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ON A NONLOCAL ELLIPTIC PROBLEM

Abstract. We study stationary solutions of the system

$$u_t = \nabla \cdot \left(\frac{m-1}{m} \nabla u^m + u \nabla \varphi \right), \qquad m > 1, \qquad \Delta \varphi = \pm u,$$

defined in a bounded domain Ω of $\mathbb{R}^n$. The physical interpretation of the above system comes from the porous medium theory and semiconductor physics.

1. Introduction. The temporal evolution of the spatial density $u(x,t)$ ($x \in \mathbb{R}^n$, $t \ge 0$) of free carriers in semiconductors or in electrolytes is described by a parabolic-elliptic system of equations [8], [12], [16], whose simplified form reads

$$u_t = \nabla \cdot (\nabla u + u \nabla \varphi) = \Delta u + \nabla u \cdot \nabla \varphi + u \Delta \varphi, \tag{1}$$

$$\Delta \varphi = -u. \tag{2}$$

Here φ is an electric potential generated by the density u.

In this model we assume that the flow of particles caused by thermal chaotic movement is proportional to the gradient ∇u of the density (Fick's law), and the velocity of each carrier is proportional to the gradient of the electric potential. The last assumption is consistent with the character of frictional forces acting on each particle.

In this paper we propose, following [13], to change the continuity equation (1) so that the term $\frac{m-1}{m} \Delta u^m$, $m > 1$, replaces Δu. This kind of term appears in the equations describing flows in porous media ([1]). In

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this way, we consider free carriers in semiconductors (ions in electrolyte, respectively) as a gas of self-interacting particles moving in a porous medium. Our problem assumes now the form

$$u_t = \nabla \cdot \left(\frac{m-1}{m} \nabla u^m + u \nabla \varphi \right), \qquad m > 1, \tag{3}$$

$$\Delta \varphi = -u. \tag{4}$$

Instead of (4) we can assume another relation between the density and potential. For example, when considering a self-gravitating system, (4) should be replaced by

$$\Delta \varphi = u, \tag{5}$$

where φ is the gravitational potential generated by u.

The natural boundary condition which guarantees the conservation of the total mass $M = \int_\Omega u(x,t)\, dx$ is the "no-flux" condition, i.e.

$$\frac{m-1}{m} \frac{\partial u^m}{\partial \nu} + u \frac{\partial \varphi}{\partial \nu} = 0. \tag{6}$$

For φ we set the zero boundary condition

$$\varphi|_{\partial \Omega} = 0, \tag{7}$$

which, in the Coulomb case, says that the boundary is grounded.

The system is supplemented with the initial condition

$$u(x,0) = u_0(x). \tag{8}$$

The problem (1)–(2) was considered in a series of papers: [3]–[11], [14], [15], [17]–[18].

It was shown that the global existence of solutions, the existence of stationary solutions and blow up phenomena depend on the character of interaction between the particles, total mass of the system and geometry of the domain Ω.

Here we are interested in the problem of existence of a stationary solution $\langle U, \Phi \rangle$ of (3), (4) (or (5)), (6)–(7) with a given total charge (mass) $\int_\Omega U = M$ of particles.

2. Stationary solutions. Stationary solutions $\langle U, \Phi \rangle$ of (3), (4) (or (5)) satisfy the system

$$\nabla \cdot \left(\frac{m-1}{m} \nabla U^m + U \nabla \Phi \right) = 0, \tag{9}$$

$$\Delta \Phi = -U \qquad (\text{resp. } \Delta \Phi = U). \tag{10}$$

The first equation can be written in the form

$$\nabla \cdot (U \exp(-(\Phi + U^{m-1})) \nabla (\exp(\Phi + U^{m-1}))) = 0. \tag{11}$$

Multiplying (11) by $\exp(\Phi + U^{m-1})$ and integrating over Ω we obtain

$$\int_\Omega U \exp(-(\Phi + U^{m-1}))|\nabla(\exp(\Phi + U^{m-1}))|^2 = 0,$$

where we used (6).

Assuming that $U > 0$ in $\overline{\Omega}$ we get the following relationship between U and Φ:

$$\Phi + U^{m-1} = C, \tag{12}$$

with some constant $C > 0$ (note that $\Phi = 0$ and $U > 0$ on the boundary $\partial\Omega$).

Putting $U = (C - \Phi)^{1/(m-1)}$ into (10) we reduce the question of the existence of stationary solutions of (3), (4) (or (5)), (6)–(7) with a given total charge (mass) M to the nonlocal elliptic problem

$$\Delta\Phi = -(C - \Phi)^\alpha, \qquad \alpha = \frac{1}{m-1} > 0, \tag{13}$$

in the Coulomb case or

$$\Delta\Phi = (C - \Phi)^\alpha, \tag{14}$$

in the gravitational case.

On the boundary $\partial\Omega$ we have

$$\Phi = 0, \tag{15}$$

and the unknown constant C is connected with $M = \int_\Omega U$ by the relation

$$\int_\Omega (C - \Phi)^\alpha = M. \tag{16}$$

For a given $C > 0$ the problem (13) ((14)), (15) will be called the *problem* (C), and Φ_C will denote its solution.

The proof of the existence of a solution to the problem (C) will be based on the theory of sub- and supersolutions of elliptic problems (cf. [19], [20]).

Recall that a function $\overline{\Phi} \in C(\overline{\Omega}) \cap C^2(\Omega)$ is called a *supersolution* of the problem

$$\Delta\Phi = f(\Phi) \quad \text{in } \Omega, \tag{17}$$

$$\Phi = 0 \qquad \text{on } \partial\Omega, \tag{18}$$

if the following inequalities hold:

$$\begin{aligned} \Delta\overline{\Phi} &\le f(\overline{\Phi}) && \text{in } \Omega, \\ \overline{\Phi} &\ge 0 && \text{on } \partial\Omega. \end{aligned}$$

Similarly $\underline{\Phi} \in C(\overline{\Omega}) \cap C^2(\Omega)$ is called a *subsolution* if it satisfies the reverse inequalities above.

The sub- and supersolutions $\underline{\Phi}, \overline{\Phi}$ are said to be *ordered* if $\underline{\Phi} \le \overline{\Phi}$ in $\overline{\Omega}$. In this case $[\underline{\Phi}, \overline{\Phi}]$ will denote the set $\{\Phi \in C(\overline{\Omega}) : \underline{\Phi} \le \Phi \le \overline{\Phi} \text{ in } \overline{\Omega}\}$.

THEOREM A ([19], Th. 7.1). *Let $\underline{\Phi}, \overline{\Phi}$ be ordered sub- and supersolutions of* (17)–(18), *and let f satisfy the condition*

$$f(\Phi_1) - f(\Phi_2) \ge -c(\Phi_1 - \Phi_2) \quad \text{for } \underline{\Phi} \le \Phi_2 \le \Phi_1 \le \overline{\Phi} \tag{19}$$

with some $c \ge 0$. Then

(i) $\{\overline{\Phi}^{(k)}\}$, *the sequence obtained from $\overline{\Phi}$ by the monotonicity method* (*as described in* [19], *Sec.* 7.1), *converges monotonically from above to a solution Φ^* of* (17)–(18), *and* $\{\underline{\Phi}^{(k)}\}$ (*similarly constructed*) *converges monotonically from below to a solution Φ_**,

(ii) $\Phi_* \le \Phi^*$ *in $\overline{\Omega}$ and any solution Φ in $[\underline{\Phi}, \overline{\Phi}]$ satisfies $\Phi_* \le \Phi \le \Phi^*$*,

(iii) *if f is nonincreasing then $\Phi_* = \Phi^*$ is a unique* (*classical*) *solution in* $[\underline{\Phi}, \overline{\Phi}]$.

The regularity assumption $f \in C^1$ immediately implies

COROLLARY ([19], Corollary 7.1). *Let $\underline{\Phi}, \overline{\Phi}$ be ordered sub- and supersolutions of* (17)–(18), *and let f be a C^1 function in $[\underline{\Phi}, \overline{\Phi}]$. Then the problem* (17)–(18) *has a solution Φ_* and a solution Φ^* such that $\underline{\Phi} \le \Phi_* \le \Phi^* \le \overline{\Phi}$ in $\overline{\Omega}$. If $f'(\Phi) \ge 0$ for Φ in $[\underline{\Phi}, \overline{\Phi}]$ then $\Phi_* = \Phi^*$ is a unique solution of* (17)–(18).

By a *weak solution* of (17)–(18) we understand a function $\Phi \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla\Phi \cdot \nabla\psi \, dx + \int_\Omega f(\Phi)\psi \, dx = 0 \quad \text{for all } \psi \in C_0^\infty(\Omega).$$

The function $\overline{\Phi} \in H^1(\Omega)$ will be called a (*weak*) *supersolution* to (17)–(18) if $\overline{\Phi} \ge 0$ on $\partial\Omega$ and

$$\int_\Omega \nabla\overline{\Phi} \cdot \nabla\psi \, dx + \int_\Omega f(\overline{\Phi})\psi \, dx \ge 0 \quad \text{for all } \psi \in C_0^\infty(\Omega),\ \psi \ge 0.$$

Similarly $\underline{\Phi} \in H^1(\Omega)$ is a (*weak*) *subsolution* to (17)–(18) if the reverse inequalities hold above.

THEOREM B ([20], Th. 2.4). *Let Ω be a smooth*, *bounded domain in $\mathbb{R}^n$ and f be a continuous function. Suppose that sub- and supersolutions $\underline{\Phi}, \overline{\Phi} \in H^1(\Omega)$ to the problem* (17)–(18) *satisfy $-\infty < \underline{c} \le \underline{\Phi} \le \overline{\Phi} \le \overline{c} < \infty$, with some constants $\underline{c}, \overline{c} \in \mathbb{R}$, almost everywhere in Ω. Then there exists a weak solution $\Phi \in H_0^1(\Omega)$ of* (17)–(18) *satisfying $\underline{\Phi} \le \Phi \le \overline{\Phi}$ almost everywhere in Ω.*

As we will see, there is a difference between the Coulomb ((13)) and the gravitational ((14)) case. In the first case the solution exists for all $M > 0$

and all domains, whereas for the gravitational interaction the existence depends on the mass M, domain Ω and exponent m.

In the Coulomb case we prove the following theorem.

THEOREM 1. *For every $M > 0$ and each domain Ω there exists a unique solution of the problem* (13), (15), (16).

Proof. We proceed as follows. First we prove, for a given $C > 0$, the existence of a unique solution Φ_C of the problem (C). Next we check the monotonicity and continuity of the function $M(C) = \int_\Omega (C - \Phi_C)^\alpha$. The theorem will be proved if we show that $M((0,\infty)) = \mathbb{R}^+$.

The proof is divided into two parts: $\alpha \ge 1$ (here we obtain classical solutions) and $0 < \alpha < 1$ (solutions will be weak). In the second case $(0 < \alpha < 1)$ we cannot, obviously, use the Corollary (f is not C^1). To use Theorem A we should verify the condition (19). Satisfying (19) is equivalent to constructing a supersolution which is strictly less than C. But, as we will see later, we can do it not for all domains Ω and constants C (see Remark 3). This is the reason why we use Theorem B which only requires the continuity of f. Obviously an application of Theorem B gives us only weak solutions, which is, however, expected for such a degenerate problem.

CASE $\alpha \ge 1$. For $\alpha \ge 1$ the existence and uniqueness of a solution to the problem (C) follows immediately from the Corollary and the fact that $\underline{\Phi} = 0$ is a subsolution and $\overline{\Phi} = C$ is a supersolution to the problem (13), (15).

Now, let $M(C) = \int_\Omega (C - \Phi_C)^\alpha$. To show the continuity of $M(C)$ note that for $\alpha > 0$ and $C > C_0$ ($C < C_0$ resp.) the function $\Phi_{C_0} + C - C_0$ (Φ_{C_0} resp.) is a supersolution to the problem (C). This implies that $\Phi_C \ge \Phi_{C_0} \ge 0$ for $C > C_0$. Hence

$$|\Phi_C - \Phi_{C_0}| = \Phi_C - \Phi_{C_0} \le \Phi_{C_0} + C - C_0 - \Phi_{C_0} = C - C_0. \tag{20}$$

Analogously for $C_0 > C$ we have $0 \le \Phi_C \le \Phi_{C_0}$, so

$$|\Phi_C - \Phi_{C_0}| = \Phi_{C_0} - \Phi_C \le \Phi_C + C_0 - C - \Phi_C = C_0 - C. \tag{21}$$

Using (20), (21) we get

$$\begin{aligned}
&|M(C) - M(C_0)| \\
&\qquad \le \int_\Omega |(C - \Phi_C)^\alpha - (C_0 - \Phi_{C_0})^\alpha| \\
&\qquad \le \alpha \int_\Omega |(C - \Phi_C) - (C_0 - \Phi_{C_0})|(|C - \Phi_C|^{\alpha-1} + |C_0 - \Phi_{C_0}|^{\alpha-1})
\end{aligned}$$

$$\begin{aligned}&\le C(\alpha, C_0)\int_\Omega |(C-C_0)-(\Phi_C-\Phi_{C_0})|\\&\le C(\alpha, C_0)\int_\Omega (|(C-C_0)|+|\Phi_C-\Phi_{C_0}|)\\&\le C(C_0,\alpha,\Omega)|C-C_0|,\end{aligned}$$

which gives the continuity of $M(C)$.

To prove the monotonicity of $M(C)$ note that for $m \in (1,2]$ ($\alpha \ge 1$) and all $C > C_0$ the function $\frac{C}{C_0}\Phi_{C_0}$ is a subsolution for the problem (C). Hence we have

$$M(C) = -\int_\Omega \Delta\Phi_C = -\int_\Omega \frac{\partial}{\partial\nu}\Phi_C > -\int_\Omega \frac{\partial}{\partial\nu}\frac{C}{C_0}\Phi_{C_0} = \frac{C}{C_0}M(C_0). \tag{22}$$

The last inequality implies that $M(C)\to\infty$ as $C\to\infty$. Note that

$$M(C) = \int_\Omega (C-\Phi_C)^\alpha \le |\Omega| C^\alpha.$$

Hence $M(C)\to 0$ as $C\to 0$, and the theorem has been proved for $\alpha \ge 1$.

Case $0<\alpha<1$. The main difficulty lies now in the fact that, for some domains Ω and constants C, we are not able to construct a supersolution $\overline{\Phi}$ such that $\overline{\Phi} < C$. We may use Theorem B for weak sub- and supersolutions, which does not require the above condition ([20]). Having weak (H^1) sub- and supersolutions we get the existence of a weak H_0^1 solution by applying Theorem B.

To do this it suffices to rewrite the above proof replacing the classical sub- and supersolutions by their weak equivalents. Only the proof of the continuity of $M(C)$ and of its unboundedness needs new arguments.

To prove the continuity of $M(C)$ note that for $\alpha\in(0,1)$ the inequality $x^\alpha - y^\alpha \le (x-y)^\alpha$ holds for all $x>y$. Thus, recalling the inequalities (20), (21) (which are valid also for $\alpha \in (0,1)$), we get

$$\begin{aligned}|M(C)-M(C_0)| &\le \int_\Omega |(C-\Phi_C)^\alpha-(C_0-\Phi_{C_0})^\alpha|\\&\le \int_\Omega |(C-C_0)-(\Phi_C-\Phi_{C_0})|^\alpha\\&\le \int_\Omega |C-C_0|^\alpha \le |\Omega|\cdot|C-C_0|^\alpha,\end{aligned}$$

which finishes the proof of the continuity.

Analogously for $\alpha \in (0,1)$ and all $C > C_0$ the function $\frac{C}{C_0}\Phi_{C_0}$ is a supersolution for the problem (C). Hence $\Phi_C \le C\Phi_1$ and

$$M(C) = \int_\Omega (C - \Phi_C)^\alpha \ge \int_\Omega (C - C\Phi_1)^\alpha = C^\alpha M(1),$$

which implies that $M(C) \to \infty$ as $C \to \infty$.

Recalling that 0 and C are sub- and supersolutions of (13), (15) respectively, we see that the density $U = (C - \Phi_C)^\alpha$ is positive.

Thus Theorem 1 is proved.

REMARK 1. To prove Theorem 1 for $\alpha = 1$, it is enough to observe that $\Phi_C(x) = C\,\Phi_1(x)$. Hence if the pair $\langle U_1, \Phi_1\rangle$ is a solution for $C = 1$ then the scaled couple $\langle kU_1, k\Phi_1\rangle$ is a solution for the mass $M = kM(1)$.

REMARK 2. For $n = 1$ and $\alpha = 1$ the solution of (13), (15) can be expressed in an explicit form. It is enough to consider the problem on $[0, R]$. The solution of the problem (C) for $C = 1$ on $[A, B]$ is given by $\Phi_1(x) = \widetilde{\Phi}_1(x - A)$ where

$$\widetilde{\Phi}_1(x) = 1 - \frac{e^R - 1}{e^R - e^{-R}}\; e^{-x} - \frac{1 - e^{-R}}{e^R - e^{-R}}\; e^{x}$$

is the solution on $[0, R]$, $R = B - A$.

REMARK 3. Assume that Ω is contained in a ball of radius R, $\Omega \subset B_R(0)$. It is easy to check that for $0 < \alpha < 1$ and $C^{\alpha-1}R^2 < 2n$ the positive function $C^\alpha \frac{R^2 - |x|^2}{2n}$ is a supersolution of the problem (C). Since $\underline{\Phi} = 0$ is a subsolution, Theorem A can be applied, which gives the existence of a classical solution of (17)–(18).

In the gravitational problem (14), (15) we consider three cases: $\alpha \in (0,1)$, $\alpha = 1$, $\alpha > 1$. In the first and second cases the existence of solutions depends on the domain Ω and mass M.

For $\alpha \in (0,1)$ the idea of the proof is exactly the same as in the Coulomb case. The only difference lies in the fact that $M((0,\infty)) = (M_0, \infty)$, where $M_0 = M_0(\Omega, m) > 0$.

For $\alpha = 1$ we will see that the existence of solutions depends only on the domain Ω. In this case we can also use the previous method; however, the result obtained is not the best possible and does not give the uniqueness for large domains. Thus we will apply another method using the particular relationship $\Phi_C = C\Phi_1$ (valid only for $\alpha = 1$).

For $\alpha > 1$ we will show the nonexistence of solutions for M large enough.

THEOREM 2. *For $m > 2$ $(\alpha \in (0,1))$, $n \ge 1$ and for any domain there exists a unique solution of* (14)–(16) *only if $M > M_0$ where M_0 is some positive constant depending on Ω and m, $M_0 = M_0(\Omega, m) > 0$.*

Proof. If $\Omega \subset B_R(0)$ and $w \geq w_C$ then the function

$$\underline{\Phi_C}(x) = -w\frac{R^2 - |x|^2}{2n}$$

is a subsolution of the problem (C). Here w_C is given by the equation

$$w_C = \left(C + \frac{w_C R^2}{2n}\right)^{\alpha}.$$

Indeed, for $w \geq w_C$ we have

$$\Delta\underline{\Phi_C} = w \geq \left(C + w\frac{R^2}{2n}\right)^{\alpha} \geq \left(C + w\frac{R^2 - |x|^2}{2n}\right)^{\alpha} = (C - \underline{\Phi_C})^{\alpha}.$$

Using the fact that 0 is a supersolution, the existence of a classical solution of (14), (15) follows from Theorem A.

To prove the uniqueness, assume that there exists a solution $\widehat{\Phi}$ different from Φ^* (the solution obtained by iteration of 0—a supersolution—by the method described in [19]). Because $\widehat{\Phi} \leq 0$ and $\widehat{\Phi}$ is a solution (subsolution) we get $\widehat{\Phi} \leq \Phi^*$ (Th. A(ii)).

Multiplying the equation $\Delta\Phi^* = (C - \Phi^*)^{\alpha}$ by $\Phi^* - \widehat{\Phi}$ and integrating over Ω we have

$$-\int_{\Omega}((C - \widehat{\Phi})^{\alpha}\Phi^* - (C - \Phi^*)^{\alpha}\widehat{\Phi}) = 0.$$

Since $\Phi^* \geq \widehat{\Phi}$ we get $(C/|\widehat{\Phi}|+1)^{\alpha} \leq (C/|\Phi^*|+1)^{\alpha}$ and $|\Phi^*|^{1-\alpha} \leq |\widehat{\Phi}|^{1-\alpha}$. Thus we have

$$\begin{aligned} 0 &= \int_{\Omega}((C - \widehat{\Phi})^{\alpha}|\Phi^*| - (C - \Phi^*)^{\alpha}|\widehat{\Phi}|) \\ &= \int_{\Omega}|\Phi^*|^{\alpha}|\widehat{\Phi}|^{\alpha}\left(\left(\frac{C}{|\widehat{\Phi}|} + 1\right)^{\alpha}|\Phi^*|^{1-\alpha} - \left(\frac{C}{|\Phi^*|} + 1\right)^{\alpha}|\widehat{\Phi}|^{1-\alpha}\right) < 0, \end{aligned}$$

which implies the uniqueness.

The proof of the continuity of $M(C)$ is similar to that for the Coulomb case.

Using the fact that for $C > C_0$ the function $\frac{C}{C_0}\Phi_{C_0}$ is a subsolution for the problem (C) we get

$$|\Phi_C - \Phi_{C_0}| \leq \left|\Phi_{C_0} - \frac{C}{C_0}\Phi_{C_0}\right| \leq \|\Phi_{C_0}\|_{\infty}|1 - C/C_0|.$$

In this way we have

$$\begin{aligned}|M(C)-M(C_0)| &= \int_\Omega |(C-\Phi_C)^\alpha-(C_0-\Phi_{C_0})^\alpha| \\ &\le C(\alpha,C_0)\int_\Omega (|C-C_0|+|\Phi_C-\Phi_{C_0}|) \\ &\le C(C_0,\alpha,\Omega)|C-C_0|,\end{aligned}$$

which implies the continuity of $M(C)$.

To prove that $M(C)$ is strictly increasing note that for $C>C_0$ the function Φ_{C_0} is a supersolution for the problem (C). Hence if $C_1<C_2$ then $\Phi_{C_1}>\Phi_{C_2}$. Thus we get

$$M(C_1)=\int_\Omega (C_1-\Phi_{C_1})^\alpha<\int_\Omega (C_2-\Phi_{C_2})^\alpha=M(C_2).$$

To complete the proof we should show that $M((0,\infty))=(M_0,\infty)$. Since $\Phi_C\le 0$ we have

$$M(C)=\int_\Omega (C-\Phi_C)^\alpha\ge |\Omega|C^\alpha,$$

which implies $M(C)\to\infty$ as $C\to\infty$.

To prove that $\lim_{C\to0}M(C)=M_0(\Omega,m)>0$ we use the particular form of sub- and supersolutions.

As shown above, for $\Omega\subset B_R(0)$, the function $-w(2n)^{-1}(R^2-|x|^2)$ is a subsolution of the problem (C) for $w\ge w_C$.

Let ψ_1 be the first eigenfunction of $-\Delta$ in Ω, $\Delta\psi_1=-\lambda_1\psi_1$, $\lambda_1>0$, such that $\psi_1\le 0$, $\|\psi_1\|_\infty=1$. Note that for $p_0=\lambda_1^{-1/(1-\alpha)}$ the function $p_0\psi_1$ is a supersolution to the problem (C) for all $C>0$. Hence for all $C>0$ we have $p_0\psi_1\ge\Phi_C$. This means that

$$M(C)=\int_\Omega (C-\Phi_C)^\alpha\ge\int_\Omega (C-p_0\psi_1)^\alpha=\mathrm{const}(\Omega).$$

So

$$\lim_{C\to0}M(C)=M_1(\Omega,p_0,\alpha)\ge\int_\Omega (-p_0\psi_1)^\alpha>0,$$

which finishes the proof.

Case $\alpha=1$. The following *maximum principle* will be needed ([2]).

Let L be an elliptic operator and Ω a bounded domain of $\mathbb{R}^n$. Applying the classical maximum principle to the operator $L_1f=L(fh)$ we have:

Suppose that there exists a nonnegative (*nonpositive*) *function* $h\in C^2(\Omega)$ *such that* $Lh\le0$ ($Lh\ge0$). *If* $f|_{\partial\Omega}\le0$ (≥0), $Lf\ge0$ (≤0) *then* $f\le0$ (≥0) *for all* $x\in\Omega$.

Theorem 3. *For $m = 2$ ($\alpha = 1$), $n \geq 1$, $M > 0$ and for any domain for which the first eigenvalue λ_1 of $-\Delta$ is greater than* 1 *there exists a unique solution of* (14)–(16). *There is no solution for domains with $\lambda_1 \leq 1$.*

Proof. Due to Remark 1 for $\alpha = 1$ (Coulomb case) we only have to consider the problem

$$\Delta\Phi_1 = 1 - \Phi_1, \tag{23}$$

$$\Phi_1|_{\partial\Omega} = 0. \tag{24}$$

We distinguish three cases:

(1) $\lambda_1 > 1$. The Fredholm alternative implies that there exists exactly one solution Φ_1 of (23), (24) in Ω. What we must show is that $\Phi_1 \leq 0$ ($1-\Phi_1$ is a density). Since for ψ_1 (the first eigenfunction) $(\Delta+\mathrm{Id})\psi_1 = (1-\lambda_1)\psi_1$, we can apply the maximum principle in Ω. The solution Φ_1 equals 0 on the boundary $\partial\Omega$ and $(\Delta + \mathrm{Id})\Phi_1 = 1 > 0$ so we get $\Phi_1 \leq 0$ in Ω.

(2) $\lambda_1 = 1$. The necessary condition for the existence of a solution of (23), (24) is not satisfied because Φ_1 and ψ (the solution of $\Delta\psi = 1$, $\psi = 0$ on $\partial\Omega$) are of constant sign and hence $\int_{\Omega_1} \Phi_1\psi \neq 0$.

(3) $\lambda_1 < 1$. Let ψ_1 be the first eigenfunction for $-\Delta$ ($\Delta\psi_1 = -\lambda_1\psi_1$, $0 < \lambda_1 < 1$). Multiplying (23) by ψ_1 and integrating over Ω by parts we get

$$(1-\lambda_1)\int_\Omega \Phi_1\psi_1 = \int_\Omega \psi_1.$$

Since $1 - \lambda_1 > 0$ and ψ_1 is of constant sign, the function Φ_1 cannot be negative.

Remark 4. For $n = 1$ the general form of solution of (14), (15) is $\Phi_1(x) = 1 + C_1\cos x + C_2 \sin x$. The boundary conditions and the nonpositivity property of the solutions are satisfied for $R < \pi$ only. In that case the solution is given by the formula

$$\Phi_1(x) = 1 - \cos x + (\sin R)^{-1}(\cos R - 1)\sin x.$$

The monotonicity of $(\cos x - 1)(\sin x)^{-1}$ on $[0, \pi)$ yields $\Phi_1 < 0$.

Remark 5. The method of sub- and supersolutions used for the function from the proof of Theorem 2 and $m = 2$, $n = 1$ gives the existence only for the interval $[0, R]$ with $R \leq 2\sqrt{2} < \pi$.

The following theorem gives some estimate of a mass $M_1(\Omega, m)$ such that for M greater than $M_1(\Omega, m)$ and $\alpha > 1$ solutions do not exist.

Theorem 4. *For $m \leq 2n/(n+2)$ ($\alpha > (n+2)/(n-2)$), $n > 2$ and a star-shaped domain there exists no solution of* (14)–(16) *with mass greater than $M_1(\Omega, m)$.*

P r o o f. To prove the nonexistence for large M we first prove the nonexistence of solutions of (14), (15) for C large enough.

Assume that for $C \geq (\lambda_1)^{1/(\alpha-1)}$ there exists a solution Φ_C in Ω. Multiplying our equation by ψ_1 ($\psi_1 \geq 0$) we get

$$\int_\Omega (-\lambda_1 \Phi_C - (C - \Phi_C)^\alpha)\psi_1 = 0.$$

But

$$\begin{aligned} -\lambda_1 \Phi_C - (C - \Phi_C)^\alpha &< \lambda_1 (C - \Phi_C) - (C - \Phi_C)^\alpha \\ &\leq (C - \Phi_C)(\lambda_1 - (C - \Phi_C)^{\alpha-1}) \\ &< (C - \Phi_C)(\lambda_1 - C^{\alpha-1}) \\ &< (C - \Phi_C)(\lambda_1 - (\lambda_1^{1/(\alpha-1)})^{\alpha-1}) \leq 0. \end{aligned}$$

Hence there is no solution of the problem (C) for $C > (\lambda_1)^{1/(\alpha-1)}$.

Assuming that Ω is a star-shaped domain in $\mathbb{R}^n$, we use the Pokhozhaev identity to show that there is no solution of (14)–(16) for M large enough.

Indeed, from the relation

$$\begin{aligned} \int_{\partial\Omega} \left|\frac{\partial \Phi_C}{\partial \nu}\right|^2 (x \cdot \nu)\, dx = 2n \frac{1}{\alpha+1} \int_\Omega ((C - \Phi_C)^{\alpha+1} - C^{\alpha+1}) \\ - (n-2)\Big(\int_\Omega (C - \Phi_C)^{\alpha+1} - CM\Big) \end{aligned}$$

we infer

$$M^2 \leq \Big(\int_{\partial\Omega} \left|\frac{\partial \Phi_C}{\partial \nu}\right|^2 (x \cdot \nu)\, dx\Big)\Big(\int_{\partial\Omega} (x \cdot \nu)^{-1}\, dx\Big).$$

Since $\int_{\partial\Omega} (x \cdot \nu)^{-1}\, dx \leq C(\Omega) d^{n-2}$, where $d =$ diam(Ω) and $C(\Omega)$ depends on the shape of Ω only (not on the size of Ω), we obtain

$$\begin{aligned} M^2 &\leq C(\Omega) d^{n-2}\Big(2n\frac{1}{\alpha+1}\int_\Omega ((C - \Phi_C)^{\alpha+1} - C^{\alpha+1}) \\ &\quad - (n-2)\int_\Omega (C - \Phi_C)^{\alpha+1} - CM\Big) \\ &\leq C(\Omega) d^{n-2}\Big(\Big(2n\frac{1}{\alpha+1} - n + 2\Big)\int_\Omega (C - \Phi_C)^{\alpha+1} + (n-2)CM\Big). \end{aligned}$$

If $m \leq \frac{2n}{n+2}$ ($n > 2$), then $2n\frac{1}{\alpha+1} - n + 2 < 0$ and hence

$$M \leq C(\Omega) d^{n-2}(n-2)C,$$

which together with the upper bound for C gives the nonexistence of solutions of (14)–(16) for $M > M_2(\Omega, \alpha) = C(\Omega) d^{n-2}(n-2)\lambda_1^{1/(\alpha-1)}$.

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