be estimated immediately:

$$\Big|\operatorname{Ma}\operatorname{Pr}\int_{\partial\widehat{\Omega}} \Delta^*(v_2\cdot\widehat{\nu})h\,d\widehat{S} - \int_{\partial\widehat{\Omega}} \Delta^*(v_2\cdot\widehat{\nu})\widehat{\nu}\cdot T(v_1,q)\widehat{\nu}\,d\widehat{S}\Big|$$
$$\leq c\|\nabla^*(v_2\chi)\|_{H^{1,2}(\widehat{\Omega})}\{\|h\|_{H^{1/2}(\partial\widehat{\Omega})} + \|v_1\|_{H^{2,2}(\widehat{\Omega})} + \|\nabla q\|_{H^{0,2}(\widehat{\Omega})}\}.$$

The remaining boundary integral gives us a positive term:

$$2\operatorname{Cr}^{-1}\operatorname{Pr}\int_{\partial\widehat{\Omega}} \Delta^*(v_2\widehat{\nu})\widetilde{H}(\eta)\,d\widehat{S}$$
$$= 2\operatorname{Cr}^{-1}\operatorname{Pr}\lambda\int_{\partial\widehat{\Omega}} \Delta^*(\eta+\widetilde{h})(\widetilde{H}(\widehat{\eta}) + \Delta^*\eta - 2\eta)\,d\widehat{S}$$
$$\geq \operatorname{Cr}^{-1}\operatorname{Pr}\lambda\int_{\partial\widehat{\Omega}} (|\Delta^*\eta|^2 + |\nabla^*\eta|^2)\,d\widehat{S}$$
$$- c\{\|\nabla^* v_2\|^2_{H^{1,2}(\widehat{\Omega})} + \lambda\|\eta\|^2_{H^{3/2,2}(\partial\widehat{\Omega})} + \lambda\|\widetilde{h}\|^2_{H^{5/2,2}(\widehat{\Omega})} + \|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}\}.$$

In analogy to the first intermediate result above, for the $v_2$-system we obtain:

Intermediate Result 1′:

$$\lambda\|\nabla^* v_2\|^2_{H^{0,2}(\widehat{\Omega})} + \|\nabla^* v_2\|^2_{H^{1,2}(\widehat{\Omega})} + 2\operatorname{Cr}^{-1}\operatorname{Pr}^2\lambda\|\eta\|^2_{H^{2,2}(\partial\widehat{\Omega})}$$
$$\leq c\{\|p - q - (p-q)_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})} + \|h\|^2_{H^{1/2}(\partial\widehat{\Omega})}$$
$$+ \|v_1\|^2_{H^{2,2}(\widehat{\Omega})} + \|\eta\|^2_{H^{3/2,2}(\partial\widehat{\Omega})} + \lambda\|\widetilde{h}\|^2_{H^{5/2,2}(\widehat{\Omega})}$$
$$+ \|\nabla q\|^2_{H^{0,2}(\widehat{\Omega})} + \|v_2\|^2_{H^{1,2}(\widehat{\Omega})} + \|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}\}.$$

In order to bound the tangential gradient of the pressure term $\nabla^*(p-q)$ we choose $\Delta^*\Phi\chi^2$ as a test function and obtain the integral equation

$$\begin{aligned}
&-\int_{\widehat{\Omega}}(p-q-(p-q)_{\widehat{\Omega}})\nabla\cdot\Delta^*\chi^2\,dx\\
&= -\lambda\int_{\widehat{\Omega}} v_2\Delta^*\Phi\chi^2\,dx-\Pr\int_{\widehat{\Omega}} D(v_2):D(\Delta^*\Phi\chi^2)\,dx+\operatorname{Ma}\Pr\int_{\partial\widehat{\Omega}}\Delta^*\Phi\cdot\widehat{\nu}h\,d\widehat{S}\\
&\quad-\int_{\partial\widehat{\Omega}}\Delta^*\Phi\cdot\widehat{\nu}\widehat{\nu}\cdot T(v_1,q)\widehat{\nu}\,d\widehat{S}+2\operatorname{Cr}^{-1}\Pr^2\int_{\partial\widehat{\Omega}}\Delta^*\Phi\cdot\widehat{\nu}\widetilde{H}(\eta)\,d\widehat{S}\\
&\quad-\int_{\widehat{\Omega}}\widehat{v}\cdot\nabla v_2\Delta^*\Phi\chi^2\,dx-\int_{\widehat{\Omega}} v_2\cdot\nabla\widehat{v}\Delta^*\Phi\chi^2\,dx=0.
\end{aligned}$$

Again only the boundary integrals are under consideration. We point out that

$$\Delta^*\Phi\cdot\widehat{\nu}=\Delta^*(\Phi\cdot\widehat{\nu})-\nabla^*\Phi\cdot\nabla^*\widehat{\nu}-\Delta^*\widehat{\nu}\cdot\Phi,$$

and the first summand vanishes identically because of the boundary condition $\Phi\cdot\widehat{\nu}=0$. Thus all boundary integrals can be estimated:

$$\begin{aligned}
&\Big|\operatorname{Ma}\Pr\int_{\partial\widehat{\Omega}}\Delta^*\Phi\cdot\widehat{\nu}h\,d\widehat{S}-\int_{\partial\widehat{\Omega}}\Delta^*\Phi\cdot\widehat{\nu}\widehat{\nu}\cdot T(v_1,q)\widehat{\nu}\,d\widehat{S}+2\operatorname{Cr}^{-1}\Pr\int_{\partial\widehat{\Omega}}\Delta^*\Phi\cdot\widehat{\nu}\widetilde{H}(\eta)\,d\widehat{S}\Big|\\
&\quad\le c\{\operatorname{Ma}\Pr\|h\|_{H^{1/2,2}(\partial\widehat{\Omega})}\|\nabla^*\nabla\Phi\|_{H^{0,2}(\widehat{\Omega})}+\|\nabla q\|_{H^{0,2}(\widehat{\Omega})}\|\nabla^*\nabla\Phi\|_{H^{0,2}(\widehat{\Omega})}\\
&\qquad+2\operatorname{Cr}^{-1}\Pr\|\eta\|_{H^{3/2,2}(\partial\widehat{\Omega})}\|\nabla^*\nabla\Phi\|_{H^{0,2}(\widehat{\Omega})}\}.
\end{aligned}$$

We are now able to write down the second intermediate result:

Intermediate Result 2′.

$$\begin{aligned}
&\|\nabla^*(p-q)\|_{H^{0,2}(\widehat{\Omega})}\\
&\qquad\le c\{\Pr\|\nabla^*v_2\|_{H^{1,2}(\widehat{\Omega})}+\|v_1\|_{H^{2,2}(\widehat{\Omega})}+\|\nabla q\|_{H^{0,2}(\widehat{\Omega})}\\
&\qquad\quad+\lambda\operatorname{Ma}\Pr\|\nabla^*v_2\|_{H^{0,2}(\widehat{\Omega})}\|h\|_{H^{1/2,2}(\partial\widehat{\Omega})}+\|v_2\|_{H^{1,2}(\widehat{\Omega})}\\
&\qquad\quad+2\operatorname{Cr}^{-1}\Pr\|\eta\|_{H^{3/2,2}(\partial\widehat{\Omega})}+\|p-q-(p-q)_{\widehat{\Omega}}\|_{H^{0,2}(\widehat{\Omega})}\}.
\end{aligned}$$

We combine the two intermediate results using the estimates obtained before:

Intermediate Result 3′.

$$\begin{aligned}
&\lambda\|\nabla^*v_2\|^2_{H^{0,2}(\widehat{\Omega})}+\|\nabla^*v_2\|^2_{H^{1,2}(\widehat{\Omega})}+\|\nabla^*(p-q)\|^2_{H^{0,2}(\widehat{\Omega})}+\lambda\|\eta\|^2_{H^{2,2}(\partial\widehat{\Omega})}\\
&\le c\{\|f\|^2_{H^{0,2}(\widehat{\Omega})}+\|g\|^2_{H^{0,2}(\widehat{\Omega})}+\|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})}+\|\eta\|^2_{H^{3/2,2}(\partial\widehat{\Omega})}+\lambda\|\widetilde{h}\|^2_{H^{3/2,2}(\widehat{\Omega})}\\
&\quad+\|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}+\|v_2\|^2_{H^{1,2}(\widehat{\Omega})}+\|p-q-(p-q)_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})}\}.
\end{aligned}$$

The second normal derivatives are obtained in the same manner as before. We write down the result.

Intermediate Result 4′.

$$\begin{aligned}\lambda\|v_2\|^2_{H^{1,2}(\widehat{\Omega})} + \|v_2\|^2_{H^{2,2}(\widehat{\Omega})} + \|\nabla(p-q)\|^2_{H^{0,2}(\widehat{\Omega})} + \lambda\|\eta\|^2_{H^{2,2}(\partial\widehat{\Omega})} \\ \le c\{\|f\|^2_{H^{0,2}(\widehat{\Omega})} + \|g\|^2_{H^{0,2}(\widehat{\Omega})} + \|h\|^2_{H^{1/2,2}(\partial\widehat{\Omega})} \\ + \lambda\|\widetilde{h}\|^2_{H^{3/2,2}(\widehat{\Omega})} + \|\eta\|^2_{H^{3/2,2}(\partial\widehat{\Omega})} + \|v_2\|^2_{H^{1,2}(\widehat{\Omega})} \\ + \|p-q-(p-q)_{\widehat{\Omega}}\|^2_{H^{0,2}(\widehat{\Omega})} + \|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}\}.\end{aligned}$$

We get higher regularity for the free boundary. The equation for $\eta$ is

$$2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,\varDelta^*\eta = \widehat{\nu}\cdot T(v_2, p-q)\widehat{\nu} - \mathrm{Ma}\,\mathrm{Pr}\,h + \widehat{\nu}\cdot T(v_1,q)\widehat{\nu}.$$

We observe that the right side is in $H^{1/2,2}(\partial\widehat{\Omega})$ and thus $\eta\in H^{5/2,2}(\partial\widehat{\Omega})$.

We now return to system (24)–(34). As mentioned the linearised subsystems satisfy the conditions formulated in [ADN], p. 78. Thus we may apply Theorem 10.5 of [ADN] to each of them. Adding the two results then gives

Theorem 3.1. *Let $m \ge 0$ and $(f,g,h) \in H^{m,2}(\widehat{\Omega}) \times H^{m,2}(\widehat{\Omega}) \times H^{m+1/2,2}(\partial\widehat{\Omega})$. Then the solution to*

$$\begin{aligned}\lambda v - \mathrm{Pr}\,\varDelta v + \widehat{v}\cdot\nabla v + v\cdot\nabla\widehat{v} + \nabla p &= f,\\ \nabla\cdot v &= 0,\\ \lambda\theta - \varDelta\theta + \widehat{v}\cdot\nabla\theta + v\cdot\nabla\widehat{\theta} &= g\end{aligned}$$

*in $\widehat{\Omega}$, together with the boundary conditions*

$$\begin{aligned}T(v,p)\widehat{\nu} - \mathrm{Ma}\,\mathrm{Pr}\,\nabla\theta &= 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,\widetilde{H}(\eta)\widehat{\nu},\\ v\cdot\widehat{\nu} &= \lambda(\eta+\widetilde{h}),\\ \widehat{\nu}\cdot\nabla\theta &= h\end{aligned}$$

*in $\partial\widehat{\Omega}$ and with zero initial values is regular, i.e.*

$$(v, p-p_{\widehat{\Omega}}, \theta, \eta) \in H^{m+2,2}(\widehat{\Omega}) \times H^{m+1,2}(\widehat{\Omega}) \times H^{m+2,2}(\widehat{\Omega}) \times H^{m+5/2,2}(\partial\widehat{\Omega})$$

*and satisfies the estimate*

$$\begin{aligned}\lambda\|v\|^2_{H^{(m+2)/2,2}(\widehat{\Omega})} + \lambda\|\theta\|^2_{H^{(m+2)/2,2}(\widehat{\Omega})} + \|v\|^2_{H^{m+2,2}(\widehat{\Omega})} + \|p-p_{\widehat{\Omega}}\|^2_{H^{m+1,2}(\widehat{\Omega})} \\ + \|\theta\|^2_{H^{m+2,2}(\widehat{\Omega})} + \lambda\|\eta\|^2_{H^{m+5/2,2}(\partial\widehat{\Omega})} \\ \le c\{\|f\|^2_{H^{m,2}(\widehat{\Omega})} + \|g\|^2_{H^{m,2}(\widehat{\Omega})} + \|h\|_{H^{m+1/2,2}(\partial\widehat{\Omega})} \\ + \lambda\|\widetilde{h}\|^2_{H^{m+5/2,2}(\partial\widehat{\Omega})} + \|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})}\}.\end{aligned}$$

## 4. The nonstationary system

**4.1.** With the help of Sections 2 and 3 we now return to the nonstationary systems (13)–(18) and (19)–(23). There are different possibilities to obtain the existence of a solution. We have chosen the method of Rothe, where the nonstationary system is approximated by a sequence of stationary systems. The stationary systems are given by the time discretisation of the space-time cylinder $\widehat{\Omega}\times(0,T)$ and the equations defined there. As the step size $h$ tends to zero, uniform estimates of the stationary solutions provide enough compactness to conclude the existence of a solution of the nonstationary system.

In Section 2 we proved estimates for (24)–(29) and (30)–(34) uniformly in $\lambda$. If we now set $\lambda = 1/h$ we may use these estimates plus a Gronwall type argument to obtain the desired estimates for the nonstationary systems.

**4.2.** We define a weak solution $(v_1, v_2, \theta, \eta)$ of (13)–(18) and (19)–(23) as an element in $H^{0,2}(0,T;\mathcal{L}(\widehat{\Omega}))\times H^{0,2}(0,T;\mathcal{L}(\widehat{\Omega}))\times H^{0,2}(0,T;H_0^{1,2}(\widehat{\Omega}))\times H_0^{0,2}(0,T;H^{1,2}(\partial\widehat{\Omega}))$ which satisfies the integral equations

$$\begin{aligned}(41)\quad &\int_0^T\int_{\widehat{\Omega}}\partial_t v_1\cdot\phi_1\,dx+\Pr\int_0^T\int_{\widehat{\Omega}}D(v_1):D(\phi_1)\,dx\,dt\\ &\quad-\mathrm{Ma}\Pr\sum_{i=1}^{2}\int_0^T\int_{\partial\widehat{\Omega}}\phi_1\cdot\tau_i\tau_i\cdot\nabla\theta+\int_0^T\int_{\widehat{\Omega}}v_1\cdot\nabla\widehat{v}\phi_1\,dx\,dt\\ &\quad+\int_0^T\int_{\widehat{\Omega}}\widehat{v}\cdot\nabla v_1\phi_1\,dx\,dt=\int_0^T\int_{\widehat{\Omega}}f\cdot\phi_1\,dx\,dt;\end{aligned}$$

$$\begin{aligned}(42)\quad &\int_0^T\int_{\widehat{\Omega}}\partial_t\theta\psi\,dx\,dt+\int_0^T\int_{\widehat{\Omega}}\nabla\theta\nabla\psi\,dx\,dt+\int_0^T\int_{\widehat{\Omega}}\widehat{v}\cdot\nabla\theta\psi\,dx\,dt\\ &\quad+\int_0^T\int_{\widehat{\Omega}}v_1\cdot\nabla\widehat{\theta}\psi\,dx\,dt=\int_0^T\int_{\widehat{\Omega}}g\psi\,dx\,dt+\int_0^T\int_{\partial\widehat{\Omega}}h\psi\,d\widehat{S}\,dt;\end{aligned}$$

$$\begin{aligned}(43)\quad &\int_0^T\int_{\widehat{\Omega}}\partial_t v_2\cdot\phi_2\,dx\,dt+\Pr\int_0^T\int_{\widehat{\Omega}}D(v_2):D(\phi_2)\,dx\,dt\\ &\quad-\mathrm{Ma}\Pr\int_0^T\int_{\partial\widehat{\Omega}}\phi_2\cdot\widehat{\nu}h\,d\widehat{S}\,dt+\int_0^T\int_{\partial\widehat{\Omega}}\phi_2\cdot\widehat{\nu}\widehat{\nu}\cdot T(v_1,q)\widehat{\nu}\,d\widehat{S}\,dt\\ &\quad-2\,\mathrm{Cr}^{-1}\Pr\int_0^T\int_{\partial\widehat{\Omega}}\phi_2\cdot\widehat{\nu}\widetilde{H}(\eta)\,d\widehat{S}\,dt+\int_0^T\int_{\widehat{\Omega}}v_2\cdot\nabla\widehat{v}\phi_2\,dx\,dt\\ &\quad+\int_0^T\int_{\widehat{\Omega}}\widehat{v}\cdot\nabla v_2\phi_2\,dx\,dt=0\end{aligned}$$

for all $(\phi_1, \phi_2, \psi) \in \mathcal{L}(\widehat{\Omega}) \times \mathcal{L}(\widehat{\Omega}) \times H^{1,2}(\widehat{\Omega})$. The data will satisfy the assumptions

$$f \in H^{0,2}(0,T; H^{0,2}(\widehat{\Omega})), \quad g \in H^{0,2}(0,T; H^{0,2}(\widehat{\Omega})), \quad h \in K^{1/2}(\widehat{\Omega} \times (0,T)).$$

We now cut the space time cylinder $Q^T = \widehat{\Omega} \times (0,T)$ with planes $t = t_k = kh$.

NOTATION. We denote by $\widehat{\Omega}_k$ the intersection of $Q^T$ with the plane $t = t_k$, $k = 1, \ldots, [T/h]$. For the quantities defined on $\widehat{\Omega}_k$ we introduce the following notation:

$$v_{1,2}(k) = v_{1,2}(x, t_k), \quad p(k) = p(x, t_k), \quad \theta(k) = \theta(x, t_k), \quad \eta(k) = \eta(x, t_k),$$

$$\delta_h v_{1,2}(k) = \frac{v_{1,2}(k) - v_{1,2}(k-1)}{h}, \quad \delta_h \theta(k) = \frac{\theta(k) - \theta(k-1)}{h},$$

$$\delta_h \eta(k) = \frac{\eta(k) - \eta(k-1)}{h},$$

$$f_h(k) = \frac{1}{h} \int\limits_{(k-1)h}^{kh} f(x,\tau)\, d\tau, \quad g_h(k) = \frac{1}{h} \int\limits_{(k-1)h}^{kh} g(x,\tau)\, d\tau,$$

$$h_h(k) = \frac{1}{h} \int\limits_{(k-1)h}^{kh} h(x,\tau)\, d\tau.$$

We now can explain the appearance of $\widetilde{h}$ in Section 2. It stands for $\eta(k-1)$ and thus may be considered as a known quantity computed one time step before. We should have introduced a similar notation for $v$ and $\theta$; however, this would not have changed the result.

The discretised integral equations are:

$$\begin{aligned} h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} \delta_h v_1(k) \cdot \phi_1(k)\, dx + \Pr h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} D(v_1(k)) : D(\phi_1(k))\, dx \\ - \operatorname{Ma} \Pr h \sum_{i=1}^{2} \sum_{k=1}^{m} \int\limits_{\partial\widehat{\Omega}} \phi_1(k) \cdot \tau_i \tau_i \cdot \nabla\theta(k)\, d\widehat{S} + h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} v_1(k) \cdot \nabla \widehat{v} \phi_1(k)\, dx \\ + h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} \widehat{v} \cdot \nabla v_1(k) \phi_1(k)\, dx = h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} f_h(k) \cdot \phi_1(k)\, dx; \end{aligned}$$

$$\begin{aligned} h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} \delta_h \theta(k) \psi(k)\, dx + h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} \nabla\theta(k) \cdot \nabla\psi(k)\, dx + h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} \widehat{v} \cdot \nabla\theta(k) \psi(k)\, dx \\ + h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} v_1(k) \cdot \nabla\widehat{\theta} \psi(k)\, dx = h \sum_{k=1}^{m} \int\limits_{\widehat{\Omega}} g_h(k) \psi(k)\, dx + h \sum_{k=1}^{m} \int\limits_{\partial\widehat{\Omega}} h_h(k) \psi(k)\, d\widehat{S}; \end{aligned}$$

$$\int_0^T \int_{\widehat{\Omega}} \partial_t v_2 \cdot \phi_2 \, dx + \Pr \int_0^T \int_{\widehat{\Omega}} D(v_2) : D(\phi_2) \, dx - \operatorname{Ma} \Pr \int_0^T \int_{\partial\widehat{\Omega}} \phi_2 \cdot \widehat{\nu} h \, d\widehat{S}$$
$$+ \int_0^T \int_{\partial\widehat{\Omega}} \phi_2 \cdot \widehat{\nu}\widehat{\nu} \cdot T(v_1, q)\widehat{\nu} d\widehat{S} - 2 \operatorname{Cr}^{-1} \Pr \int_0^T \int_{\partial\widehat{\Omega}} \phi_2 \cdot \widehat{\nu} \widetilde{H}(\eta) \, d\widehat{S}$$
$$+ \int_0^T \int_{\widehat{\Omega}} v_2 \cdot \nabla \widehat{v} \phi_2 \, dx + \int_0^T \int_{\widehat{\Omega}} \widehat{v} \cdot \nabla v_2 \phi_2 \, dx = 0.$$

We now proceed exactly as in Section 2 to arrive at the following a priori estimate:

LEMMA 4.1. *A weak solution of the nonstationary system satisfies the a priori estimate*

$$\int_{\widehat{\Omega}} |v(m)|^2 \, dx + \int_{\widehat{\Omega}} |\theta(m)|^2 \, dx + \|\eta(m)\|^2_{H^{1,2}(\partial\widehat{\Omega})} + h \sum_{k=1}^m \|v(k)\|^2_{H^{1,2}(\widehat{\Omega})}$$
$$+ h \sum_{k=1}^m \|\theta(k)\|^2_{H^{1,2}(\widehat{\Omega})} + h \sum_{k=1}^m \|(q - q_{\widehat{\Omega}})(k)\|^2_{H^{0,2}(\widehat{\Omega})} + h \sum_{k=1}^m \|\eta(k)\|_{H^{1,2}(\partial\widehat{\Omega})}$$
$$\le ch \sum_{k=1}^m \{ \|f_h(k)\|^2_{H^{0,2}(\widehat{\Omega})} + \|g_h(k)\|^2_{H^{0,2}(\widehat{\Omega})}$$
$$+ \|h_h(k)\|^2_{H^{1/2,2}(\partial\widehat{\Omega})} + \|H(\widehat{\eta})\|^2_{H^{0,2}(\partial\widehat{\Omega})} \}.$$

Here we have also used a discrete version of Gronwall's inequality (see e.g. [Kacur], Lemma 1.3.19(ii), p. 29). Instead of presenting the details we just mention that we may now extract a subsequence for which we have weak convergence in the spaces chosen. Furthermore the limit satisfies the integral equations (41)–(43). Finally we have to recover the pressure.

We write down the result:

THEOREM 4.1. *The linearised nonstationary system is uniquely solvable for any given time interval* $(0, T)$ *with*

$$v \in H^{0,2}(0, T; H^{1,2}(\widehat{\Omega})), \quad p - p_{\widehat{\Omega}} \in H^{0,2}(0, T; H^{0,2}(\widehat{\Omega})),$$
$$\theta \in H^{0,2}(0, T; H^{1,2}(\widehat{\Omega})), \quad \eta \in H^{0,2}(0, T; H^{1,2}(\partial\widehat{\Omega})).$$

*We have the estimate*

$$\|v\|_{H^{0,2}(0,T;H^{1,2}(\widehat{\Omega}))} + \|\theta\|_{H^{0,2}(0,T;H^{1,2}(\widehat{\Omega}))}$$
$$+ \|p - p_{\widehat{\Omega}}\|_{H^{0,2}(0,T;H^{0,2}(\widehat{\Omega}))} + \|\eta\|_{H^{0,2}(0,T;H^{1,2}(\partial\widehat{\Omega}))}$$
$$\le c\{ \|f\|_{H^{0,2}(0,T;H^{0,2}(\widehat{\Omega}))} + \|g\|_{H^{0,2}(0,T;H^{0,2}(\widehat{\Omega}))}$$
$$+ \|h\|_{H^{0,2}(0,T;H^{1/2,2}(\partial\widehat{\Omega}))} + \|H(\widehat{\eta})\|_{H^{0,2}(\widehat{\Omega})} \}.$$

**4.3.** The regularity problem can be treated as in Section 3. The only difference is that the estimate for $\lambda\|v_1\|_{H^{0,2}(\widehat{\Omega})}$ now turns into an estimate for the first time derivative for $v_1$. Similarly for $v_2$ and $\eta$.

THEOREM 4.2. *For the linearised system*

$$\begin{aligned}\partial_t v - \operatorname{Pr} \Delta v + \widehat{v}\cdot\nabla v + v\cdot\nabla\widehat{v} + \nabla p = f,\\ \nabla\cdot v = 0,\\ \partial_t\theta - \Delta\theta + \widehat{v}\cdot\nabla\theta + v\cdot\nabla\widehat{\theta} = g\end{aligned}$$

*in* $\widehat{\Omega}\times(0,T)$, *with the boundary conditions*

$$\begin{aligned}T(v,p)\widehat{\nu} - \operatorname{Ma}\operatorname{Pr}\nabla\theta = 2\operatorname{Cr}^{-1}\operatorname{Pr}\widetilde{H}(\eta)\widehat{\nu},\\ v\cdot\widehat{\nu} = \partial_t\eta,\\ \widehat{\nu}\cdot\nabla\theta = h,\end{aligned}$$

*assume*

$$f\in K^0(\widehat{\Omega}\times(0,T)),\qquad g\in K^0(\widehat{\Omega}\times(0,T)),\qquad h\in K^{1/2}(\partial\widehat{\Omega}\times(0,T))$$

*and zero initial values. Then the weak solution is regular*, *i.e.*

$$\begin{aligned}v(x,t)\in K^2(\widehat{\Omega}\times(0,T)),\quad \theta(x,t)\in K^2(\widehat{\Omega}\times(0,T)),\\ \nabla p\in K^0(\widehat{\Omega}\times(0,T)),\qquad \eta\in K^{5/2}(\partial\widehat{\Omega}\times(0,T)).\end{aligned}$$

*The quantities are bounded by constants which only depend on the data of the system.*

We obtain higher regularity.

THEOREM 4.3. *For the linearised problem let* $f,g\in K^{r-2}(\widehat{\Omega}\times(0,T))$ *and* $h\in K^{r-3/2}(\partial\widehat{\Omega}\times(0,T))$. *Then*

$$\begin{aligned}(v,\nabla p,\theta,\eta)\in K^r(\widehat{\Omega}\times(0,T))\times K^{r-2}(\widehat{\Omega}\\ \times(0,T))\times K^r(\widehat{\Omega}\times(0,T))\times K^{r+1/2}(\partial\widehat{\Omega}\times(0,T)).\end{aligned}$$

*The quantities are bounded by a constant which only depends on the data of the system.*

**4.4.** We do not treat the nonlinear system in detail, since it can be done as in [Beale2]. Instead we just give the main steps, leading to the existence theorem.

We are now concerned with the fully nonlinear system, transformed to a ball. On the left side we put the linearisation, while all the terms of higher order will be written on the right side.

The highest derivatives appearing on the right side are: third spatial derivatives of the transformation, mixed spatial and time derivatives of the transformation, second spatial derivatives of the velocity field and of the temperature and first derivatives of the pressure.

Terms on the right side are products of such derivatives.

We use the fact that $K^r$ is an algebra if $r$ is large enough (see p. 332, Lemma 5.1 in [Beale2]):

LEMMA 4.2. *Let $r > 5/2$. Then elements of $K^r(\widehat{\Omega} \times (0,T))$ are continuous functions on the closure of the domain.*

*If $f \in K^s(\widehat{\Omega} \times (0,T))$ with $r > s \geq 0$ and $g \in K^r(\widehat{\Omega} \times (0,T))$, then*

$$\|g \cdot f\|_{K^s} \leq c\|g\|_{K^r}\|g\|_{K^r}.$$

Next we extend $\eta$ in the interior of the domain. For that we use an extension theorem in [L&M] (Theorem 1.4.2). As a result we gain half a derivative, thus for the extension we have $\nabla\widetilde{\eta} \in K^r(\widehat{\Omega} \times (0,T))$. Now choose $r > 5/2$.

As a consequence, for the right side of the Navier–Stokes system ($= F_0(\eta, v, \nabla p, \theta)$) and for the right side of the heat equation ($= F_1(\eta, v, \nabla p, \theta)$) we obtain an estimate

$$\|F_{0,1}(\eta, v, \nabla p, \theta)\|_{K^{r-2}} \leq c\{\|\eta\|_{K^{r+1/2}} + \|v\|_{K^r} + \|\theta\|_{K^r} + \|\nabla p\|_{K^{r-2}} + K_r\}.$$

Here $K_r$ denotes the norm of the forces. We get the same estimate for the right side of the boundary equations ($= F_3, F_4, F_5, F_6$). This was the first step.

We now write our system in the form

$$Lz = F(z)$$

where $z = (\eta, v, p, \theta)$, $F = (F_1, F_2, F_3, F_4, F_5, F_6)$ and $L$ denotes the linearisation.

Let $X^r$ be the space of all $z$ and $Y^{r-2}$ the image of $L$. The estimates for the $F_i$ can now be written as

$$|F(z)|_{Y^{r-2}} \leq c|z|^2_{X^r}.$$

From the system we also have the inequality

$$|F(z) - F(\widetilde{z})|_{Y^r} \leq c|z - \widetilde{z}|_{X^r}(|z|_{X^r} + |\widetilde{z}|_{X^r}).$$

Looking at our transformation $\Phi$ and using the fact that $r > 5/2$ we see that the transformation is in fact a $C^1$-diffeomorphism (as long as the norm of $\eta$ is small). Thus already because of this observation we may expect existence only for short time.

In the third step it is shown that the solution operator is a contraction. We record the final result.

THEOREM 4.4. *The free boundary problem*

$$\begin{aligned}
\partial_t v - \Pr \Delta v + v \cdot \nabla v + \nabla p &= f, \\
\nabla \cdot v &= 0, \\
\partial_t \theta - \Delta\theta + v \cdot \nabla\theta &= g,
\end{aligned}$$

*with the boundary conditions*

$$T(v,p)\nu - \mathrm{Ma}\,\mathrm{Pr}\,\nabla\theta = 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,H\nu,$$
$$v\cdot\nu = \partial_t\eta,$$
$$\nu\cdot\nabla\theta = h$$

*and sufficiently large* Pr *number and small* Ma *number*, *is uniquely solvable for* $f, g \in K^r(\Omega\times(0,T))$, $h \in K^{r+1/2}(\partial\Omega\times(0,T))$ $(r > 1/2)$ *and for small time* $0 < t < T' < T$, *where* $T'$ *depends on the norms of the forces.*

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