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BEHAVIOUR OF GLOBAL SOLUTIONS FOR A SYSTEM OF REACTION-DIFFUSION EQUATIONS FROM COMBUSTION THEORY

Abstract. We are concerned with the boundedness and large time behaviour of the solution for a system of reaction-diffusion equations modelling complex consecutive reactions on a bounded domain under homogeneous Neumann boundary conditions. Using the techniques of E. Conway, D. Hoff and J. Smoller [3] we also show that the bounded solution converges to a constant function as $t \to \infty$. Finally, we investigate the rate of this convergence.

1. Introduction. In this paper we investigate the asymptotic behaviour of global solutions for the following reaction-diffusion system:

$$\frac{\partial Y_1}{\partial t} = d_0 \Delta Y_1 - d_1 Y_1 Y_2 f_1(T), \qquad x \in \Omega,\ t > 0, \tag{1.1}$$

$$\begin{aligned}\frac{\partial Y_2}{\partial t} &= d_2 \Delta Y_2 + d_3 Y_1 Y_2 f_1(T)\\ &\quad - d_4 Y_2 f_2(T) - d_5 Y_2 - d_6 Y_2^2, \qquad x \in \Omega,\ t > 0,\end{aligned} \tag{1.2}$$

$$\begin{aligned}\frac{\partial T}{\partial t} &= d_7 \Delta T + d_8 Y_1 Y_2 f_1(T)\\ &\quad + d_9 Y_2 f_2(T) + d_{10} Y_2 + d_{11} Y_2^2, \qquad x \in \Omega,\ t > 0,\end{aligned} \tag{1.3}$$

$$\frac{\partial Y_1}{\partial \nu} = \frac{\partial Y_2}{\partial \nu} = \frac{\partial T}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.4}$$

$$(Y_1, Y_2, T)(x, 0) = (Y_{10}, Y_{20}, T_0)(x), \qquad x \in \Omega, \tag{1.5}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, such that $\partial\Omega$ is a C^m hypersurface separating Ω from $\mathbb{R}^n/\overline{\Omega}$ $(m \geq 1)$, d_j $(j = 0, 1, \ldots, 11)$ are positive constants, f_i $(i = 1, 2)$ are given by the Arrhenius law

$$f_i(T) = B_i \exp(-E_i/T),$$

where B_i, E_i are constants, and E_i denotes the activation energy.

This system of reaction-diffusion equations arises as a model of chain branching and chain breaking kinetics of reactions with complex chemistry. Here Y_1 is the concentration of fuel, Y_2 is the concentration of radicals, and T is the dimensionless temperature. Y_1, Y_2 and T depend on x and t where $(x, t) \in \Omega \times \mathbb{R}^+$.

Under suitable conditions (see (CD) in Section 3), it is expected that (1.1)–(1.5) has a unique global solution (Y_1, Y_2, T) and this solution tends to an equilibrium state uniformly in x as $t \to \infty$.

We will show that $(Y_1(t), Y_2(t), T(t))$ approaches an equilibrium state $(0, 0, T_\infty)$ in $C^\mu(\overline{\Omega})^3$ as $t \to \infty$ for every $\mu \in [0, 2)$, where T_∞ is a constant, and we will consider the rate of this convergence, by means of integral equations, fractional powers of operators, Poincaré's inequality and some imbedding theorems.

2. Preliminary results. We state some results needed in the sequel.

LEMMA 2.1. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two real Banach spaces with continuous inclusion $E \subset F$. Let A be a linear operator generating a strongly continuous semigroup $G(t)$ in E such that*:

(i) *$G(t)E \subset F$ for all $t > 0$,*
(ii) *there exists $\theta \in [0, 1)$ such that $\|G(t)\varphi\|_F \leq ct^{-\theta}\|\varphi\|_E$ for all $t > 0$.*

Moreover, let $p > 1/(1-\theta)$, $f \in L^p_{\mathrm{loc}}(\mathbb{R}^+, E)$ and $\sup_{t\geq 0} \|f\|_{L^p(t,t+1;E)} < \infty$. Let u be a mild solution on $\mathbb{R}^+$ of

$$\frac{du}{dt} = Au(t) + f(t).$$

If $u \in L^\infty(0, \infty; E)$, then $u(t) \in F$ for all $t > 0$ and $u \in C_B(\delta, \infty; F)$ for all $\delta > 0$, where $C_B(\delta, \infty; F)$ is the space of all continuous functions $u : (\delta, \infty) \to F$ such that $\sup\{\|u(t)\|_F : t \geq \delta\} < \infty$.

For the proof, see [4].

LEMMA 2.2. *Let $G(t)$ be the semigroup generated by the operator $d\Delta$ in $L^p(\Omega)$. Then for all $1 \leq p < q \leq \infty$ and all $\varphi \in L^p(\Omega)$ we have $G(t)\varphi \in L^q(\Omega)$ and*

$$\|G(t)\varphi\|_q \leq c(p, q)t^{-(n/2)(1/p-1/q)}\|\varphi\|_p.$$

For the proof, see [2].

3. Global existence and positivity. Throughout this paper, the following assumptions are in force:

(CD) (i) d_j $(j = 0, 1, \ldots, 11)$ are positive constants,
(ii) Y_{10}, Y_{20} and T_0 are nonnegative measurable functions such that $0 \le Y_{10}(x), Y_{20}(x), T_0(x) \le M_0$ for almost every $x \in \Omega$, for some positive constant M_0.

THEOREM 3.1. *Assume* (CD). *Then there exists a unique nonnegative global solution* (Y_1, Y_2, T) *for* (1.1)–(1.5) *which is smooth in* $\overline{\Omega} \times (0, \infty)$.

Proof. For each $1 < p < \infty$ and $j \in \{1, 2, 3\}$ define the linear operator $A_{j,p}$ on $L^p(\Omega)$ by

$$\begin{aligned} &D(A_{j,p}) = \{u \in W^{2,p}(\Omega) : (\partial u/\partial \nu)|\partial\Omega = 0\}, \\ &A_{1,p}u = d_0 \Delta u, \quad A_{2,p}u = d_2 \Delta u, \quad A_{3,p}u = d_7 \Delta u, \end{aligned} \tag{3.1}$$

where $W^{2,p}(\Omega)$ is the usual Sobolev space. It is well known that $A_{j,p}$ generates a compact, analytic contraction semigroup $G_{j,p}(t)$, $t \ge 0$, of bounded linear operators on $L^p(\Omega)$ (see, e.g., Amann [2]).

For the local existence we write (1.1)–(1.3) as a system of integral equations via the variation of constants formula. For simplicity we set

$$\begin{aligned} F_1(Y_1, Y_2, T)(t)(\cdot) &= -d_1 Y_1(t) Y_2(t) f_1(T(t))(\cdot), \\ F_2(Y_1, Y_2, T)(t)(\cdot) &= (d_3 Y_1(t) Y_2(t) f_1(T(t)) - d_4 Y_2(t) f_2(T(t)) \\ &\quad - d_5 Y_2(t) - d_6 Y_2^2(t))(\cdot), \\ F_3(Y_1, Y_2, T)(t)(\cdot) &= (d_8 Y_1(t) Y_2(t) f_1(T(t)) + d_9 Y_2(t) f_2(T(t)) \\ &\quad + d_{10} Y_2(y) + d_{11} Y_2^2(t))(\cdot), \end{aligned}$$

for $x \in \Omega$, $t > 0$; we then have

$$Y_1(t) = G_{1,p}(t) Y_{10} + \int_0^t G_{1,p}(t - \tau) F_1(Y_1(\tau), Y_2(\tau), T(\tau))\, d\tau, \tag{3.2}$$

$$Y_2(t) = G_{2,p}(t) Y_{20} + \int_0^t G_{2,p}(t - \tau) F_2(Y_1(\tau), Y_2(\tau), T(\tau))\, d\tau, \tag{3.3}$$

$$T(t) = G_{3,p}(t) T_0 + \int_0^t G_{3,p}(t - \tau) F_3(Y_1(\tau), Y_2(\tau), T(\tau))\, d\tau. \tag{3.4}$$

For each $\alpha > 0$ define the operator $B_{j,p} = I - A_{j,p}$. Then the fractional powers $B_{j,p}^{-\alpha} = (I - A_{j,p})^{-\alpha}$ exist and are injective, bounded linear operators on $L^p(\Omega)$ (see Pazy [8]). Let $B_{j,p}^{\alpha} = (B_{j,p}^{-\alpha})^{-1}$ and $X_{j,p}^{\alpha} = D(B_{j,p}^{\alpha})$, the domain of $B_{j,p}^{\alpha}$. Then $X_{j,p}^{\alpha}$ is a Banach space with the graph norm $\|u\|_\alpha = \|B_{j,p}^{\alpha} w\|_p$, and for $\alpha > \beta \ge 0$, $X_{j,p}^{\alpha}$ is a dense subspace of X_p^{β} with the inclusion $X^\alpha_{j,p} \subset X^\beta_{j,p}$ compact (we use the convention $X^0_p = L^p(\Omega)$). Also if $0 \le \alpha < 1$ we have

$$X^\alpha_{j,p} \subset C^\mu(\overline{\Omega}) \quad \text{for every } 0 \le \mu < m\alpha - n/p. \tag{3.5}$$

Note that this inclusion is valid even if $p = 1$ (see Henry [5], p. 39).

In addition, $G_{j,p}$ and $B^\alpha_{j,p}$ have the properties summarised in the following lemma.

Lemma 3.2. *The operators G_p and B^α_p satisfy*

(i) $G_{j,p}(t) : L^p(\Omega) \to X^\alpha_{j,p}$ *for all* $t > 0$,
(ii) $G_{j,p}(t)B^\alpha_{j,p}u = B^\alpha_{j,p}G_{j,p}(t)u$ *for every* $u \in X^\alpha_{j,p}$,
(iii) $\|G_{j,p}(t)u\|_\alpha \le C_1(\alpha)t^{-\alpha}e^{-t}\|u\|_p$ *for every* $t > 0$ *and* $u \in L^p(\Omega)$,
(iv) $\|(G_{j,p}(t) - I)u\|_p \le C_2(\alpha)t^\alpha\|u\|_\alpha$ *for* $0 < \alpha \le 1$ *and* $u \in X^\alpha_{j,p}$.

The proof can be found in Pazy [8].

Select $0 < \alpha < 1$ and $p > 1$ so that (3.5) holds, and use the techniques of Pazy [8] to show that there exists a unique noncontinuable solution (Y_1, Y_2, T) to (3.2)–(3.4) for $Y_{10} \in X^\alpha_{1,p}$, $Y_{20} \in X^\alpha_{2,p}$ and $T_0 \in X^\alpha_{3,p}$. The solution satisfies

$$Y_1 \in C([0,\delta]; X^\alpha_{1,p}) \cap C^1((0,\delta); L^p(\Omega)),$$
$$Y_2 \in C([0,\delta]; X^\alpha_{2,p}) \cap C^1((0,\delta); L^p(\Omega)),$$
$$T \in C([0,\delta]; X^\alpha_{3,p}) \cap C^1((0,\delta); L^p(\Omega)),$$

for some $\delta > 0$; and we have $\|Y_1(t)\|_\infty + \|Y_2(t)\|_\infty + \|T(t)\|_\infty \to \infty$ as $t \to t_{\max}$ if $t_{\max} < \infty$.

Suppose now that $(Y_{10}, Y_{20}, T_0) \in L^\infty(\Omega)^3$ and let $\{Y^k_{10}\}^\infty_{k=1}$ be a sequence in $X^\alpha_{1,p}$, $\{Y^k_{20}\}^\infty_{k=1}$ a sequence in $X^\alpha_{2,p}$ and $\{T^k_0\}^\infty_{k=1}$ a sequence in $X^\alpha_{3,p}$ such that $Y^k_{10}, Y^k_{20}, T^k_0 \ge 0$ and $\|Y^k_{10} - Y_{10}\|_p \to 0$, $\|Y^k_{20} - Y_{20}\|_p \to 0$ and $\|T^k_0 - T_0\|_p \to 0$ as $t \to \infty$. Using the equation (3.2) and the properties of $A_{1,p}$ stated in Lemma 3.2, it follows for $\alpha \le \beta < 1$ that

$$\|Y^k_1\|_\beta \le C_\beta t^{-\beta}\|Y^k_{10}\|_p + \int_0^t C_\beta(t-\tau)^{-\beta}\|F_1(Y^k_1(\tau), Y^k_1(\tau), Y^k_1(\tau))\|_p\, d\tau$$

for all $t \in [0, t^k_{\max})$, where $t^k_{\max}$ is the maximal time of existence for the system (1.1)–(1.5) with initial conditions $0 \le (Y^k_{10}, Y^k_{20}, T^k_0) \in X^\alpha_{1,p} \times X^\alpha_{2,p} \times X^\alpha_{3,p}$. From these estimates one can deduce the existence of a $\overline{C}_\beta$ such that

$$\max\{\|Y^k_1(t)\|_\beta, \|Y^k_2(t)\|_\beta, \|T^k(t)\|_\beta\} \le \overline{C}_\beta t^{-\beta}$$

for all $t \in [0,\delta]$, $k \ge 1$; thus $\{(Y^k_1(t), Y^k_2(t), T^k(t))\}^\infty_{k=1}$ is contained in a bounded subset of $X^\beta_{1,p} \times X^\beta_{2,p} \times X^\beta_{3,p}$ for each $t \in (0,\delta]$. So by the compact imbedding of $X^\beta_{j,p}$ in $X^\alpha_{j,p}$ $(j = 1,2,3)$ for $\alpha < \beta < 1$, we see that for each $t \in (0,\delta]$ the sequences $\{Y_1^k(t)\}_{k=1}^{\infty}, \{Y_2^k(t)\}_{k=1}^{\infty}$ and $\{T^k(t)\}_{k=1}^{\infty}$ contain convergent subsequences $\{Y_1^{k,i}(t)\}_{i=1}^{\infty}, \{Y_2^{k,i}(t)\}_{i=1}^{\infty}$ and $\{T^{k,i}(t)\}_{i=1}^{\infty}$ in $X_{1,p}^{\alpha}, X_{2,p}^{\alpha}$ and $X_{3,p}^{\alpha}$ respectively.

Now define

$$Y_1(t) = \lim_{i\to\infty} Y_1^{k,i}(t), \qquad Y_2(t) = \lim_{i\to\infty} Y_2^{k,i}(t), \qquad T(t) = \lim_{i\to\infty} T^{k,i}(t)$$

for each $t \in [0,\delta]$. Then $(Y_1(t), Y_2(t), T(t))$ satisfies (3.2)–(3.4) for each $t \in [0,\delta]$. Replacing $[0, t_{\max})$ with $[\delta, t_{\max})$ and (Y_{10}, Y_{20}, T_0) by $(Y_1(\delta), Y_2(\delta), T(\delta))$ and using the results already established when $(Y_{10}, Y_{20}, T_0) \in X_{1,p}^{\alpha} \times X_{2,p}^{\alpha} \times X_{3,p}^{\alpha}$, we find that there is a unique, classical noncontinuable solution $(Y_1(t), Y_2(t), T(t))$ on $\Omega \times [0, t_{\max})$, for every $(Y_{10}, Y_{20}, T_0) \in (L^{\infty}(\Omega))^3$.

Since $F_1(0, Y_2, T) \ge 0, F_2(Y_1, 0, T) \ge 0$ and $F_3(Y_1, Y_2, 0) \ge 0$ it follows that $Y_1(t)$, $Y_2(t)$ and $T(t)$ have nonnegative values on Ω (see [10]), and by the maximum principle we have

$$\|Y_1(t)\|_{\infty} \le \|Y_{10}\|_{\infty} \quad \text{for all } t \in [0, t_{\max}). \tag{3.6}$$

Multiplying (1.2) by Y_2^{p-1} and integrating the result over $\Omega \times (0,t)$ we obtain

$$\frac{1}{n}\frac{d}{dt}\int_{\Omega} Y_2^p\,dx \le c\int_{\Omega} Y_2^p\,dx,$$

where $c = d_3\|Y_{10}\|_{\infty}\|f_1(T(t))\|_{\infty}$, hence

$$\int_{\Omega} Y_2^p\,dx \le |\Omega|\,\|Y_{20}\|_{\infty}e^{npt} \quad \text{for all } t < t_{\max}.$$

We can then deduce

$$\|Y_2(t)\|_{\infty} \le e^{ct}\|Y_{20}\|_{\infty} \quad \text{for all } t < t_{\max}. \tag{3.7}$$

From the expression of $F_3(Y_1, Y_2, T)$ and (3.7) we can find two positive numbers c_1 and c_2 such that

$$\|F_3(Y_1(T), Y_2(T), T(t))\|_{\infty} \le e^{ct}(c_1 + c_2e^{ct}) \quad \text{for all } t < t_{\max}, \tag{3.8}$$

where $c_1 = B_1d_8\|Y_{10}\|_{\infty} + d_9B_2 + d_{10}$ and $c_2 = d_{11}\|Y_{20}\|_{\infty}$.

From (3.4) and (3.8) we obtain

$$\|T(t)\|_{\infty} \le \|T_0\|_{\infty} + \int_0^t e^{c\tau}(c_1 + c_2e^{c\tau})\,d\tau,$$

from which we have

$$\|T(t)\|_{\infty} \le \|T_0\|_{\infty} + \frac{c_1}{c}(e^{ct} - 1) + \frac{c_2}{2c}(e^{2ct} - 1) \quad \text{for all } t < t_{\max}. \tag{3.9}$$

Inequalities (3.6), (3.7) and (3.9) contradict the fact that $t_{\max} < \infty$, hence $t_{\max} = \infty$.

4. Boundedness of the solution. In fact, the solution obtained in Theorem 3.1 is uniformly bounded over $\Omega \times (0,\infty)$.

THEOREM 4.1. *Assume* (CD). *Then there exists a positive number* M *such that*

$$0 \le Y_1(x,t) \le \|Y_{10}\|_\infty \quad \text{for } x \in \Omega,\ t \ge 0, \tag{4.1}$$

$$0 \le Y_2(x,t),\ T(x,t) \le M \quad \text{for } x \in \Omega,\ t \ge 0. \tag{4.2}$$

Proof. The function Y_1 is uniformly bounded by $\|Y_{10}\|_\infty$ by the maximum principle.

Let $B(x,t) = d_3 Y_1(x,t) f_1(T(x,t)) - d_4 f_2(T(x,t)) - d_5 - d_6 Y_2(x,t)$. Then we can write

$$\frac{\partial Y_2}{\partial t} = d_2 \Delta Y_2 + B(x,t) Y_2$$

with $B(x,t) \le a$ (for example $a = d_3 \|Y_{10}\| B_1$) and $B(x,t)$ is locally Lipschitz in (x,t). Moreover, $Y_2 \in L^\infty(\mathbb{R}^+, L^1(\Omega))$. In fact, integrating (1.1) over $\Omega \times (0,t)$ we obtain

$$\int_\Omega Y_1(x,t)\,dx = \int_\Omega Y_{10}(x)\,dx - d_1 \int_0^t \int_\Omega Y_1(x,\tau) Y_2(x,\tau) f_1(T(x,\tau))\,dx\,d\tau, \tag{4.3}$$

which implies

$$\int_0^t \int_\Omega (Y_1 Y_2 f_1(T))(x,\tau)\,dx\,d\tau \le \frac{|\Omega|}{d_1} \|Y_{10}\|_\infty \quad \text{for all } t \ge 0, \tag{4.4}$$

where $|\Omega|$ is the Lebesgue measure of Ω. Similarly, we get

$$\int_\Omega Y_2(x,t)\,dx \le \int_\Omega Y_{20}(x)\,dx + d_3 \int_0^t \int_\Omega (Y_1 Y_2 f_1(T))(x,\tau)\,dx\,d\tau \quad \text{for all } t \ge 0. \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$\|Y_2(t)\|_1 \le |\Omega| \left(\|Y_{20}\|_\infty + \frac{d_3}{d_1} \|Y_{10}\|_\infty \right) \quad \text{for all } t \ge 0. \tag{4.6}$$

An application of the result of Alikakos ([1], §3) shows that $Y_2(t)$ is uniformly bounded over $\Omega \times (0,\infty)$:

$$\|Y_2(t)\|_\infty \le K \quad \text{for all } t \ge 0, \tag{4.7}$$

for some $K > 0$.

Now, integrating (1.3) over $\Omega \times (0,t)$ we obtain

$$\int_\Omega T(x,t)\,dx = \int_\Omega T_0(x)\,dx + d_8 \int_0^t \int_\Omega (Y_1 Y_2 f_1(T))(x,\tau)\,dx\,d\tau + d_9 \int_0^t \int_\Omega (Y_2 f_2(T))(x,\tau)\,dx\,d\tau + d_{10} \int_0^t \int_\Omega Y_2(x,\tau)\,dx\,d\tau + d_{11} \int_0^t \int_\Omega Y_2^2(x,\tau)\,dx\,d\tau. \tag{4.8}$$

Integrating (1.2) over $\Omega \times (0,t)$ we obtain

$$\int_\Omega Y_2(x,t)\,dx + d_4 \int_0^t \int_\Omega (Y_2 f_2)(x,\tau)\,dx\,d\tau + d_5 \int_0^t \int_\Omega Y_2(x,\tau)\,dx\,d\tau + d_6 \int_0^t \int_\Omega Y_2^2(x,\tau)\,dx\,d\tau = d_3 \int_0^t \int_\Omega (Y_1 Y_2)(x,\tau)\,dx\,d\tau + \int_\Omega Y_{20}(x)\,dx, \tag{4.9}$$

from which we deduce that

$$\int_0^\infty \int_\Omega (Y_2 f_2)(x,\tau)\,dx\,d\tau < \infty \quad \text{and} \quad \int_0^\infty \int_\Omega Y_2^2(x,\tau)\,dx\,d\tau < \infty. \tag{4.10}$$

From (4.4)–(4.7) and (4.10) in (4.8) we obtain

$$\int_\Omega T(x,t)\,dx \le C \quad \text{for all } t \ge 0, \tag{4.11}$$

i.e., $T \in L^\infty(\mathbb{R}^+, L^1(\Omega))$.

To prove that $T \in L^\infty(\mathbb{R}^+, L^\infty(\Omega))$ we distinguish two cases. We define $S_p(t) \equiv G_{3,p}(t)$.

Case 1: $n = 1$, i.e., $\Omega = (a,b) \subset \mathbb{R}$. In this case we take $E := L^1(\Omega)$ and $F = C(\overline{\Omega})$. Then Lemma 2.2 shows that

$$\|S_1(t)\varphi\|_\infty \le ct^{-1/2}\|\varphi\|_1 \quad \text{for all } \varphi \in L^1(\Omega). \tag{4.12}$$

Take $\alpha = 3/4$; from Lemma 2.2 and (3.5) we have $S_1(t)L^1(\Omega) \subset C(\overline{\Omega})$. Applying Lemma 2.1, we conclude that $T \in C_B(\delta, \infty; C(\overline{\Omega}))$ for all $\delta > 0$, hence from the result concerning the local existence we obtain

$$\|T(t)\|_\infty \le C \quad \text{for all } t \ge 0.$$

Case 2: $n \ge 2$. Let $q_1 = 1$, $q_r = n/(n-r)$ and $E = L^{q_r}(\Omega)$, $F = L^{q_{r+1}}(\Omega)$ for $r \in \{1, \ldots, n-1\}$. We have $T \in C_B(\mathbb{R}^+, L^{q_1}(\Omega))$, $S_{q_1}(t)L^{q_1}(\Omega) \subset L^{q_2}(\Omega)$ and $\|S_{q_1}(t)\varphi\|_{q_2} \le ct^{-1/2}\|\varphi\|_{q_1}$. Application of Lemma 2.1 gives $T \in C_B(\mathbb{R}^+, L^{q_2}(\Omega))$. Next we take $E = L^{q_2}(\Omega)$ and $F = L^{q_3}(\Omega)$ to obtain $T \in C_B(\mathbb{R}^+, L^{q_3}(\Omega))$. Continuing this process we finally have $T \in$

$C_B(\mathbb{R}^+, L^n(\Omega))$. In the last iteration we take $E = L^n(\Omega)$ and $F = C(\overline{\Omega})$. As $S_n(t)L^n(\Omega) \subset X^\alpha_{3,n}$ and $\|S_n(t)\varphi\|_\infty \le ct^{-1/2}\|\varphi\|_n$ for all $\varphi \in L^n(\Omega)$ and $T \in C_B(\mathbb{R}^+, L^n(\Omega))$, from Lemma 2.1 we conclude that $T \in C_B(\mathbb{R}^+; C(\overline{\Omega}))$.

5. Asymptotic behaviour. First, let us establish a preparatory lemma. Consider the problem

$$\text{(P)} \qquad \begin{cases} \partial u/\partial t + Au = \varphi(t), \\ u(0) = u_0, \end{cases}$$

where $-A$ generates an analytic semigroup $G(t)$ in a Banach space $(X, \|\cdot\|)$ with $\operatorname{Re}\sigma(A) > a > 0$. We have the following lemma.

LEMMA 5.1. *Let X be a Banach space. If $\varphi \in L^\infty(\mathbb{R}^+, X)$ and the problem* (P) *has a bounded global solution $u \in L^\infty(\mathbb{R}^+, X)$ then for all $0 < \alpha < 1$ we have*

(A) $\sup_{t\ge\delta} \|A^\alpha u(t)\| \le C(\alpha, \delta)$ *for any $\delta > 0$, and*

(B) *the function $t \mapsto A^\alpha u(t)$ is Hölder continuous from $[\delta, \infty)$ to X for any $\delta > 0$.*

Proof. The solution u of (P) satisfies the integral equation

$$u(t) = G(t)u_0 + \int_0^t G(t-\tau)\varphi(\tau)\,d\tau, \qquad t > 0.$$

Applying A^α to both sides yields

$$\|A^\alpha u(t)\| \le \|A^\alpha G(t)u_0\| + \int_0^t \|A^\alpha G(t-\tau)\varphi(\tau)\|\,d\tau.$$

From this and Lemma 3.2, we obtain

$$\begin{aligned}\|A^\alpha u(t)\|_p &\le C_1(\alpha)t^{-\alpha}e^{-at}\|u_0\| + \int_0^t C_1(\alpha)(t-\tau)^{-\alpha}e^{-a(t-\tau)}\|\varphi(\tau)\|\,d\tau \\ &\le C_1(\alpha)\|u_0\| + C_1(\alpha)M\Gamma(1-\alpha)a^{\alpha-1}.\end{aligned}$$

Here Γ is the gamma function of Euler. Hence $\|A^\alpha u(t)\|$ is uniformly bounded on $[\delta, \infty)$ for any $\delta > 0$.

To prove (B), we have

$$\begin{aligned}\|A^\alpha u(t+h) - A^\alpha u(t)\| \le{}& \|(G(h)-I)A^\alpha G(t)u_0\| \\ &+ \int_t^{t+h} \|A^\alpha G(t+h-\tau)\varphi(\tau)\|\,d\tau \\ &+ \int_0^t \|(G(h)-I)A^\alpha G(t-\tau)\varphi(\tau)\|\,d\tau.\end{aligned}$$

Set

$$I_1 = \|(G(h) - I)A^\alpha G(t)u_0\|,$$

$$I_2 = \int_t^{t+h} \|A^\alpha G(t + h - \tau)\varphi(\tau)\| \, d\tau,$$

$$I_3 = \int_0^t \|(G(h) - I)A^\alpha G(t - \tau)\varphi(\tau)\| \, d\tau.$$

From the inequalities of Lemma 3.2, there exist two constants $C_1(\alpha), C_2(\alpha)$ such that

$$I_1 \le C_1(\alpha + \beta)C_2(\alpha)t^{-1}e^{-at}\|u_0\|h^\beta,$$

$$I_2 \le MC_1(\alpha)h^{1-\alpha},$$

$$I_3 \le MC_1(\alpha + \beta)C_2(\beta)\Gamma(1 - \alpha - \beta)a^{\alpha+\beta-1}h^\beta,$$

where $M = \sup_{t\ge0} \|\varphi(t)\|_p$ for every $0 < \beta < 1$. Taking $\beta < 1 - \alpha$, we then have for all $t \ge \delta$,

$$\|A^\alpha u(t + h) - A^\alpha u(t)\| \le C(\alpha, \|u_0\|) \max\{h^\beta, h^{1-\alpha}\}.$$

REMARK. As a consequence of this lemma, the function $t \mapsto A^\alpha u(t)$ is uniformly continuous.

The following proposition is also useful in the sequel.

PROPOSITION 5.2. *For any $\delta > 0$, the family $\{Y_1(t) : t \ge \delta\}$ is relatively compact in $C(\overline{\Omega})$.*

Proof. We have $\partial Y_1/\partial t = d_0\Delta Y_1 + F_1(Y_1, Y_2, T)$ where $F_1(Y_1, Y_2, T) = -d_1Y_1Y_2f_1(T)$. There is a positive constant N such that $\|F_1(Y_1, Y_2, T)\|_\infty \le N$ for all $t \ge 0$. Let $0 < \varepsilon < 1$ and $t > \varepsilon$. Then we can write $Y_1(t) = G_{1,\infty}(\varepsilon)Y_1(t-\varepsilon)+[Y_1(t)-G_{1,\infty}(\varepsilon)Y_1(t-\varepsilon)]$, where $G_{1,\infty}(t)$ is the semigroup generated by $d_0\Delta$ with homogeneous Neumann boundary conditions in the Banach space $C(\overline{\Omega})$. We set

$$Y_{1\varepsilon}(t) = G_{1,\infty}(\varepsilon)Y_1(t - \varepsilon) \quad \text{and} \quad \overline{Y}_{1\varepsilon}(t) = Y_1(t) - G_{1,\infty}(\varepsilon)Y_1(t - \varepsilon).$$

Then $\{Y_{1\varepsilon}(t) : t \ge \delta\}$ is relatively compact in $C(\overline{\Omega})$ since $\{Y_1(t - \varepsilon) : t \ge \delta\}$ is bounded and $G_{1,\infty}(\delta)$ is a compact operator. Also,

$$\|\overline{Y}_{1\varepsilon}(t)\|_\infty = \Big\| \int_{t-\varepsilon}^t G_{1,\infty}(t - s)F_1(Y_1, Y_2, T)(s) \, ds \Big\|_\infty \le \varepsilon N,$$

therefore $\{Y_1(t) : t \ge 1\}$ is totally bounded, hence $\{Y_1(t) : t \ge 1\}$ is relatively compact in $C(\overline{\Omega})$. As $\{Y_1(t) : \delta \le t \le 1\}$ is compact in $C(\overline{\Omega})$, it follows that $\{Y_1(t) : t \ge \delta\}$ is relatively compact in $C(\overline{\Omega})$. The same holds true for $\{Y_2(t) : t \ge \delta\}$ and $\{T(t) : t \ge \delta\}$.

THEOREM 5.3. *Under the assumptions* (CD) *we have*

$$\lim_{t\to\infty} \|Y_1(t)\|_\infty = 0, \quad \lim_{t\to\infty} \|Y_2(t)\|_\infty = 0 \tag{5.1}$$

and there exists a positive constant T_∞ *such that*

$$\lim_{t\to\infty} \|T(t) - T_\infty\|_\infty = 0. \tag{5.2}$$

Proof. From (1.1) we have

$$\frac{d}{dt}\int_\Omega Y_1(x,t)\,dx = -d_1\int_\Omega (Y_1(t)Y_2(t)f_1(T(t)))(x)\,dx \le 0, \tag{5.3}$$

hence the function $t \mapsto \int_\Omega Y_1(x,t)\,dx$ is nonincreasing. Let $\overline{Y_1}$ be a constant such that

$$\lim_{t\to\infty}\int_\Omega Y_1(x,t)\,dx = \overline{Y_1}. \tag{5.4}$$

From (1.2) we have

$$\frac{d}{dt}\int_\Omega Y_2(x,t)\,dx = \int_\Omega (d_3Y_1Y_2f_1(T) - d_4Y_2f_2(T) - d_5Y_2 - d_6Y_2^2)(x,t)\,dx. \tag{5.5}$$

From (5.3) and (5.5) we deduce

$$\begin{aligned}\frac{d}{dt}\int_\Omega &\left(\frac{1}{d_1}Y_1 + \frac{1}{d_3}Y_2\right)(x,t)\,dx \\ &= -\int_\Omega \left(\frac{d_4}{d_3}Y_2f_2(T) + \frac{d_5}{d_3}Y_2 + \frac{d_6}{d_3}Y_2^2\right)(x,t)\,dx \le 0,\end{aligned} \tag{5.6}$$

from which we infer that there is a constant K such that

$$\frac{1}{d_1}\int_\Omega Y_1(x,t)\,dx + \frac{1}{d_3}\int_\Omega Y_2(x,t)\,dx \to K \quad \text{as } t\to\infty. \tag{5.7}$$

Combining (5.1) and (5.7) we conclude that there is a positive constant $\overline{Y_2}$ such that

$$\lim_{t\to\infty}\int_\Omega Y_2(x,t)\,dx = \overline{Y_2}. \tag{5.8}$$

Integrating (5.6) over $(0,\infty)$ we conclude that there is a constant C such that

$$\int_0^\infty\int_\Omega Y_2(x,\tau)\,dx\,d\tau \le C. \tag{5.9}$$

Combining (5.8) and (5.9) we find that $\overline{Y_2} = 0$, whence

$$\lim_{t\to\infty}\int_\Omega Y_2(x,t)\,dx = 0. \tag{5.10}$$

As $Y_2(x,t) \geq 0$, the invariance principle of La Salle [5] and (5.10) imply $\lim_{t\to\infty} \|Y_2(t)\|_\infty = 0$.

Multiplying (1.1) by Y_1 and integrating over Ω and using Poincaré's inequality we obtain

$$\frac{d}{dt}\int_\Omega Y_1^2(x,t)\,dx \leq -c\int_\Omega Y_1^2(x,t)\,dx$$

for some positive constant $c > 0$, from which we deduce

$$\|Y_1(t)\|_2^2 \leq e^{-ct}\|Y_{10}\|_2^2. \tag{5.11}$$

Also, as a consequence of the maximum principle we have

$$\|Y_1(t)\|_\infty \leq \|Y_1(s)\|_\infty \quad \text{for } t \geq s > 0. \tag{5.12}$$

According to Proposition 5.2, $\{Y_1(t) : t \geq \delta\}$ is relatively compact in $C(\overline{\Omega})$ for all $\delta > 0$; so from this, (5.11) and (5.12) we have

$$\lim_{t\to\infty} \|Y_1(t)\|_\infty = 0. \tag{5.13}$$

Multiplying (1.2) by Y_2 and integrating over $\Omega \times (0,t)$ we have

$$\begin{aligned} \|Y_2(t)\|_2^2 &+ 2d_2\int_0^t \|\nabla Y_2(\tau)\|_2^2\,d\tau + 2d_4\int_0^t\int_\Omega Y_2^2 f_2(T)\,dx\,d\tau \\ &+ 2d_5\int_0^t \|Y_2(\tau)\|_2^2\,d\tau + 2d_6\int_0^t\int_\Omega Y_2^3\,dx\,d\tau \\ &= \|Y_{20}\|_2^2 + 2d_3\int_0^t\int_\Omega Y_1Y_2^2 f_1(T)\,dx\,d\tau. \end{aligned} \tag{5.14}$$

Similarly for (1.3),

$$\begin{aligned} \|T(t)\|_2^2 &+ 2d_7\int_0^t \|\nabla T(\tau)\|_2^2\,d\tau \\ &= \|T_0\|_2^2 + 2d_8\int_0^t\int_\Omega Y_1Y_2Tf_1(T)\,dx\,d\tau \\ &+ 2d_9\int_0^t\int_\Omega Y_2Tf_2(T)\,dx\,d\tau \\ &+ 2d_{10}\int_0^t\int_\Omega Y_2T\,dx\,d\tau + 2d_{11}\int_0^t\int_\Omega Y_2^2T\,dx\,d\tau. \end{aligned} \tag{5.15}$$

By (4.4) and as Y_1, Y_2 and T are uniformly bounded, it follows from (5.14) and (5.15) that $\nabla Y_2, \nabla T \in L^2(\mathbb{R}, L^2(\Omega))$, i.e.

$$\int_0^\infty \|\nabla Y_1(\tau)\|_2^2\,d\tau < \infty, \quad \int_0^\infty \|\nabla Y_2(\tau)\|_2^2\,d\tau < \infty, \quad \int_0^\infty \|\nabla T(\tau)\|_2^2\,d\tau < \infty. \tag{5.16}$$

For the equation (1.1) for example, we define the operator B_p as follows:

$$D(B_p) = \{u \in W^{2,p}(\Omega) : (\partial u/\partial\nu)|\partial\Omega = 0\}, \qquad B_p u = (-d_0\Delta + a)u,$$

with a fixed positive real number $a > 0$. It is well known that $-B_p$ generates an analytic semigroup and $\operatorname{Re}\sigma(B_p) > a > 0$. Also, if we set $\varphi(t) = aY_1(t) + F_1(Y_1, Y_2, T)(t)$, then $\varphi \in L^\infty(\mathbb{R}^+, L^p(\Omega))$. Application of Lemma 5.1 then implies

$$\sup_{t\ge\delta} \|B_p^\alpha Y_1(t)\|_p \le C(p,\alpha,\delta) \quad \text{for any } \delta > 0, \tag{5.17}$$

and

$$t \mapsto B_p^\alpha Y_1(t) \text{ is uniformly continuous from } [\delta,\infty) \text{ to } L^p(\Omega) \text{ for any } \delta > 0. \tag{5.18}$$

The same holds for Y_2 and T.

By (5.18) we find that $t \mapsto \|\nabla Y_1(t)\|_2$, $t \mapsto \|\nabla Y_2(t)\|_2$ and $t \mapsto \|\nabla T(t)\|_2$ are uniformly continuous on $[\delta,\infty)$ by choosing $p = 2$ and suitable $\alpha \in (0,1)$ and m. From this and (5.16), Lemma 5.1 gives

$$\lim_{t\to\infty} \|\nabla Y_1(t)\|_2 = 0, \qquad \lim_{t\to\infty} \|\nabla Y_2(t)\|_2 = 0, \qquad \lim_{t\to\infty} \|\nabla T(t)\|_2 = 0. \tag{5.19}$$

The interested reader can see [7] for details.

Since $\{T(t) : t \ge \delta\}$ is compact in $C(\overline{\Omega})$ it follows that there is a sequence $\{t_k\}$ such that

$$\lim_{t_k\to\infty} T(t_k) = T_\infty \quad \text{in } C(\overline{\Omega}),$$

where T_∞ is a constant. Owing to the Poincaré inequality (see [11]) we have

$$\lambda \Big\| T(t) - |\Omega|^{-1} \int_\Omega T(x,t)\,dx \Big\|_2^2 \le \|\nabla T(t)\|_2^2.$$

Here λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary conditions on $\partial\Omega$. Since the limit T_∞ is uniquely determined we have

$$\lim_{t\to\infty} T(t) = T_\infty \quad \text{in } C(\overline{\Omega}).$$

6. Rates of convergence. In this section we study the rates of convergence obtained in Theorem 5.3.

THEOREM 6.1. *Assume* (CD). *Then for given* $\mu \in [0,2)$, *there exist* $K_1(\mu), K_2(\mu), K(\mu) > 0$ *and* $\varrho, \sigma, \omega > 0$ *such that*

$$\|Y_1(t)\|_{C^\mu(\Omega)} \le K_1(\mu)e^{-\varrho(t-t^*)},$$
$$\|Y_2(t)\|_{C^\mu(\Omega)} \le K_2(\mu)e^{-\sigma(t-t^*)},$$
$$\|T(t)\|_{C^\mu(\Omega)} \le K(\mu)e^{-\omega(t-t^*)},$$

for some $t^* > 0$, *as* $t \to \infty$, *where* $0 < \sigma < d_5$, $\varrho = \min\{\sigma, d_0\lambda\}$, $\omega = \min\{\sigma, d_7\lambda\}$ *and* λ *is the smallest positive eigenvalue of* $-\Delta$ *with homogeneous Neumann boundary condition on* $\partial\Omega$.

Let us recall the following two lemmas.

Lemma 6.2. *For* $1 < p < \infty$ *and* $d > 0$, *let* L_p *be the operator defined by* $D(L_p) = \{u \in W^{2,p}(\Omega) : (\partial u/\partial\nu)|\partial\Omega = 0\}$, $L_p u = -d\Delta u$, *and let the operators* $Q_0, Q_+ : L^p(\Omega) \to L^p(\Omega)$ *be defined by*

$$Q_0 u = \frac{1}{|\Omega|}\int\limits_\Omega u(x)\,dx, \qquad Q_+ u = u - Q_0 u.$$

Define the operator L_{p+} *as* $L_{p+} \equiv L_p|Q_+L^p(\Omega)$, *the restriction of* L_p *to* $Q_+L^p(\Omega)$. *Then there exists a constant* $C_3(\alpha) > 0$ *such that for* $u \in L^p(\Omega)$ *and* $t > 0$,

$$\|L^\alpha_{p+}e^{-tL_{p+}}Q_+u\|_p \le C_3(\alpha)q(t)^{-\alpha}e^{-d\lambda t}\|Q_+u\|_p,$$

where $q(t) = \min\{t, 1\}$ *and* λ *is the smallest positive eigenvalue of* $-\Delta$ *with homogeneous Neumann boundary conditions on* $\partial\Omega$.

Lemma 6.2 is proved by Rothe [9].

Lemma 6.3. *For* $\alpha \in [0, 1)$ *and* $\beta > 0$, *there exists a constant* $C(\alpha, \beta) > 0$ *such that*

$$\int\limits_0^t q(\xi)^{-\alpha}e^{\beta\xi}\,d\xi \le C(\alpha, \beta)e^{\beta t}.$$

For the proof, see [6].

Proof of Theorem 6.1. For $1 < p < \infty$ we take the operators

$$D(A_p) = D(B_p) = D(R_p) = \{u \in W^{2,p}(\Omega) : (\partial u/\partial\nu)|\partial\Omega = 0\},$$
$$A_p u = -d_0\Delta u, \qquad B_p = -(d_2\Delta - d_5)u, \qquad R_p = -d_7\Delta u.$$

By Theorem 5.3 we already know that $Y_1(t) \to 0$ and $Y_2(t) \to 0$ in $C(\overline{\Omega})$ as $t \to \infty$, hence for any $\varepsilon > 0$ there exists a constant $t^* > 0$ such that

$$d_3Y_1f_1(T) < \varepsilon \quad \text{for all } t \ge t^*. \tag{6.1}$$

We take $0 < \varepsilon < d_5$.

(I) *The decay rate of* $\|Y_2(t)\|_p$. From (1.2) and (6.1) we get

$$\frac{\partial Y_2}{\partial t} \le d_2\Delta Y_2 - (d_5 - \varepsilon)Y_2, \qquad t > t^*. \tag{6.2}$$

Multiplying both sides by Y_2^{p-1} for $p \in [1,\infty)$, integrating over Ω and using Green's formula, we obtain

$$\frac{d}{dt}\|Y_2(t)\|_p^p \le -p(d_5-\varepsilon)\|Y_2(t)\|_p^p \quad \text{for } t > t^*, \tag{6.3}$$

which leads to

$$\|Y_2(t)\|_p \le \|Y_2(t^*)\|_p e^{-(d_5-\varepsilon)(t-t^*)} \quad \text{for } t > t^*.$$

The Hölder inequality then yields

$$\|Y_2(t)\|_p \le M|\Omega|^{1/p} e^{-(d_5-\varepsilon)(t-t^*)} \quad \text{for } t > t^*, \tag{6.4}$$

where M is the positive number appearing in (4.2).

(II) *The decay rate of* $\|Y_2(t)\|_{C^\mu(\overline{\Omega})}$. To investigate the decay rate of $\|Y_2(t)\|_{C^\mu(\overline{\Omega})}$, we treat the following integral equation which is equivalent to (1.2) with (1.4) for $t > t^*$:

$$\frac{\partial Y_2}{\partial t} = d_2 \Delta Y_2 - d_5 Y_2 + F(Y_1, Y_2, T),$$

where $F(Y_1,Y_2,T) = d_3 Y_1 Y_2 f_1(T) - d_4 Y_2 f_2(T) - d_6 Y_2^2$. Let $G_p(t)$ be the semigroup generated by $-B_p$. Then

$$Y_2(t) = G_p(t-t^*)Y_2(t^*) + \int_{t^*}^{t} G_p(t-\tau)F(\tau)\,d\tau, \quad t > t^*. \tag{6.5}$$

From Lemma 3.2(iii), we obtain

$$\|B_p^\alpha Y_2(t)\|_p \le C_1(\alpha) q(t-t^*)^{-\alpha} e^{-d_5(t-t^*)}\|Y_2(t^*)\|_p + d_3 M B_1 J_1(t), \tag{6.6}$$

where

$$J_1(t) = \int_{t^*}^{t} \|G_p(t-\tau)\|_{L^p(\Omega)\to L^p(\Omega)} \|Y_2(\tau)\|_p\,d\tau.$$

It is sufficient to estimate $J_1(t)$. By (6.4) and Lemmas 3.2 and 6.3 we have

$$\begin{aligned} J_1(t) &\le C_1(\alpha) M|\Omega|^{1/p} \int_0^{t-t^*} q(t-t^*-\tau)^{-\alpha} e^{-(d_5-\varepsilon)\tau}\,d\tau \\ &\le C_1(\alpha) M|\Omega|^{1/p} e^{-(d_5-\varepsilon)(t-t^*)} \quad \text{for } t \ge t^*. \end{aligned} \tag{6.7}$$

Consequently, the imbedding $D(B_p^\alpha) \subset C^\mu(\overline{\Omega})$ ensures the existence of a constant $K_2(\mu) > 0$ such that for every $0 < \sigma < d_5$ there is $t^* > 0$ such that

$$\|Y_2(t)\|_{C^\mu(\overline{\Omega})} \le K_2(\mu) e^{-\sigma(t-t^*)} \quad \text{for } t > t^*. \tag{6.8}$$

(III) *The decay rate of* $\|Y_1(t)\|_{C^\mu(\overline{\Omega})}$. First we write $Y_1(t) = Q_0Y_1(t) + Q_+Y_1(t)$, where

$$Q_0Y_1(t) = \frac{1}{|\Omega|}\int_\Omega Y_1(x,t)\,dx, \qquad Q_+Y_1(t) = Y_1(t) - Q_0Y_1(t).$$

We see from (4.3) and the fact that $Y_1 \to 0$ as $t \to \infty$ in $C(\overline{\Omega})$ that

$$\begin{aligned} Q_0Y_1(t) &\equiv \frac{1}{|\Omega|}\int_\Omega Y_1(x,t)\,dx \\ &= \frac{1}{|\Omega|}\int_\Omega Y_{10}(x,t)\,dx - \frac{d_1}{|\Omega|}\int_0^t\int_\Omega (Y_1Y_2f_1)(x,\tau)\,dx\,d\tau \\ &= \frac{d_1}{|\Omega|}\int_t^\infty\int_\Omega (Y_1Y_2f_1)(x,\tau)\,dx\,d\tau \\ &\le \frac{d_1}{|\Omega|}B_1M_0\int_t^\infty\int_\Omega Y_2(x,\tau)\,dx\,d\tau \\ &\le d_1B_1M_0\int_t^\infty e^{-\sigma(\tau-t^*)}\,d\tau \\ &\le \frac{1}{\varrho}d_1B_1M_0e^{-\varrho(t-t^*)} \quad \text{for } t > t^*. \end{aligned} \tag{6.9}$$

Next, we study the decay of $Q_+Y_1(t)$. We consider the integral equation associated with (1.1) and apply $A^\alpha_{p+}Q_+$ to get

$$A^\alpha_{p+}Q_+Y_1(t) = G_{1,p+}(t-t^*)Q_+Y_1(t^*) - d_1\int_{t^*}^t G_{1,p+}(t-\tau)(Q_+Y_1Y_2f_1)(\tau)\,d\tau$$

for $t > t^*$. By Lemma 6.2, we get

$$\begin{aligned} \|A^\alpha_{p+}Q_+Y_1(t)\|_p &\le C_3(\alpha)q(t-t^*)^{-\alpha}e^{-d_0\lambda(t-t^*)}\|Q_+Y_1(t^*)\|_p \\ &\quad + MB_1d_1C_3(\alpha)\|Q_+\|J_2(t) \quad \text{for } t > t^*, \end{aligned} \tag{6.10}$$

where $J_2(t) = \int_{t^*}^t q(t-\tau)^{-\alpha}e^{-d_0\lambda(t-\tau)}\|Y_2(\tau)\|_p\,d\tau$ for $t > t^*$ and $\|Q_+\|$ is the norm of the linear operator $Q_+ : L^p(\Omega) \to L^p(\Omega)$. Here $Q_+Y_1(t^*) \in D(A^\alpha_{p+})$ because $t > t^*$ and by the smoothness of $Y_1(t^*)$. By Lemma 6.3,

$$\begin{aligned} J_2(t) &= \int_{t^*}^t q(t-\tau)^{-\alpha}e^{-d_0\lambda(t-\tau)}\|Y_2(\tau)\|_p\,d\tau \\ &\le M|\Omega|^{1/p}\int_0^{t-t^*} q(\xi)^{-\alpha}e^{-d_0\lambda\xi}\,d\xi \\ &\le C(\alpha,d_0\lambda)e^{-d_0\lambda(t-t^*)} \quad \text{for } t > t^*. \end{aligned} \tag{6.11}$$

Let $\varrho = \min\{\sigma, d_0\lambda\}$. Now we take p, m and α as in (3.5), so that by combining (6.10)–(6.12) we get the decay rate

$$\|Y_1(t)\|_{C^\mu(\overline{\Omega})} \le K_1(\mu) e^{-\varrho(t-t^*)} \quad \text{for } t > t^*. \tag{6.12}$$

(IV) *The decay rate of* $\|T(t) - T_\infty\|_{C^\mu(\overline{\Omega})}$. The argument in (III) can also be used to investigate this decay rate. We write $T(t) - T_\infty = (Q_0 T(t) - T_\infty) + Q_+ T(t)$. We see from (4.7) that

$$|Q_0 T(t) - T_\infty| \le D|\Omega|^{-1} N \int_t^\infty \int_\Omega Y_2(x,\tau)\, dx\, d\tau,$$

where $D = \max\{d_8, d_9, d_{10}, d_{11}\}$ and $N = M_0 B_1 + B_2 + M + 1$. From the estimate of $\|Y_2(t)\|_{C^\mu(\overline{\Omega})}$, we obtain

$$|Q_0 T(t) - T_\infty| \le \frac{1}{\sigma} DNC(0) e^{-\sigma(t-t^*)} \quad \text{for } t > t^*. \tag{6.13}$$

Next we study the decay of $Q_+ T(t)$. We consider the integral equation associated with (1.3) and apply $R^\alpha_{p+} Q_+$ with $\alpha \in (0,1)$, to get

$$R^\alpha_{p+} Q_+ T(t) = R^\alpha_{p+} S_p(t-t^*) Q_+ T(t^*) + \int_{t^*}^t R^\alpha_{p+} S_p(t-\tau) Q_+ F_3(Y_1, Y_2, T)(\tau)\, d\tau$$

for $t > t^*$. By Lemma 6.2,

$$\begin{aligned} \|R^\alpha_{p+} Q_+ T(t)\|_p &\le C_3(0) e^{-d_7\lambda(t-t^*)} \|Q_+ T(t^*)\|_p \\ &\quad + N C_3(\alpha) \Big\| Q_+ \Big\| \int_{t^*}^t q(t-\tau)^{-\alpha} e^{-d_7\lambda(t-\tau)} \Big\| Y_2(\tau) \Big\|_p\, d\tau \end{aligned}$$

for $t > t^*$. The estimate of $\|Y_2(t)\|_{C^\mu(\overline{\Omega})}$ yields

$$\begin{aligned} \|R^\alpha_{p+} Q_+ T(t)\|_p &\le C(0) \|Q_+ T(t^*)\|_p e^{-d_7\lambda(t-t^*)} \\ &\quad + N' \int_{t^*}^t q(t-\tau)^{-\alpha} e^{-d_7\lambda(t-\tau)} e^{-\sigma(\tau-t^*)}\, d\tau \end{aligned}$$

for $t > t^*$, where $N' = MN|\Omega|^{1/p} C(\alpha) \|Q_+\|$. By Lemma 6.3 we get

$$\begin{aligned} \|R^\alpha_{p+} Q_+ T(t)\|_p &\le C(0) \|Q_+ T(t^*)\|_p e^{-d_7\lambda(t-t^*)} \\ &\quad + N' C(\alpha, d_7\lambda) e^{-d_7\lambda(t-t^*)} \quad \text{for } t > t^*. \end{aligned} \tag{6.14}$$

Let $\omega = \min\{\sigma, d_7\lambda\}$ and take p and α as in (3.5). By combining (6.12) and (6.13) we get

$$\|T(t) - T_\infty\|_{C^\mu(\overline{\Omega})} \le K(\mu) e^{-\omega(t-t^*)} \quad \text{for } t > t^*.$$

Remark. As a final remark, note that if

$$d_3 + d_8 \le d_1, \quad d_9 \le d_4, \quad d_{10} \le d_5, \quad d_{11} \le d_6, \tag{6.15}$$

then

$$T_\infty \le \frac{1}{|\Omega|}(\|Y_{10}\|_1 + \|Y_{20}\|_1 + \|T_0\|_1). \tag{6.16}$$

Indeed, from equations (1.1)–(1.3) and the boundary conditions (1.4)–(1.5) we have

$$\begin{aligned}\int_\Omega Y_1\,dx + \int_\Omega Y_2\,dx + \int_\Omega T\,dx &= \int_\Omega Y_{10}\,dx + \int_\Omega Y_{20}\,dx + \int_\Omega T_0\,dx \\ &+ (d_3 + d_8 - d_1)\int_0^t\int_\Omega Y_1Y_2f_1(T)\,dx\,d\tau + (d_9 - d_4)\int_0^t\int_\Omega Y_2f_2(T)\,dx\,d\tau \\ &+ (d_{10} - d_5)\int_0^t\int_\Omega Y_2\,dx\,d\tau + (d_{11} - d_6)\int_0^t\int_\Omega Y_2^2\,dx\,d\tau.\end{aligned} \tag{6.17}$$

The estimate (6.16) follows from (6.15) and (6.17).

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