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ROBUST BAYESIAN ESTIMATION IN A NORMAL MODEL WITH ASYMMETRIC LOSS FUNCTION

Abstract. The problem of robust Bayesian estimation in a normal model with asymmetric loss function (LINEX) is considered. Some uncertainty about the prior is assumed by introducing two classes of priors. The most robust and conditional Γ-minimax estimators are constructed. The situations when those estimators coincide are presented.

1. Introduction and notation. In Bayesian statistical inference the goal of research are optimal decisions under a specified loss function and a prior distribution over the parameter space. However the arbitrariness of a unique prior distribution is a permanent problem. Robust Bayesian inference deals with the problem of expressing uncertainty of the prior information using a class Γ of priors and of measuring the range of a posterior quantity while the prior distribution Π runs over the class Γ. It is interesting not only in calculating the range but also in constructing optimal procedures.

In the problem of estimation of an unknown parameter two concepts of optimality are considered: the idea of conditional Γ-minimax estimators (see DasGupta and Studden [4], Betro and Ruggeri [1]) and the idea of stable estimators developed in Męczarski and Zieliński [6] and Boratyńska and Męczarski [3]. The first concept is connected with the problem of efficiency of the estimator with respect to the posterior risk when the priors run over Γ. The second one is connected with the problem of finding an estimator with the smallest oscillation of the posterior risk when the priors run over Γ. Sometimes those two estimators coincide (see Męczarski [5] and Boratyńska [2]).

In all papers mentioned above the quadratic loss function was considered. However in many situations a quadratic loss function seems inappro-

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priate in that it assigns the same loss to overestimates as to equal underestimates.

In this paper we estimate an unknown parameter θ and consider the asymmetric loss function (LINEX)

$$L(\theta, d) = \exp(a(\theta - d)) - a(\theta - d) - 1,$$

where a is a known parameter and $a \neq 0$. Exhaustive motivations to use LINEX are presented in Varian [7] and Zellner [8]. We find the conditional Γ-minimax estimators and the stable estimators, and present conditions when those estimators coincide, in a normal model with two classes of conjugate priors given below.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with normal $N(\theta, b^2)$ distribution where θ is unknown and b^2 is known. Set $X = (X_1, \ldots, X_n)$. Let $\Pi_{\mu_0,\sigma_0} = N(\mu_0, \sigma_0^2)$ be a fixed prior distribution of θ.

Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad v_n = \frac{n(\overline{X} - \mu_0)}{b^2}, \qquad \lambda = \lambda(\sigma) = \left(\frac{1}{\sigma^2} + \frac{n}{b^2}\right)^{-1},$$

$$m = m(\mu) = \mu\left(1 - \frac{n}{b^2}\left(\frac{1}{\sigma_0^2} + \frac{n}{b^2}\right)^{-1}\right),$$

$$w_n = \left(\frac{a}{2} + \frac{n\overline{X}}{b^2}\right)\left(\frac{1}{\sigma_0^2} + \frac{n}{b^2}\right)^{-1}.$$

If $X = x$ then the posterior distribution is the normal distribution

$$N(\mu_0 + v_n\lambda_0, \lambda_0) = N(m_0 + w_n - a\lambda_0/2, \lambda_0),$$

where $\lambda_0 = \lambda(\sigma_0)$ and $m_0 = m(\mu_0)$. The posterior risk of an estimator $\widehat{\theta}$ with LINEX loss function is equal to

$$Ee^{a(\theta - \widehat{\theta})} - aE\theta + a\widehat{\theta} - 1,$$

where $Ey(\theta)$ denotes the expected value of a function $y(\theta)$ when θ has the posterior distribution. Thus under the prior Π_{μ_0,σ_0},

$$Ee^{a\theta} = \exp(a\mu_0 + (a^2/2 + av_n)\lambda_0) = \exp(am + aw_n)$$

and

$$E\theta = \mu_0 + v_n\lambda_0 = m_0 + w_n - a\lambda_0/2.$$

The minimum of the posterior risk as a function of θ is reached for

$$\widehat{\theta} = \frac{1}{a}\ln Ee^{a\theta}.$$

Thus the Bayes estimator with LINEX loss function is given by the formula

$$\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0} = \frac{1}{a}\ln Ee^{a\theta} = \mu_0 + (a/2 + v_n)\lambda_0 = m_0 + w_n.$$

Now suppose that the prior distribution is not exactly specified and consider two classes of prior distributions of θ:

$$\mathit{\Gamma}_{\mu_0} = \{\mathit{\Pi}_{\mu_0,\sigma} : \mathit{\Pi}_{\mu_0,\sigma} = N(\mu_0, \sigma^2),\ \sigma \in (\sigma_1, \sigma_2)\},$$

where $\sigma_1 < \sigma_2$ are fixed and $\sigma_0 \in (\sigma_1, \sigma_2)$, and

$$\mathit{\Gamma}^*_{\sigma_0} = \{\mathit{\Pi}_{\mu,\sigma_0} : \mathit{\Pi}_{\mu,\sigma_0} = N(\mu, \sigma_0^2),\ \mu \in (\mu_1, \mu_2)\},$$

where $\mu_1 < \mu_2$ are fixed and $\mu_0 \in (\mu_1, \mu_2)$. The classes $\mathit{\Gamma}_{\mu_0}$ and $\mathit{\Gamma}^*_{\sigma_0}$ express two types of uncertainty about the elicited prior.

Let $R_x(\mu, \sigma, \widehat{\theta}\,)$ denote the posterior risk of the estimator $\widehat{\theta}$ when the prior is normal $N(\mu, \sigma^2)$. The posterior risk can be expressed by two formulas as a function of λ and m:

$$\begin{aligned} R(\mu_0, \sigma, \widehat{\theta}) &= \varrho_{\mu_0}(\lambda, \widehat{\theta}) \\ &= \exp(-a\widehat{\theta} + a\mu_0 + (a^2/2 + av_n)\lambda) - a(\mu_0 + \lambda v_n) + a\widehat{\theta} - 1 \end{aligned}$$

and

$$\begin{aligned} R(\mu, \sigma_0, \widehat{\theta}) &= \varrho^*_{\sigma_0}(m, \widehat{\theta}) \\ &= \exp(-a\widehat{\theta} + am + aw_n) - a(m + w_n) + a^2\lambda_0/2 + a\widehat{\theta} - 1. \end{aligned}$$

Observe that λ is an increasing function of σ and therefore if $\sigma \in (\sigma_1, \sigma_2)$ then $\lambda \in (\lambda_1, \lambda_2)$, where $\lambda_i = \lambda(\sigma_i)$, $i = 1, 2$. Similarly, m is an increasing function of μ and therefore if $\mu \in (\mu_1, \mu_2)$ then $m \in (m_1, m_2)$, where $m_i = m(\mu_i)$, $i = 1, 2$. The ranges of the posterior risk of the estimator $\widehat{\theta}$ when the prior runs over $\mathit{\Gamma}_{\mu_0}$ and $\mathit{\Gamma}^*_{\sigma_0}$ are

$$r_{\mu_0}(\widehat{\theta}) = \sup_{\lambda \in (\lambda_1, \lambda_2)} \varrho_{\mu_0}(\lambda, \widehat{\theta}) - \inf_{\lambda \in (\lambda_1, \lambda_2)} \varrho_{\mu_0}(\lambda, \widehat{\theta})$$

and

$$r^*_{\sigma_0}(\widehat{\theta}) = \sup_{m \in (m_1, m_2)} \varrho^*_{\sigma_0}(m, \widehat{\theta}) - \inf_{m \in (m_1, m_2)} \varrho^*_{\sigma_0}(m, \widehat{\theta}),$$

respectively.

2. The range of the posterior risk for the Bayes estimator. Consider the prior $\mathit{\Pi}_{\mu_0,\sigma_0}$, note that $\mathit{\Pi}_{\mu_0,\sigma_0} \in \mathit{\Gamma}_{\mu_0}$ and $\mathit{\Pi}_{\mu_0,\sigma_0} \in \mathit{\Gamma}^*_{\sigma_0}$, and consider the Bayes estimator

$$\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0} = \mu_0 + (a/2 + v_n)\lambda_0 = m_0 + w_n.$$

The posterior risk of this estimator under an arbitrary prior $\mathit{\Pi}_{\mu_0,\sigma} \in \mathit{\Gamma}_{\mu_0}$ is

$$\varrho_{\mu_0}(\lambda, \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0}) = \exp((a^2/2 + av_n)(\lambda - \lambda_0)) - av_n(\lambda - \lambda_0) + a^2\lambda_0/2 - 1.$$

Denote it by $f(\lambda)$. Now computations lead to the following form of the oscillation of ϱ_{μ_0} for $\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0}$ while λ runs over (λ_1,λ_2):

$$r_{\mu_0}(\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0}) = \begin{cases} f(\lambda_2)-f(\lambda_1) & \text{if } -a/2\le v_n<0 \text{ and } a>0, \text{ or} \\ & 0<v_n\le -a/2 \text{ and } a<0, \text{ or } \widehat{\lambda}<\lambda_1, \\ f(\lambda_2)-f(\widehat{\lambda}) & \text{otherwise,} \end{cases}$$

where

$$\widehat{\lambda} = \lambda_0 + (a^2/2+av_n)^{-1}\ln\frac{v_n}{a/2+v_n}.$$

Thus

$$r_{\mu_0}(\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0}) = \begin{cases} e^{z(\lambda_1-\lambda_0)}[e^{z\delta}-1]-av_n\delta & \text{if } -a/2<v_n<0 \text{ and } a>0, \text{ or} \\ & 0<v_n\le -a/2 \text{ and } a<0, \text{ or } \widehat{\lambda}<\lambda_1, \\ a^2\delta/2 & \text{if } v_n=-a/2, \\ e^{z(\lambda_2-\lambda_0)}+av_n(\widehat{\lambda}-\lambda_2-1/z) & \text{otherwise,} \end{cases}$$

where $z=a^2/2+av_n$ and $\delta=\lambda_2-\lambda_1$.

Consider the class $\Gamma^*_{\sigma_0}$. The posterior risk of this estimator under an arbitrary prior $\Pi_{\mu,\sigma_0}\in\Gamma^*_{\sigma_0}$ is

$$\varrho^*_{\sigma_0}(m,\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0}) = e^{-a(m_0-m)}+a(m_0-m)+a^2\lambda_0/2-1$$

and the oscillation of $\varrho^*_{\sigma_0}$ is equal to

$$r^*_{\sigma_0}(\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_0}) = \begin{cases} e^{-a(m_0-m_2)}+a(m_0-m_2)-1 & \text{for } m_0\le\widehat{m}, \\ e^{-a(m_0-m_1)}+a(m_0-m_1)-1 & \text{for } m_0>\widehat{m}, \end{cases}$$

where

$$\widehat{m} = m_1+\frac{1}{a}\ln\frac{\exp(am_2-am_1)-1}{a(m_2-m_1)}.$$

3. Most stable and conditional Γ-minimax estimators. Now the problem is to find most stable estimators $\widehat{\theta}_{\mu_0}$ and $\widehat{\theta}^*_{\sigma_0}$, i.e. those satisfying

$$\inf_{\widehat{\theta}} r_{\mu_0}(\widehat{\theta}) = r_{\mu_0}(\widehat{\theta}_{\mu_0}) \quad\text{and}\quad \inf_{\widehat{\theta}} r^*_{\sigma_0}(\widehat{\theta}) = r^*_{\sigma_0}(\widehat{\theta}^{\,*}_{\sigma_0})$$

and to find the *conditional Γ-minimax estimators* $\widetilde{\theta}_{\mu_0}$ and $\widetilde{\theta}^*_{\sigma_0}$, i.e. those satisfying

$$\inf_{\widehat{\theta}}\sup_{\sigma\in[\sigma_1,\sigma_2]} R_x(\mu_0,\sigma,\widehat{\theta}) = \sup_{\sigma\in[\sigma_1,\sigma_2]} R_x(\mu_0,\sigma,\widetilde{\theta}_{\mu_0})$$

and

$$\inf_{\widehat{\theta}}\sup_{\mu\in[\mu_1,\mu_2]} R_x(\mu,\sigma_0,\widehat{\theta}) = \sup_{\mu\in[\mu_1,\mu_2]} R_x(\mu,\sigma_0,\widetilde{\theta}^*_{\sigma_0}).$$

We use the following theorem proved by Męczarski [5].

THEOREM 1 (Męczarski [5]). *Let $\Gamma = \{\Pi_\alpha : \alpha \in [\alpha_1, \alpha_2]\}$ be a set of prior distributions, where α is a real parameter. Let $\varrho(\alpha, d)$ be the posterior risk of a decision d based on an observation x when the prior is Π_α. Assume that the function $\varrho(\alpha, d)$ satisfies the following conditions*:

1. *$\varrho(\alpha, \cdot)$ is a strictly convex function for any α*;
2. *for any d the minimum point $\alpha_{\min}(d)$ of $\varrho(\cdot, d)$ is unique and $\alpha_{\min}$ is a strictly monotone function of d*;
3. *for any $\overline{\alpha}$ and $\overline{d}$ such that $\alpha_{\min}(\overline{d}) = \overline{\alpha}$ we have*

$$\forall d_1 < d_2 \le \overline{d} \qquad \frac{\varrho(\overline{\alpha}, d_2) - \varrho(\overline{\alpha}, d_1)}{d_2 - d_1} < \frac{\varrho(\alpha_{\min}(d_2), d_2) - \varrho(\alpha_{\min}(d_1), d_1)}{d_2 - d_1}$$

and

$$\forall d_2 > d_1 \ge \overline{d} \qquad \frac{\varrho(\overline{\alpha}, d_2) - \varrho(\overline{\alpha}, d_1)}{d_2 - d_1} > \frac{\varrho(\alpha_{\min}(d_2), d_2) - \varrho(\alpha_{\min}(d_1), d_1)}{d_2 - d_1};$$

4. *the function $\varrho(\alpha_1, d) - \varrho(\alpha_2, d)$ is a monotone function of d.*

Then

(i) *if there exists $\widehat{d}$ such that*

$$\sup_{\alpha \in [\alpha_1, \alpha_2]} \varrho(\alpha, \widehat{d}) = \varrho(\alpha_1, \widehat{d}) = \varrho(\alpha_2, \widehat{d})$$

then $\widehat{d}$ is the most stable;

(ii) *if $\widehat{d}$ satisfying* (i) *belongs to $\mathcal{L}_\Gamma = \{d : \forall x \in \mathcal{X}\ \exists \alpha \in [\alpha_1, \alpha_2]\ d(x) = d_\alpha^{\mathrm{Bay}}(x)\}$ then $\widehat{d}$ is conditional Γ-minimax.* ■

We now prove our results.

THEOREM 2. *If the class of priors is $\Gamma^*_{\sigma_0}$ then*

$$\widehat{\theta}^*_{\sigma_0} = \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_1, \sigma_0} + \frac{1}{a} \ln \frac{\exp[a(m_2 - m_1)] - 1}{a(m_2 - m_1)}$$

and $\widetilde{\theta}^*_{\sigma_0} = \widehat{\theta}^*_{\sigma_0}$ for all values x of the random variable X.

Proof. Let us check the conditions of Theorem 1 for

$$\varrho^*_{\sigma_0}(m, \widehat{\theta}) = \exp(-a\widehat{\theta} + am + aw_n) - a(m + w_n) + a^2\lambda_0/2 + a\widehat{\theta} - 1.$$

The function $\varrho^*_{\sigma_0}(m, \cdot)$ is convex and

$$\frac{\partial \varrho^*_{\sigma_0}(m, \widehat{\theta})}{\partial m} = a \exp(-a\widehat{\theta} + am + aw_n) - a,$$

thus the minimum point $m_{\min}(\widehat{\theta}) = \widehat{\theta} - w_n$, and $m_{\min}$ is an increasing function of $\widehat{\theta}$.

To check condition 3 it is enough to show the inequalities

$$\forall\theta_1 < \theta_2 \le \widehat{\theta} \qquad e^{a\widehat{\theta}}\,\frac{e^{-a\theta_2} - e^{-a\theta_1}}{\theta_2 - \theta_1} < -a$$

and

$$\forall\theta_2 > \theta_1 \ge \widehat{\theta} \qquad e^{a\widehat{\theta}}\,\frac{e^{-a\theta_2} - e^{-a\theta_1}}{\theta_2 - \theta_1} > -a.$$

These hold by the Lagrange formula. The last condition of Theorem 1 is also true, thus $\widehat{\theta}^*_{\sigma_0}$ is a solution of the equation

$$\varrho^*_{\sigma_0}(m_1, \widehat{\theta}) = \varrho^*_{\sigma_0}(m_2, \widehat{\theta}).$$

To obtain the conditional Γ-minimax estimator note that for all values x of the random variable X we have $\widehat{\theta}^*_{\sigma_0}(x) \in [\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_1,\sigma_0}(x), \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_2,\sigma_0}(x)]$. ■

Theorem 3. *Let the class of priors be Γ_{μ_0}. Then the most stable estimator $\widehat{\theta}_{\mu_0}$ of θ in the class of all estimators of θ exists only for the values of X satisfying*

$$v_n(v_n + a/2) > 0 \quad \text{or} \quad v_n = -a/2.$$

For $v_n(v_n + a/2) > 0$,

$$\widehat{\theta}_{\mu_0} = \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_1} + \frac{1}{a}\ln\frac{e^{(\lambda_2-\lambda_1)(a^2/2+av_n)} - 1}{av_n(\lambda_2 - \lambda_1)}.$$

For $v_n = -a/2$ the range of the posterior risk does not depend on the value of $\widehat{\theta}$.

The conditional Γ-minimax estimator is

$$\widetilde{\theta}_{\mu_0} = \begin{cases} \widehat{\theta}_{\mu_0} & \text{if } v_n(v_n + a/2) > 0 \text{ and} \\ & \exp[(\lambda_1 - \lambda_2)(a^2/2 + av_n)] + av_n(\lambda_2 - \lambda_1)) \ge 1, \\ \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2} & \text{otherwise.} \end{cases}$$

The most stable estimator in the class

$$\mathcal{L} = \{\widehat{\theta} : \forall x\ \exists\sigma \in [\sigma_1, \sigma_2]\ \widehat{\theta}(x) = \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma}(x)\}$$

is equal to the conditional Γ-minimax estimator in the class of all estimators.

Proof. Let us check the conditions of Theorem 1 for

$$\varrho_{\mu_0}(\lambda, \widehat{\theta}) = \exp(-a\widehat{\theta} + a\mu_0 + (a^2/2 + av_n)\lambda) - a(\mu_0 + \lambda v_n) + a\widehat{\theta} - 1.$$

The function $\varrho_{\mu_0}(\lambda, \cdot)$ is convex and

$$\frac{\partial\varrho_{\mu_0}(\lambda, \widehat{\theta})}{\partial\lambda} = (a^2/2 + av_n)\exp(-a\widehat{\theta} + a\mu_0 + \lambda(a^2/2 + av_n)) - av_n.$$

Thus the minimum point is

$$\lambda_{\min}(\widehat{\theta}) = \frac{a\widehat{\theta} - a\mu_0 + \ln \frac{v_n}{a/2+v_n}}{a^2/2 + av_n}$$

and $\lambda_{\min}$ exists iff $v_n(v_n + a/2) > 0$.

For v_n satisfying $v_n(v_n + a/2) \le 0$ the function $\varrho_{\mu_0}(\cdot, \widehat{\theta})$ is an increasing function of λ and the oscillation of the posterior risk

$$\begin{aligned} r_{\mu_0}(\widehat{\theta}) = & - av_n(\lambda_2 - \lambda_1) + \exp(-a\widehat{\theta} + a\mu_0 + (a^2/2 + av_n)\lambda_1) \\ & \times [\exp((a^2/2 + av_n)(\lambda_2 - \lambda_1)) - 1] \end{aligned}$$

is a monotone function of $\widehat{\theta}$ (decreasing for $a > 0$ and $-a/2 < v_n \le 0$, constant for $v_n = -a/2$ and increasing for $a < 0$ and $0 \le v_n < -a/2$). Thus the most stable estimator does not exist for $v_n(v_n + a/2) \le 0$ and $v_n \ne -a/2$. For $v_n = -a/2$ the oscillation $r_{\mu_0}(\widehat{\theta}) = a^2(\lambda_2 - \lambda_1)/2$ does not depend on the value of $\widehat{\theta}$. The conditional Γ-minimax estimator $\widetilde{\theta}_{\mu_0}$ is equal to $\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2}$.

Let us consider the situation when $v_n(v_n + a/2) > 0$. The minimum point $\lambda_{\min}$ and the function $\varrho_{\mu_0}(\lambda_2, \cdot) - \varrho_{\mu_0}(\lambda_1, \cdot)$ are monotone functions of $\widehat{\theta}$. Condition 3 of Theorem 1 is similar to that in Theorem 2 so we obtain the most stable estimator as a solution of the equation

$$\varrho_{\mu_0}(\lambda_1, \widehat{\theta}_{\mu_0}) = \varrho_{\mu_0}(\lambda_2, \widehat{\theta}_{\mu_0}).$$

To find the conditional Γ-minimax estimator we check when $\widehat{\theta}_{\mu_0} \in \mathcal{L}$.

For $v_n + a/2 > 0$ we have $\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_1} < \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2}$. Solving the inequalities

$$\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_1} \le \widehat{\theta}_{\mu_0} \le \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2}$$

we obtain the condition

$$(*) \qquad \exp[(\lambda_1 - \lambda_2)(a^2/2 + av_n)] + av_n(\lambda_2 - \lambda_1) \ge 1.$$

For $v_n + a/2 < 0$ we have $\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_1} > \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\,\sigma_2}$. Solving the inequalities

$$\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_1} \ge \widehat{\theta}_{\mu_0} \ge \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2}$$

we also obtain $(*)$. Thus if $v_n(v_n + a/2) > 0$ and $(*)$ is true then $\widetilde{\theta}_{\mu_0} = \widehat{\theta}_{\mu_0}$.

If $v_n + a/2 > 0$ and $v_n > 0$ and $(*)$ is not true then

$$\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_1} < \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2} < \widehat{\theta}_{\mu_0}$$

and

$$\sup_{\lambda \in [\lambda_1, \lambda_2]} \varrho_{\mu_0}(\lambda, \widehat{\theta}) = \begin{cases} \varrho_{\mu_0}(\lambda_2, \widehat{\theta}) & \text{if } \widehat{\theta} \le \widehat{\theta}_{\mu_0}, \\ \varrho_{\mu_0}(\lambda_1, \widehat{\theta}) & \text{if } \widehat{\theta} \ge \widehat{\theta}_{\mu_0}, \end{cases}$$

and the oscillation $r_{\mu_0}(\widehat{\theta})$ is a decreasing function for $\widehat{\theta} < \widehat{\theta}_{\mu_0}$.

If $v_n + a/2 < 0$ and $v_n < 0$ and $(*)$ is not true then

$$\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_1} > \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2} > \widehat{\theta}_{\mu_0}$$

and

$$\sup_{\lambda\in[\lambda_1,\lambda_2]} \varrho_{\mu_0}(\lambda,\widehat{\theta}) = \begin{cases} \varrho_{\mu_0}(\lambda_1,\widehat{\theta}) & \text{if } \widehat{\theta} \le \widehat{\theta}_{\mu_0}, \\ \varrho_{\mu_0}(\lambda_2,\widehat{\theta}) & \text{if } \widehat{\theta} \ge \widehat{\theta}_{\mu_0}, \end{cases}$$

and the oscillation $r_{\mu_0}(\widehat{\theta})$ is an increasing function for $\widehat{\theta} > \widehat{\theta}_{\mu_0}$.

Thus if $v_n(v_n+a/2) > 0$ and $(*)$ is not true then $\widetilde{\theta}_{\mu_0} = \widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2}$ and $\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2}$ is the most stable estimator in the class $\mathcal{L}$.

The monotonicity of the function r_{μ_0} shows that $\widehat{\theta}^{\,\mathrm{Bay}}_{\mu_0,\sigma_2}$ is also the most stable estimator in the class $\mathcal{L}$ for $v_n(v_n + a/2) \le 0$. ∎

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