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OPTIMALITY OF THE REPLICATING STRATEGY
FOR AMERICAN OPTIONS

Abstract. The aim of this paper is to study the problem of optimality of replicating strategies associated with pricing of American contingent claims in the Cox–Ross–Rubinstein model with proportional transaction costs. We show that a replication of the option is always possible. We give sufficient conditions for the existence of a replicating strategy which is optimal, and also show an example of an optimal replicating strategy that is not optimal in the global sense.

1. Introduction. Pricing of derivatives is an important problem of modern finance. There are various market models for which this problem is considered (see [3]). In continuous time setting, the results obtained for pricing with proportional transaction costs seem to be unacceptable from the economical point of view [1], [2]. This is one of the reasons (besides computational feasibility and simplicity) for which discrete time models with transaction costs are particularly important. In [4] and [5] the problem of optimality of replicating strategies for European options in the Cox–Ross–Rubinstein model with transaction costs was studied. The above paper deals with the same problem for American options.

The paper consists of four sections. In Section 2 we present the model. In Section 3 we formulate sufficient conditions for the existence of a replication of European option which is optimal then we derive similar ones for the case of American option. The problem of equivalence of American and European call options with the same payment functions and the case of small costs are also briefly discussed. In Section 4 we consider an example in which replication is not an optimal way of hedging an option.

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2. The model. Consider a discrete time market with two assets, a risky stock and a riskless bond. Assume that the assets are infinitely divisible. Let s_n denote the price of the stock at time n. Assume that s_n satisfies the following recursive formula:

$$(1) \qquad s_{n+1} = (1 + \varrho_n)s_n, \quad n = 0, 1, 2, \ldots,$$

where ϱ_n is a sequence of i.i.d. random variables which take with positive probability only two values, a and b, where $a, b \in (-1, \infty)$. The bond earns interest with a constant rate r such that $r > -1$ and $a < r < b$. The above recursive formula (1) for the price of the stock characterizes the so-called Cox–Ross–Rubinstein model. Consider now a financial instrument called an *option*, that is, a pair $(f_1(s), f_2(s))$ where $f_1(s), f_2(s)$ denote amounts of money that are paid to the buyer of the claim in bonds and stock, respectively, assuming that the exercise price of the stock is s.

When transferring money from stocks to bonds we pay proportional transaction costs. The cost of buying one share of stock at time n is $(1+\lambda)s_n$, where $\lambda \in [0, \infty)$, and the amount received for selling one share at time n is $(1 - \mu)s_n$, with $\mu \in [0, 1)$. Let x_n, y_n denote the amounts of money invested in bonds and stocks, respectively at time n (before transaction at this moment). Analogous post-transaction quantities will be denoted by x'_n, y'_n, respectively. Let also l_n, m_n be amounts of money transferred at time n from bonds to shares and from shares to bonds, respectively. We assume that l_n, m_n are nonnegative and measurable with respect to the σ-field $\mathcal{F}_n = \sigma(s_0, \ldots, s_n)$.

Taking into account (1), the interest rate r, and x_n, y_n defined above, we have for $n = 0, 1, 2, \ldots,$

$$(2) \qquad \begin{aligned} x_{n+1} &= (1 + r)(x_n - (1 + \lambda)l_n + (1 - \mu)m_n), \\ y_{n+1} &= (1 + \varrho_n)(y_n + l_n - m_n). \end{aligned}$$

Every sequence $_k\phi_n = (x_n, y_n)$, $n = k, k + 1, \ldots,$ for which there exists nonnegative and $\mathcal{F}_n$-measurable l_n, m_n such that (2) holds will be called a *trading strategy*. If $k = 0$ we simply write ϕ_n.

In what follows, if the option is not exercised, we do not allow simultaneous buying and selling, since it does not make sense from the economical point of view, and therefore we assume that $l_n m_n = 0$.

Every pair (x, y), where x, y are expressed in units of cash, will be called a *portfolio*.

In this paper, by a *cone*, we mean the intersection of two closed upper half planes. The intersection of three or more closed upper half planes that

is not a cone will be called a *polyhedron*. It is easy to see that at any given moment the set of all portfolios which can be used to pay the buyer's demand is a cone bounded by two half lines with slopes $-1/(1-\mu)$ and $-1/(1+\lambda)$. We give such a cone a special name.

DEFINITION 1. Let
$$C = \left\{ (x, y) \in \mathbb{R}^2 : y \geq \max\left\{ \frac{-1}{1+\lambda}x, \frac{-1}{1-\mu}x \right\} \right\}.$$
Any translate of the cone C is called a *basic cone*.

Let $C_{(x,y)}$ denote the basic cone with vertex at (x, y).

By *hedging* the contingent claim we mean covering with possible excess of the liability connected with this claim at each moment n, $n = 0, 1, \ldots, T$, which means that $C_{(x_n, y_n)} \subseteq C_{(f_1(s_n), f_2(s_n))}$ for $n = 0, \ldots, T$.

Denote by $C_f(s_n, k)$ the set of all portfolios that guarantee at time n hedging the claim $f(s_{n+k}) = (f_1(s_{n+k}), f_2(s_{n+k}))$ at time $n + k$. If $k = 0$, we write simply $C_f(s_n)$. The analogous set of post-transaction portfolios will be denoted by $C'_f(s_n, k)$.

Let $_AC_f(s_n, k)$ denote the set of all portfolios that at time n guarantee hedging the claim $(f_1(s_{n+k},), f_2(s_{n+k}))$ at time $n+k$, for $k = 0, 1, \ldots, T-n$.

We now introduce the definition of a replicating strategy:

DEFINITION 2. A strategy $_k\phi_n = (x_n, y_n), n = k, k+1, \ldots$, starting at time k is *replicating* for an American contingent claim $f(s)$ if $C_{(x_n, y_n)} \subseteq C_{(f_1(s_n), f_2(s_n))}$ for $n = k, k+1, \ldots$, and there exists a stopping time τ, $k \leq \tau \leq T$, such that (x_τ, y_τ) lies on the boundary of $C_f(s_\tau)$, which means that after selling or buying assets at time τ (we do not allow simultaneous buying and selling at time τ) we obtain $(x'_\tau, y'_\tau) = (f_1(s_\tau), f_2(s_\tau))$.

The following questions are important for the seller of the contingent claim:

What is the minimal price of the option (assuming it exists) and what is a hedging strategy concerned with this price?

When there are transaction costs, there is a problem of defining the price of the option, because it may turn out to be impossible to compare two hedging strategies. But in some cases we may point out a hedging strategy which is optimal from the point of view of the buyer of the contingent claim. Let us introduce the following definition:

DEFINITION 3. A hedging strategy $\phi_n = (x_n, y_n)$ is *optimal* if for any other hedging strategy $\widehat{\phi}_n = (\widehat{x}_n, \widehat{y}_n)$ we have $C_{(\widehat{x}_0, \widehat{y}_0)} \subseteq C_{(x_0, y_0)}$.

Analogously, a strategy ϕ_n is *optimal in a given set of strategies* (e.g. all the replicating strategies) if for any other $\widehat{\phi}_n = (\widehat{x}_n, \widehat{y}_n)$ belonging to this set we have $C_{(\widehat{x}_0, \widehat{y}_0)} \subseteq C_{(x_0, y_0)}$.

It is easy to see that an optimal hedging strategy exists if and only if $_A C(s_0, T)$ is a basic cone.

When we make the assumption that no transaction costs are paid at time 0 for buying and selling stocks, there is no longer any problem with defining the option price. The price of an option is the minimal amount of money that invested in the market allows the seller to compensate his payments associated with the option (therefore, it is sometimes called the seller's price).

When we resign from the assumption on the lack of transaction costs at time 0, we may define the seller's price as a minimal amount of money either in bonds or in shares of stock for which there exists a hedging strategy against the claim considered. If there exists an optimal hedging strategy, it is cheapest for each of the above approaches.

3. Sufficient conditions for the existence of a globally optimal replicating strategy. In this section we give conditions on an American option which assure that there exists a replicating strategy which is optimal. We first prove the existence of a replicating strategy.

Let us define two transformations:

DEFINITION 4. Let $P_a, P_b : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows:

$$P_a(x,y) = ((1+r)x, (1+a)y), \qquad P_b(x,y) = ((1+r)x, (1+b)y).$$

PROPOSITION 1. *For every contingent claim $f(s)$ there exists a replicating strategy ϕ_n.*

P r o o f. Consider pricing in one step.

One can check that $P_a^{-1}(C_f((1+a)s_1))$ is a cone bounded by two half lines with slopes

$$\frac{-(1+r)}{(1-\mu)(1+a)} \quad \text{and} \quad \frac{-(1+r)}{(1+\lambda)(1+a)},$$

and similarly $P_b^{-1}(C_f((1+b)s_1))$ is a cone bounded by two half lines with slopes

$$\frac{-(1+r)}{(1-\mu)(1+b)} \quad \text{and} \quad \frac{-(1+r)}{(1+\lambda)(1+b)}.$$

It is easy to see that the set of all post-transaction portfolios which at time 0 guarantee replicating the claim at time 1 is the intersection of the boundaries of the cones $P_a^{-1}(C_f((1+a)s_0))$ and $P_b^{-1}(C_f((1+b)s_0))$. Because the upper arm of $P_a^{-1}(C_f((1+a)s_1))$ is steeper than the upper arm of $P_b^{-1}C_f((1+b)s_1)$ and the lower arm of $P_a^{-1}(C_f((1+a)s_1))$ is steeper than the lower arm of $P_b^{-1}C_f((1+b)s_1)$, the intersection considered is nonempty.

Consequently, there exists a basic cone such that all portfolios of its boundary, at time 0, guarantee replicating the relevant claim at time 1. Denote this cone by $C_f^R(s_0, 1)$.

We have three possible cases:

1. $C_f(s_0) \subseteq C_f^R(s_0, 1)$. In this case if we put $(x_0, y_0) = (f_1(s_0), f_2(s_0))$, we will replicate our claim at time 0.

2. $C_f^R(s_0, 1) \subseteq C_f(s_0)$. In this case if we take as (x_0, y_0) the vertex of $C_f^R(s_0, 1)$ we will replicate our claim at time 1.

3. $C_f(s_0) \nsubseteq C_f^R(s_0, 1)$ and $C_f^R(s_0, 1) \nsubseteq C_f(s_0)$. In this case the intersection of the boundaries of $C_f(s_0)$ and $C_f^R(s_0, 1)$ is nonempty (it is just one point) and some strategy starting from this intersection is replicating.

So, the proposition is true for pricing in one step.

Suppose that for every contingent claim a replicating strategy ϕ_n exists for pricing in N steps, and consider pricing in $N + 1$ steps.

By the above considerations, there exists a family of replicating strategies $\widehat{\phi}_n(s_1) = (\widehat{x}_n(s_1), \widehat{y}_n(s_1))$, $n = 1, 2, \ldots$, depending on s_1, and starting at time 1.

It is easy to see that there exists a portfolio $(\widetilde{x}_0, \widetilde{y}_0)$ which at time 0 guarantees replicating the claim $(\widehat{x}_1(s_1), \widehat{y}_1(s_1))$ at time 1. As in pricing in one step, we have three possible cases:

1. $C_f(s_0) \subseteq C_{(\widetilde{x}_0, \widetilde{y}_0)}$. In this case if we put $(x_0, y_0) = (f_1(s_0), f_2(s_0))$, we will replicate our claim at time 0.

2. $C_{(\widetilde{x}_0, \widetilde{y}_0)} \subseteq C_f(s_0)$. In this case if we take as (x_0, y_0) the vertex of $C_{(\widetilde{x}_0, \widetilde{y}_0)}$ we will replicate the claim $(\widehat{x}_1(s_1), \widehat{y}_1(s_1))$ at time 1.

3. $C_f(s_0) \nsubseteq C_{(\widetilde{x}_0, \widetilde{y}_0)}$ and $C_{(\widetilde{x}_0, \widetilde{y}_0)} \nsubseteq C_f(s_0)$. In this case the intersection of the boundaries of $C_f(s_0)$ and $C_{(\widetilde{x}_0, \widetilde{y}_0)}$ is nonempty (it is just one point) and some strategy starting from this intersection is replicating.

Summarizing, in each case we have a replicating strategy for pricing in $N + 1$ steps, which by induction completes the proof. ∎

Let
$$I_1(s) = \frac{f_2((1 + b)s)}{(1 + b)} - \frac{f_2((1 + a)s)}{(1 + a)} - \frac{f_1((1 + a)s) - f_1((1 + b)s)}{(1 + b)(1 + \lambda)},$$
$$I_2(s) = \frac{f_1((1 + a)s) - f_1((1 + b)s)}{(1 + a)(1 - \mu)} - \frac{f_2((1 + b)s)}{(1 + b)} + \frac{f_2((1 + a)s)}{(1 + a)}.$$

By [5, Theorem 1] (see also [4] for slightly stronger conditions), the following conditions are sufficient for the existence of a replication of a European option which is optimal:

$$I_1(s_{T-1}) \geq 0 \text{ and } I_2(s_{T-1}) \geq 0 \quad \text{for each admissible } s_{T-1}.$$

Below we derive sufficient conditions for the existence of a replicating strategy which is optimal for an American option.

We first give three lemmas which are useful in the proof of the theorem.

To shorten the formulae, define

$$c_1(s) = \frac{f_1(s)}{1 + \lambda} + f_2(s), \quad c_2(s) = \frac{f_1(s)}{1 - \mu} + f_2(s).$$

LEMMA 1. *For a contingent claim $f(s)$ the following equivalence holds*: $I_1(s_0) \geq 0$ *and* $I_2(s_0) \geq 0$ *if and only if* $C'_f(s_0, 1)$, *the set of all post-transaction portfolios that guarantee at time 0 hedging the claim at time 1, is a convex cone bounded by two half lines with slopes*

$$\frac{-(1 + r)}{(1 - \mu)(1 + a)} \quad and \quad \frac{-(1 + r)}{(1 + \lambda)(1 + b)}.$$

P r o o f. A post-transaction portfolio (x'_0, y'_0) at time 0, in order to assure hedging at time 1, must satisfy the following inequalities:

(i$_1$) $$y'_0 \geq \frac{-(1 + r)}{(1 - \mu)(1 + a)} x'_0 + \frac{1}{1 + a} c_2((1 + a)s_0),$$

(i$_2$) $$y'_0 \geq \frac{-(1 + r)}{(1 + \lambda)(1 + a)} x'_0 + \frac{1}{1 + a} c_1((1 + a)s_0),$$

(i$_3$) $$y'_0 \geq \frac{-(1 + r)}{(1 - \mu)(1 + b)} x'_0 + \frac{1}{1 + b} c_2((1 + b)s_0),$$

(i$_4$) $$y'_0 \geq \frac{-(1 + r)}{(1 + \lambda)(1 + b)} x'_0 + \frac{1}{1 + b} c_1((1 + b)s_0).$$

Generally the set of all portfolios satisfying these conditions is a polyhedron.

With any of the above inequalities we can associate a line which is the set of points for which we have equality. We denote these lines by $\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle$, respectively. The following facts are equivalent:

1. The set of all post-transaction portfolios guaranteeing at time 0 hedging our claim at time 1 is a cone bounded by two half lines with slopes

$$\frac{-(1 + r)}{(1 - \mu)(1 + a)} \quad and \quad \frac{-(1 + r)}{(1 + \lambda)(1 + b)}.$$

2. The intersection of lines $\langle 1 \rangle$ and $\langle 4 \rangle$ does not lie below line $\langle 2 \rangle$ or below line $\langle 3 \rangle$.

Let p denote the intersection point of $\langle 1 \rangle$ and $\langle 4 \rangle$. One can check that p lies on or above line $\langle 2 \rangle$ if and only if $I_1(s_0) \geq 0$. Analogously p lies on or above line $\langle 3 \rangle$ if and only if $I_2(s_0) \geq 0$.

This completes the proof. ∎

LEMMA 2. *Assume that for the options $f'(s)$, $f''(s)$ we have $I_1(s_0) \geq 0$ and $I_2(s_0) \geq 0$ for some s_0. Let $f^v(s)$ denote the vertex of the basic cone $C_{f'}(s) \cap C_{f''}(s)$. Then for the option $f^v(s)$ we also have $I_1(s_0) \geq 0$ and $I_2(s_0) \geq 0$.*

P r o o f. By Lemma 1 it is sufficient to prove that $C'_{f^v}(s_0, 1)$ is a cone bounded by two half-lines with slopes $-(1+r)/((1-\mu)(1+a))$ and $-(1+r)/((1+\lambda)(1+b))$. From Lemma 1 both cones $C'_{f'}(s_0, 1)$ and $C'_{f''}(s_0, 1)$ have the above property, and consequently so does $C'_{f^v}(s_0, 1)$ as their intersection.

We complete the proof by applying Lemma 1 to the claim $f^v(s)$. ∎

LEMMA 3. *Consider pricing in two steps. Let the claim $f(s_2)$ satisfy $I_1(s_1) \geq 0$ and $I_2(s_1) \geq 0$ for each admissible s_1. Then $C_f(s_1, 1)$, the set of all portfolios that at time 1 guarantee hedging $f(s_2)$ at time 2, is a basic cone and each portfolio from the boundary of $C_f(s_1, 1)$ at time 1 guarantees replicating the claim at time 2. Moreover, if $z(s_1)$ is the vertex of $C_f(s_1, 1)$, then for the claim $z(s_1)$ we have $I_1(s_0) \geq 0$ and $I_2(s_0) \geq 0$.*

P r o o f (compare also [4] and [5]). To distinguish between $I_1(s)$, $I_2(s)$ connected with $f(s)$ and the ones connected with $z(s)$, we write $I_1^f(s)$, $I_2^f(s)$ and $I_1^z(s)$, $I_2^z(s)$, respectively.

One can check that $C_f(s_1, 1)$ is a basic cone with vertex $z(s_1) = (z_1(s_1), z_2(s_1))$ where

$$z_1(s_1) = \frac{(1-\mu)(1+\lambda)[(1+b)c_2((1+a)s_0) - (1+a)c_2((1+b)s_0)]}{(1+r)[(1+\lambda)(1+b) - (1-\mu)(1+a)]},$$

$$z_2(s_1) = \frac{-(1+r)}{(1+\lambda)(1+b)}z_1(s_1) + \frac{1}{1+b}c_1((1+b)s_1).$$

We have

$$z(s_1) \in P_a^{-1}(C_f((1+a)s_1)) \cap P_b^{-1}(C_f((1+b)s_1))$$

and hence $z(s_1)$ at time 1 guarantees replicating the claim at time 2 (see proof of Proposition 1).

By straightforward calculations we find that

$$I_1^z(s_0) = \frac{(1+\lambda)(1+r) - (1-\mu)(1+a)}{(1+r)[(1+\lambda)(1+b) - (1-\mu)(1+a)]}I_1^f((1+b)s_0)$$
$$+ \frac{(r-a)(1-\mu)}{(1+r)[(1+\lambda)(1+b) - (1-\mu)(1+a)]}I_2^f((1+a)s_0),$$

$$I_2^z(s_0) = \frac{(b-r)(1+\lambda)}{(1+r)[(1+\lambda)(1+b) - (1-\mu)(1+a)]}I_1^f((1+b)s_0)$$
$$+ \frac{(1+b)(1+\lambda) - (1+r)(1-\mu)}{(1+r)[(1+\lambda)(1+b) - (1-\mu)(1+a)]}I_2^f((1+a)s_0).$$

Because $a < r < b$, $\mu \in [0,1)$, $\lambda \in [0,\infty)$ all the coefficients of I_1^f and I_2^f are positive and therefore $I_1^z(s_0) \geq 0$ and $I_2^z(s_0) \geq 0$, which completes the proof. ∎

THEOREM 1. *Assume that for an option $(f_1(s), f_2(s))$ we have $I_1(s_n) \geq 0$ and $I_2(s_n) \geq 0$ for $n = 1, \ldots, T-1$. Then there exists a replicating strategy ϕ_n which is optimal. Moreover, $_AC_f(s_{T-n}, n)$ is a basic cone for $n = 1, \ldots, T$.*

P r o o f. Consider pricing in one step at time $T-1$. Because $I_1(s_{T-1}) \geq 0$ and $I_2(s_{T-1}) \geq 0$ for each s_{T-1} by Lemma 3, it follows that the set of all portfolios that guarantee at time $T-1$ hedging the claim $f(s_T)$, that is, $C_f(s_{T-1}, 1)$, is a basic cone. Denote its vertex by $z(s_{T-1})$.

Moreover by Lemma 3 for $z(s_{T-1})$ we have $I_1(s_{T-2}) \geq 0$ and $I_2(s_{T-2}) \geq 0$ for any admissible s_{T-2}. Since $_AC_f(s_{T-1}, 1) = C_f(s_{T-1}) \cap C_f(s_{T-1}, 1)$ the cone $_AC_f(s_{T-1}, 1)$ is a basic cone (as an intersection of basic cones). Denote its vertex by $f^1(s_{T-1})$. Let (x_{T-1}, y_{T-1}) be a portfolio lying on the boundary of $_AC_f(s_{T-1}, 1)$. If $(x_{T-1}, y_{T-1}) \in C_f(s_{T-1}, 1)$ then we make an appropriate transaction so as to get $(x'_{T-1}, y'_{T-1}) = z(s_{T-1})$ and by Lemma 3 at the next moment we can have the claim $f(s_T)$ replicated. On the other hand, if $(x_{T-1}, y_{T-1}) \in C_f(s_{T-1})$, then by a suitable transaction we obtain $(x'_{T-1}, y'_{T-1}) = (f_1(s_{T-1}), f_2(s_{T-1}))$ and consequently we have a replication of $f(s_{T-1})$ at time $T-1$. From the above considerations we see that all portfolios from the boundary of $_AC_f(s_{T-1})$ are replicating. Moreover, for $f^1(s_{T-1})$ by Lemma 2 we have $I_1(s_{T-2}) \geq 0$ and $I_2(s_{T-2}) \geq 0$.

Consider the following induction hypothesis:

1. $_AC_f(s_{T-n}, n)$ is a basic cone and all the strategies starting from its boundary at time $T-n$ are replicating.

2. For $f^n(s_{T-n}) = (f_1^n(s_{T-n}), f_2^n(s_{T-n}))$, where $f^n(s_n)$ is the vertex of $_AC_f(s_{T-n}, n)$, we have $I_1(s_{T-(n+1)})$ and $I_2(s_{T-(n+1)}) \geq 0$.

By the considerations above we see that for $n = 1$ the induction hypothesis holds.

Suppose it is true for some n.

Because for $f^n(s_{T-n})$ we have $I_1(s_{T-(n+1)}) \geq 0$ and $I_2(s_{T-(n+1)}) \geq 0$, by Lemma 3 the set $C_{f^n}(s_{T-(n+1)}, 1)$ of all portfolios that guarantee at time $T-(n+1)$ hedging the claim $f^n(s_{T-n})$ at time $T-n$ is a basic cone. Denote its vertex by $w(s_{T-(n+1)})$.

Moreover, by Lemma 3 for $w(s_{T-(n+1)})$ we have $I_1(s_{T-(n+2)}) \geq 0$ and $I_2(s_{T-(n+2)}) \geq 0$ for any $s_{T-(n+2)}$. Since $_AC_f(s_{T-(n+1)}, n+1) = C_f(s_{T-1}) \cap C_{f^n}(s_{T-(n+1)}, 1)$ the cone $_AC_f(s_{T-(n+1)}, n+1)$ is a basic cone (as an intersection of basic cones). Denote its vertex by $f^{n+1}(s_{T-(n+1)})$. Let $(x_{T-(n+1)}, y_{T-(n+1)})$ be a portfolio lying on the boundary of $_AC_f(s_{T-(n+1)}, n+1)$.

If $(x_{T-(n+1)}, y_{T-(n+1)}) \in C_{f^n}(s_{T-(n+1)}, 1)$ then we make an appropriate transaction so as to get $(x'_{T-(n+1)}, y'_{T-(n+1)}) = w(s_{T-(n+1)})$ and by Lemma 3 at the next moment we can have the claim $f^n(s_{T-n})$ replicated. On the other hand, if $(x_{T-(n+1)}, y_{T-(n+1)}) \in C_f(s_{T-(n+1)})$, then by a suitable transaction we obtain $(x'_{T-(n+1)}, y'_{T-(n+1)}) = (f_1(s_{T-(n+1)}), f_2(s_{T-(n+1)}))$ and consequently we have a replication of $f(s_{T-(n+1)})$ at time $T - (n+1)$. From the above considerations we see that all strategies starting at time $T - (n+1)$ from the boundary of $_AC_f(s_{T-(n+1)})$ are replicating. Moreover, for $f^{n+1}(s_{T-1})$ by Lemma 2 we have $I_1(s_{T-(n+2)}) \geq 0$ and $I_2(s_{T-(n+2)}) \geq 0$.

Hence our hypothesis is true for $n = 1, \ldots, T$.

A replicating strategy starting at time 0 from the vertex of $_AC(s_0, T)$, that is, from $(f_1^T(s_0), f_2^T(s_0))$ is optimal. ∎

Below we show some examples of contingent claims which satisfy the condition imposed in Theorem 1.

EXAMPLE 1 (Long call option with delivery). When the stock price is q or greater, a holder of the option buys one share of stock for the price q. We have

$$f_1(s) = -q1_{s \geq q}, \quad f_2(s) = s1_{s \geq q}.$$

EXAMPLE 2 (Long call option with delivery and cash settlement). As in Example 1, a holder buys one share of stock for the price q, but he does it when possible cash settlement is nonnegative. If it is negative he does not exercise the option. We have

$$f_1(s) = -q1_{s \geq q/(1-\mu)}, \quad f_2(s) = s1_{s \geq q/(1-\mu)}.$$

EXAMPLE 3 (Long call option with delivery and settlement in shares of stock). This case is similar to the preceding one. However, now the decision of buying one share of stock at the price q is made when the holder's settlement in shares of stock is nonnegative. We have

$$f_1(s) = -q1_{s \geq q/(1+\lambda)}, \quad f_2(s) = s1_{s \geq q/(1+\lambda)}.$$

EXAMPLE 4 (Long put option). When the stock price is q or lower a holder of the option sells one share of stock for the price q. We have

$$f_1(s) = q1_{s \leq q}, \quad f_2(s) = -s1_{s \leq q}.$$

When there are no transaction costs, and $r \geq 0$, American and European call options have the same price (see [3]).

A problem arises when transaction costs are incurred.

First, we introduce a definition of equivalence of the American and European versions of an option.

DEFINITION 4. An American option $f(s)$ is *equivalent* to a European option with the same payment function $f(s)$ if

$$_A C_f(s_n, T - n) = C_f(s_n, T - n) \quad \text{for } n = 0, 1, \ldots, T.$$

It turns out that when $a \leq 0$ and $r \geq 0$ the American options from Examples 1–3 are equivalent to their European versions.

In case $a > 0$ we have equivalence for Example 2, while for Examples 1 and 3 it occurs if and only if:

$$(1 + r)(1 - \mu) \geq 1 + a \qquad \text{for Example 1,}$$
$$(1 + r)(1 - \mu) \geq (1 + a)(1 + \lambda) \quad \text{for Example 3.}$$

Consider now the case of small transaction costs.

As in the case of European options (see [4] and [5]), if the transaction costs are sufficiently small, i.e.

$$\frac{1 + r}{1 + a} \geq \frac{1 + \lambda}{1 - \mu} \quad \text{and} \quad \frac{1 + b}{1 + r} \geq \frac{1 + \lambda}{1 - \mu}$$

for any American contingent claim, there exists a replicating strategy which is optimal.

4. Numerical example. In this section we consider an example for which an optimal replicating strategy is not an optimal hedging strategy. Our contingent claim will be the following:

EXAMPLE 4 (Call option with cash settlement). The situation is different from Example 2. Now we do not have delivery. When the stock price is greater than q, a holder of the option gets an amount $(s - q)^+$ in cash. We have

$$f_1(s) = (s - q)^+, \quad f_2(s) = 0.$$

We set the parameters of our model:

$$s_0 = 1, \qquad b = 0.2,$$
$$q = 1.2, \qquad r = 0,$$
$$a = -0.2, \quad \mu = \lambda = 0.1.$$

Assume the pricing will be in two steps.

At time 2, we have three possibilities:

1. $s_2 = (1 + a)^2 s_0 = 0.64$, $f_1(s_2) = f_2(s_2) = 0$,
2. $s_2 = (1 + a)(1 + b)s_0 = 0.96$, $f_1(s_2) = f_2(s_2) = 0$,
3. $s_2 = (1 + b)^2 s_0 = 1.44$, $f_1(s_2) = 0.24$, $f_2(s_2) = 0$.

At time 1, two cases are possible:

1. $s_1 = (1 + a)s_0 = 0.8$, $f_1(s_1) = f_2(s_1) = 0$.

In this case $C_f(s_2)$ is, independently of the admissible choices of s_2, a basic cone with vertex $(0,0)$. So, by a simple calculation $C_f(s_1, 1)$ is also a basic cone with the same vertex. The same is obviously true for $C_f(s_1)$.

We have the equality

$$_A C_f(s_1, 1) = C_f(s_1, 1) \cap C_f(s_1).$$

Hence $_A C_f(s_1, 1)$ is a basic cone with vertex $(0,0)$. Moreover, it is easy to see that all portfolios from the boundary of $_A C_f(s_1, 1)$ guarantee at time 1 replication of the relevant contingent claim at time 2.

2. $s_1 = (1 + b)s_0 = 1.2$, $f_1(s_1) = f_2(s_1) = 0$.

Now, we have $I_1(s_1) > 0$ and $I_2(s_1) < 0$.

$P_a^{-1}(C_f((1 + a)s_1))$ is a cone with vertex $(0,0)$, bounded by half lines with slopes $-1/((1 - \mu)(1 + a))$ and $-1/((1 + \lambda)(1 + a))$.

$P_b^{-1}(C_f((1 + b)s_1))$ is a cone with vertex $(0.24, 0)$, bounded by half lines with slopes $-1/((1 - \mu)(1 + b))$ and $-1/((1 + \lambda)(1 + b))$.

The intersection $P_a^{-1}(C_f((1 + a)s_1)) \cap P_b^{-1}(C_f((1 + b)s_1))$, which is the set of all post-transaction portfolios that guarantee hedging the claim at time 2, will be a polyhedron with two vertices: $(-0.48, 0.(6))$ and $(0.24, 0)$, where $0.(6)$ means $0.66\ldots$.

It is easy to see that the set of all portfolios that at time 1 guarantee hedging our option at time 2 is a polyhedron with the same vertices but the upper half line of its boundary has slope $-1/(1 - \mu)$ and its lowest boundary has slope $-1/(1 + \lambda)$.

Then $C_f(s_1)$ is a basic cone with vertex $(0,0)$, and we have $C_f(s_1, 1) \subset C_f(s_1)$.

So, $_A C_f(s_1, 1) = C_f(s_1, 1)$.

The only post-transaction portfolio that at time 1 guarantees replication of the option at time 2, lies at the intersection of the boundaries of $P_a^{-1}(C_f((1 + a)s_1))$ and $P_b^{-1}(C_f((1 + b)s_1))$. This point is $(-0.48, 0.(6))$. So in that case, the set of all portfolios that at time 1 guarantee replication of the option at time 2 is the boundary of a basic cone with vertex $(-0.48, 0.(6))$.

Now consider time 0.

$P_a^{-1}(_A C_f((1 + a)s_0))$ is a cone with vertex at $(0,0)$ bounded by half lines with slopes $-1/((1 - \mu)(1 + a))$ and $-1/((1 + \lambda)(1 + a))$.

$P_b^{-1}(_A C_f((1 + b)s_1))$ is a polyhedron with two vertices $(-0.48, 0.(5))$ and $(0.24, 0)$. The upper half line of its boundary has slope $-1/((1 - \mu)(1 + b))$ and its lowest boundary has slope $-1/((1 + \lambda)(1 + b))$.

The intersection $P_a^{-1}(_A C_f((1 + a)s_0)) \cap P_b^{-1}(_A C_f((1 + b)s_0))$ will be a polyhedron with two vertices: $(-0.3, 0.41(6))$ and $(0.24, 0)$. The upper half line of its boundary has slope $-1/((1 - \mu)(1 + a))$ and its lowest boundary has

slope $-1/((1+\lambda)(1+b))$. This intersection is the set of all post-transaction portfolios that at time 0 guarantee hedging of our option at times 1 and 2.

It is easy to see that the set of all portfolios that at time 0 guarantee hedging our option at times 1 and 2 is a basic cone with vertex $(-0.3, 0.41(6))$. This cone is contained in $C_f(s_0)$. So $_AC_f(s_0, 2)$ is a basic cone with vertex $(-0.3, 0.41(6))$ and every strategy starting from this point is optimal.

Now, we will find the initial portfolio of an optimal replicating strategy. Let $C_{(x,y)}$ be defined as in Section 1.

By the above considerations, if $s_1 = (1+a)s_0 = 0.8$, then the set of all portfolios that at time 1 guarantee replication of our contingent claim at that moment or later is the boundary of $C_{(0,0)}$.

On the other hand, if $s_1 = (1+b)s_0 = 1.2$, then the set of all portfolios that at time 1 guarantee replication of our claim at that moment or later is the boundary of $C_{(-0.48, 0.(6))}$.

The set of all post-transaction portfolios that at time 0 guarantee replication of the option at time 1 or 2 is the intersection of the boundaries of the cones $P_a^{-1}(C_{(0,0)})$ and $P_b^{-1}(C_{(-0.48, 0.(6))})$. This intersection is the point $(-0.608, 0.8(4))$.

Because $_AC_f(s_0, 2)$ is strictly contained in $C_f(s_0)$, replication at time 0 is impossible, and we see that each replicating strategy starting at time 0 must begin from the boundary of the basic cone with vertex $(-0.608, 0.8(4))$.

So, an optimal replicating strategy starts from $(-0.608, 0.8(4))$, and an optimal hedging strategy starts from $(-0.3, 0.41(6))$.

We have $C_{(-0.608, 0.8(4))} \subset C_{(-0.3, 0.41(6))}$, and we see that from the point of view of the buyer $(-0.3, 0.41(6))$ is a better portfolio than $(-0.608, 0.8(4))$.

If there are no transaction costs at time 0 the cost of the starting portfolio of the optimal replicating strategy is $0.236(4)$, and the analogous portfolio of the optimal hedging strategy costs $0.11(6)$.

So, if someone wants to have a possibility of buying one share of stock at the price 1.2 at one of the moments $0, 1, 2$ he has to pay $0.11(6)$. If he wants his claim to be perfectly replicated (it is just a theoretical consideration) he has to pay $0.236(4)$.

References

[1]   J. Cvitanić, H. Pham and N. Touzi, *A closed-form solution to the problem of super-replication under transaction costs*, Finance Stochastics 3 (1999), 35–54.

[2]   S. Levental and A. V. Skorohod, *On the possibility of hedging options in the presence of transaction costs*, Ann. Appl. Probab. 7 (1997), 410-443.

[3]   M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer, Berlin, 1997.

[4]   M. Rutkowski, *Optimality of replication in the CRR model with transaction costs*, Appl. Math. (Warsaw) 25 (1998), 29–53.

[5]   Ł. S t e t t n e r, *Option pricing in the CRR model with proportional transaction costs*: *A cone transformation approach*, ibid. 24 (1997), 475–514.

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