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ROBUST BAYESIAN ESTIMATION IN A NORMAL MODEL
WITH ASYMMETRIC LOSS FUNCTION

Abstract. The problem of robust Bayesian estimation in a normal model
with asymmetric loss function (LINEX) is considered. Some uncertainty
about the prior is assumed by introducing two classes of priors. The most
robust and conditional $\Gamma$-minimax estimators are constructed. The situ-
tations when those estimators coincide are presented.

1. Introduction and notation. In Bayesian statistical inference the
goal of research are optimal decisions under a specified loss function and a
prior distribution over the parameter space. However the arbitrariness of a
unique prior distribution is a permanent problem. Robust Bayesian inference
deals with the problem of expressing uncertainty of the prior information
using a class $\Gamma$ of priors and of measuring the range of a posterior quantity
while the prior distribution $\Pi$ runs over the class $\Gamma$. It is interesting not
only in calculating the range but also in constructing optimal procedures.

In the problem of estimation of an unknown parameter two concepts
of optimality are considered: the idea of conditional $\Gamma$-minimax estimators
(see DasGupta and Studden [4], Betro and Ruggeri [1]) and the idea of
stable estimators developed in Męczarski and Zieliński [6] and Boratyńska
and Męczarski [3]. The first concept is connected with the problem of ef-
ficiency of the estimator with respect to the posterior risk when the priors
run over $\Gamma$. The second one is connected with the problem of finding an
estimator with the smallest oscillation of the posterior risk when the priors
run over $\Gamma$. Sometimes those two estimators coincide (see Męczarski [5] and
Boratyńska [2]).

In all papers mentioned above the quadratic loss function was consid-
ered. However in many situations a quadratic loss function seems inappro-

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appropriate in that it assigns the same loss to overestimates as to equal under-
estimates.

In this paper we estimate an unknown parameter $\theta$ and consider the asymmetrical loss function (LINEX)

$$L(\theta, d) = \exp(a(\theta - d)) - a(\theta - d) - 1,$$

where $a$ is a known parameter and $a \neq 0$. Exhaustive motivations to use LINEX are presented in Varian [7] and Zellner [8]. We find the conditional $I$-minimax estimators and the stable estimators, and present conditions when those estimators coincide, in a normal model with two classes of conjugate priors given below.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with normal $N(\theta, b^2)$ distribution where $\theta$ is unknown and $b^2$ is known. Let $H_{\mu_0, \sigma_0} = N(\mu_0, \sigma_0^2)$ be a fixed prior distribution of $\theta$.

Define

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad v_n = \frac{n(\bar{X} - \mu_0)}{b^2}, \quad \lambda = \lambda(\sigma) = \left( \frac{1}{\sigma^2} + \frac{n}{b^2} \right)^{-1},$$

$$m = m(\mu) = \mu \left( 1 - \frac{n}{b^2} \left( \frac{1}{\sigma_0^2} + \frac{n}{b^2} \right)^{-1} \right),$$

$$w_n = \left( \frac{a}{2} + \frac{n\bar{X}}{b^2} \right) \left( \frac{1}{\sigma_0^2} + \frac{n}{b^2} \right)^{-1}.$$

If $X = x$ then the posterior distribution is the normal distribution

$$N(\mu_0 + v_n\lambda_0, \lambda_0) = N(m_0 + w_n - a\lambda_0/2, \lambda_0),$$

where $\lambda_0 = \lambda(\sigma_0)$ and $m_0 = m(\mu_0)$. The posterior risk of an estimator $\hat{\theta}$ with LINEX loss function is equal to

$$Ee^{a(\theta - \hat{\theta})} - aE\theta + a\hat{\theta} - 1,$$

where $Ey(\theta)$ denotes the expected value of a function $y(\theta)$ when $\theta$ has the posterior distribution. Thus under the prior $H_{\mu_0, \sigma_0}$,

$$E e^{a\theta} = \exp(a\mu_0 + (a^2/2 + av_n)\lambda_0) = \exp(am + aw_n)$$

and

$$E\theta = \mu_0 + v_n\lambda_0 = m_0 + w_n - a\lambda_0/2.$$

The minimum of the posterior risk as a function of $\theta$ is reached for

$$\hat{\theta} = \frac{1}{a} \ln E e^{a\theta}.$$

Thus the Bayes estimator with LINEX loss function is given by the formula

$$\hat{\theta}_{\mu_0, \sigma_0} = \frac{1}{a} \ln E e^{a\theta} = \mu_0 + (a/2 + v_n)\lambda_0 = m_0 + w_n.$$
Now suppose that the prior distribution is not exactly specified and consider two classes of prior distributions of $\theta$:

$$
\Gamma_{\mu_0} = \{ \Pi_{\mu_0, \sigma} : \Pi_{\mu_0, \sigma} = N(\mu_0, \sigma^2), \sigma \in (\sigma_1, \sigma_2) \},
$$

where $\sigma_1 < \sigma_2$ are fixed and $\sigma_0 \in (\sigma_1, \sigma_2)$, and

$$
\Gamma_{\sigma_0} = \{ \Pi_{\mu, \sigma_0} : \Pi_{\mu, \sigma_0} = N(\mu, \sigma_0^2), \mu \in (\mu_1, \mu_2) \},
$$

where $\mu_1 < \mu_2$ are fixed and $\mu_0 \in (\mu_1, \mu_2)$. The classes $\Gamma_{\mu_0}$ and $\Gamma_{\sigma_0}$ express two types of uncertainty about the elicited prior.

Let $R_\pi(\mu, \sigma, \hat{\theta})$ denote the posterior risk of the estimator $\hat{\theta}$ when the prior is normal $N(\mu, \sigma^2)$. The posterior risk can be expressed by two formulas as a function of $\lambda$ and $m$:

$$
R(\mu_0, \sigma, \hat{\theta}) = \varrho_{\mu_0}(\lambda, \hat{\theta}) = \exp(-a\hat{\theta} + a\mu_0 + (a^2/2 + aw_n)\lambda) - a(\mu_0 + \lambda w_n) + a\hat{\theta} - 1
$$

and

$$
R(\mu, \sigma_0, \hat{\theta}) = \varrho_{\sigma_0}(\lambda, \hat{\theta}) = \exp(-a\hat{\theta} + am + aw_n) - a(m + w_n) + a^2\lambda_0/2 + a\hat{\theta} - 1.
$$

Observe that $\lambda$ is an increasing function of $\sigma$ and therefore if $\sigma \in (\sigma_1, \sigma_2)$ then $\lambda \in (\lambda_1, \lambda_2)$, where $\lambda_i = \lambda(\sigma_i), i = 1, 2$. Similarly, $m$ is an increasing function of $\mu$ and therefore if $\mu \in (\mu_1, \mu_2)$ then $m \in (m_1, m_2)$, where $m_i = m(\mu_i), i = 1, 2$. The ranges of the posterior risk of the estimator $\hat{\theta}$ when the prior runs over $\Gamma_{\mu_0}$ and $\Gamma_{\sigma_0}$ are

$$
r_{\mu_0}(\hat{\theta}) = \sup_{\lambda \in (\lambda_1, \lambda_2)} \varrho_{\mu_0}(\lambda, \hat{\theta}) - \inf_{\lambda \in (\lambda_1, \lambda_2)} \varrho_{\mu_0}(\lambda, \hat{\theta})
$$

and

$$
r_{\sigma_0}(\hat{\theta}) = \sup_{m \in (m_1, m_2)} \varrho_{\sigma_0}(m, \hat{\theta}) - \inf_{m \in (m_1, m_2)} \varrho_{\sigma_0}(m, \hat{\theta}),
$$

respectively.

2. The range of the posterior risk for the Bayes estimator.

Consider the prior $\Pi_{\mu_0, \sigma_0}$, note that $\Pi_{\mu_0, \sigma_0} \in \Gamma_{\mu_0}$ and $\Pi_{\mu_0, \sigma_0} \in \Gamma_{\sigma_0}$, and consider the Bayes estimator

$$
\hat{\theta}_B^{\mu_0, \sigma_0} = \mu_0 + (a/2 + v_n)\lambda_0 = m_0 + w_n.
$$

The posterior risk of this estimator under an arbitrary prior $\Pi_{\mu_0, \sigma} \in \Gamma_{\mu_0}$ is

$$
\varrho_{\mu_0}(\lambda, \hat{\theta}_B^{\mu_0, \sigma_0}) = \exp((a^2/2 + aw_n)(\lambda - \lambda_0)) - a\lambda_0(\lambda - \lambda_0) + a^2\lambda_0/2 - 1.
$$

Denote it by $f(\lambda)$. Now computations lead to the following form of the
satisfying and to find the conditional

\[
\rho_{\hat{\theta}}(r, \hat{\theta}) = \begin{cases} 
  f(\lambda) - f(\lambda_1) & \text{if } -a/2 \leq v_n < 0 \text{ and } a > 0, \text{ or } 0 < v_n \leq -a/2 \text{ and } a < 0, \text{ or } \hat{\lambda} < \lambda_1, \\
  f(\lambda) - f(\hat{\lambda}) & \text{otherwise},
\end{cases}
\]

where

\[
\hat{\lambda} = \lambda_0 + (a^2/2 + av_n)^{-1} \ln \frac{v_n}{a^2 + v_n}.
\]

Thus

\[
r_{\rho_{\mu_0}}(\hat{\theta}_{\mu_0, \sigma_0}) = \begin{cases} 
  e^{z(\lambda_1 - \lambda_0)}[e^{z\delta} - 1] - av_n \delta & \text{if } -a/2 < v_n < 0 \text{ and } a > 0, \text{ or } 0 < v_n \leq -a/2 \text{ and } a < 0, \text{ or } \hat{\lambda} < \lambda_1, \\
  a^2\delta/2 & \text{if } v_n = -a/2, \\
  e^{z(\lambda_2 - \lambda_0)} + av_n(\hat{\lambda} - \lambda_2 - 1/z) & \text{otherwise},
\end{cases}
\]

where \( z = a^2/2 + av_n \) and \( \delta = \lambda_2 - \lambda_1 \).

Consider the class \( \Gamma_{\sigma_0} \). The posterior risk of this estimator under an arbitrary prior \( \Pi_{\mu_0, \sigma_0} \in \Gamma_{\sigma_0} \) is

\[
\rho_{\sigma_0}(m, \hat{\theta}_{\mu_0, \sigma_0}) = e^{-a(m_0 - m)} + a(m_0 - m) + a^2\lambda_0/2 - 1
\]

and the oscillation of \( \rho_{\sigma_0}^* \) is equal to

\[
r_{\sigma_0}^*(\hat{\theta}_{\mu_0, \sigma_0}) = \begin{cases} 
  e^{-a(m_0 - m_2)} + a(m_0 - m_2) - 1 & \text{for } m_0 \leq \hat{m}, \\
  e^{-a(m_0 - m_1)} + a(m_0 - m_1) - 1 & \text{for } m_0 > \hat{m},
\end{cases}
\]

where

\[
\hat{m} = m_1 + \frac{1}{a} \ln \frac{\exp(an_2 - am_1) - 1}{a(m_2 - m_1)}.
\]

3. Most stable and conditional \( \Gamma \)-minimax estimators. Now the problem is to find most stable estimators \( \hat{\theta}_{\mu_0} \) and \( \hat{\theta}_{\sigma_0}^* \), i.e. those satisfying

\[
\inf_{\hat{\theta}} r_{\rho_{\mu_0}}(\hat{\theta}) = r_{\rho_{\mu_0}}(\hat{\theta}_{\mu_0}) \quad \text{and} \quad \inf_{\hat{\theta}} r_{\rho_{\sigma_0}}^*(\hat{\theta}) = r_{\rho_{\sigma_0}}^*(\hat{\theta}_{\sigma_0}^*)
\]

and to find the conditional \( \Gamma \)-minimax estimators \( \hat{\theta}_{\mu_0} \) and \( \hat{\theta}_{\sigma_0}^* \), i.e. those satisfying

\[
\inf \sup_{\sigma \in [\sigma_1, \sigma_2]} R_x(\mu_0, \sigma, \hat{\theta}) = \sup_{\sigma \in [\sigma_1, \sigma_2]} R_x(\mu_0, \sigma, \hat{\theta}_{\mu_0})
\]

and

\[
\inf \sup_{\mu \in [\mu_1, \mu_2]} R_x(\mu, \sigma_0, \hat{\theta}) = \sup_{\mu \in [\mu_1, \mu_2]} R_x(\mu, \sigma_0, \hat{\theta}_{\sigma_0}^*).
\]

We use the following theorem proved by Męczarski [5].
Theorem 1 (Męczarski [5]). Let $\Gamma = \{\Pi_\alpha : \alpha \in [\alpha_1, \alpha_2]\}$ be a set of prior distributions, where $\alpha$ is a real parameter. Let $\varrho(\alpha, d)$ be the posterior risk of a decision $d$ based on an observation $x$ when the prior is $\Pi_\alpha$. Assume that the function $\varrho(\alpha, d)$ satisfies the following conditions:

1. $\varrho(\alpha, \cdot)$ is a strictly convex function for any $\alpha$;
2. for any $d$ the minimum point $\alpha_{\min}(d)$ of $\varrho(\cdot, d)$ is unique and $\alpha_{\min}$ is a strictly monotone function of $d$;
3. for any $\alpha$ and $d$ such that $\alpha_{\min}(d) = \alpha$ we have
   \[
   \forall d_1 < d_2 \leq \alpha \quad \frac{\varrho(\alpha, d_2) - \varrho(\alpha, d_1)}{d_2 - d_1} < \frac{\varrho(\alpha_{\min}(d_2), d_2) - \varrho(\alpha_{\min}(d_1), d_1)}{d_2 - d_1},
   \]
   and
   \[
   \forall d_2 > d_1 \geq \alpha \quad \frac{\varrho(\alpha, d_2) - \varrho(\alpha, d_1)}{d_2 - d_1} > \frac{\varrho(\alpha_{\min}(d_2), d_2) - \varrho(\alpha_{\min}(d_1), d_1)}{d_2 - d_1};
   \]
4. the function $\varrho(\alpha_1, d) - \varrho(\alpha_2, d)$ is a monotone function of $d$.

Then

(i) if there exists $\hat{d}$ such that
   \[
   \sup_{\alpha \in [\alpha_1, \alpha_2]} \varrho(\alpha, \hat{d}) = \varrho(\alpha_1, \hat{d}) = \varrho(\alpha_2, \hat{d})
   \]
then $\hat{d}$ is the most stable;

(ii) if $\hat{d}$ satisfying (i) belongs to $L_\Gamma = \{d : \forall x \in X \exists \alpha \in [\alpha_1, \alpha_2] d(x) = d_{\text{Bay}}(x)\}$ then $\hat{d}$ is conditional $\Gamma$-minimax. ■

We now prove our results.

Theorem 2. If the class of priors is $\Gamma^{*}_{\sigma_0}$ then

\[
\hat{\theta}^{*}_{\sigma_0} = \hat{\theta}^{\text{Bay}}_{\mu_1, \sigma_0} + \frac{1}{a} \ln \frac{\exp[a(m_2 - m_1)] - 1}{a(m_2 - m_1)}
\]

and $\hat{\theta}^{*}_{\sigma_0} = \hat{\theta}^{*}_{\sigma_0}$ for all values $x$ of the random variable $X$.

Proof. Let us check the conditions of Theorem 1 for $\varrho^{*}_{\sigma_0}(m, \cdot) = \exp(-a\hat{\theta} + am + aw) - a(m + w) + a^2 \lambda_0/2 + a\hat{\theta} - 1$.

The function $\varrho^{*}_{\sigma_0}(m, \cdot)$ is convex and

\[
\frac{\partial \varrho^{*}_{\sigma_0}(m, \hat{\theta})}{\partial m} = a \exp(-a\hat{\theta} + am + aw) - a,
\]

thus the minimum point $m_{\min}(\hat{\theta}) = \hat{\theta} - w$, and $m_{\min}$ is an increasing function of $\hat{\theta}$. 
To check condition 3 it is enough to show the inequalities
\[
\forall \theta_1 < \theta_2 \leq \hat{\theta} \quad e^{a\hat{\theta}} \frac{e^{-a\theta_2} - e^{-a\theta_1}}{\theta_2 - \theta_1} < -a
\]
and
\[
\forall \theta_2 > \theta_1 \geq \hat{\theta} \quad e^{a\hat{\theta}} \frac{e^{-a\theta_2} - e^{-a\theta_1}}{\theta_2 - \theta_1} > -a.
\]
These hold by the Lagrange formula. The last condition of Theorem 1 is also true, thus \( \hat{\theta}_{\sigma_0}^* \) is a solution of the equation
\[
\varrho_{\sigma_0}^*(m_1, \hat{\theta}) = \varrho_{\sigma_0}^*(m_2, \hat{\theta}).
\]
To obtain the conditional \( \Gamma \)-minimax estimator note that for all values \( x \) of the random variable \( X \) we have \( \hat{\theta}_{\sigma_0}^*(x) \in [\hat{\theta}_{\mu_1, \sigma_0}^{\text{Bay}}(x), \hat{\theta}_{\mu_2, \sigma_0}^{\text{Bay}}(x)] \).

**Theorem 3.** Let the class of priors be \( \Gamma_{\mu_0} \). Then the most stable estimator \( \hat{\theta}_{\mu_0}^* \) of \( \theta \) in the class of all estimators of \( \theta \) exists only for the values of \( X \) satisfying
\[
v_n(v_n + a/2) > 0 \quad \text{or} \quad v_n = -a/2.
\]
For \( v_n(v_n + a/2) > 0 \),
\[
\hat{\theta}_{\mu_0}^* = \hat{\theta}_{\mu_0, \sigma_1}^{\text{Bay}} + \frac{1}{a} \ln \frac{e^{(\lambda_2 - \lambda_1)(a^2/2 + av_n)} - 1}{av_n(\lambda_2 - \lambda_1)}.
\]
For \( v_n = -a/2 \) the range of the posterior risk does not depend on the value of \( \hat{\theta} \).

The conditional \( \Gamma \)-minimax estimator is
\[
\hat{\theta}_{\mu_0}^* = \begin{cases} 
\hat{\theta}_{\mu_0} & \text{if } v_n(v_n + a/2) > 0 \text{ and } \exp[(\lambda_1 - \lambda_2)(a^2/2 + av_n)] + av_n(\lambda_2 - \lambda_1) \geq 1, \\
\hat{\theta}_{\mu_0, \sigma_2}^{\text{Bay}} & \text{otherwise.}
\end{cases}
\]

The most stable estimator in the class
\[
\mathcal{L} = \{ \hat{\theta} : \forall \sigma \exists \sigma_1, \sigma_2 \ [\hat{\theta}(x) = \hat{\theta}_{\mu_0, \sigma}^{\text{Bay}}(x)] \}
\]
is equal to the conditional \( \Gamma \)-minimax estimator in the class of all estimators.

**Proof.** Let us check the conditions of Theorem 1 for
\[
\varrho_{\mu_0}(\lambda, \hat{\theta}) = \exp(-a\hat{\theta} + a\mu_0 + (a^2/2 + av_n)\lambda) - a(\mu_0 + \lambda v_n) + a\hat{\theta} - 1.
\]
The function \( \varrho_{\mu_0}(\lambda, \cdot) \) is convex and
\[
\frac{\partial \varrho_{\mu_0}(\lambda, \hat{\theta})}{\partial \lambda} = (a^2/2 + av_n) \exp(-a\hat{\theta} + a\mu_0 + \lambda(a^2/2 + av_n)) - av_n.
\]
Thus the minimum point is
\[
\lambda_{\text{min}}(\hat{\theta}) = \frac{a\hat{\theta} - a\mu_0 + \ln \frac{v_n}{a^2/2 + av_n}}{a^2/2 + av_n}
\]
and \( \lambda_{\text{min}} \) exists iff \( v_n(v_n + a/2) > 0 \).

For \( v_n \) satisfying \( v_n(v_n + a/2) \leq 0 \) the function \( q_{\mu_0}(\cdot, \hat{\theta}) \) is an increasing function of \( \lambda \) and the oscillation of the posterior risk
\[
r_{\mu_0}(\hat{\theta}) = -av_n(\lambda_2 - \lambda_1) + \exp\left(-a\hat{\theta} + a\mu_0 + (a^2/2 + av_n)\lambda_1\right) \\
\times [\exp((a^2/2 + av_n)(\lambda_2 - \lambda_1)) - 1]
\]
is a monotone function of \( \hat{\theta} \) (decreasing for \( a > 0 \) and \( -a/2 < v_n \leq 0 \), constant for \( v_n = -a/2 \) and increasing for \( a < 0 \) and \( 0 \leq v_n < -a/2 \)). Thus the most stable estimator does not exist for \( v_n(v_n + a/2) \leq 0 \) and \( v_n \neq -a/2 \). For \( v_n = -a/2 \) the oscillation \( r_{\mu_0}(\hat{\theta}) = a^2(\lambda_2 - \lambda_1)/2 \) does not depend on the value of \( \hat{\theta} \). The conditional \( \Gamma \)-minimax estimator \( \hat{\theta}_{\mu_0} \) is equal to \( \hat{\theta}_{\mu_0, \sigma_2} \).

Let us consider the situation when \( v_n(v_n + a/2) > 0 \). The minimum point \( \lambda_{\text{min}} \) and the function \( q_{\mu_0}(\lambda_2, \cdot) - q_{\mu_0}(\lambda_1, \cdot) \) are monotone functions of \( \hat{\theta} \). Condition 3 of Theorem 1 is similar to that in Theorem 2 so we obtain the most stable estimator as a solution of the equation
\[
q_{\mu_0}(\lambda_1, \hat{\theta}_{\mu_0}) = q_{\mu_0}(\lambda_2, \hat{\theta}_{\mu_0}).
\]

To find the conditional \( \Gamma \)-minimax estimator we check when \( \hat{\theta}_{\mu_0} \in \mathcal{L} \).

For \( v_n + a/2 > 0 \) we have \( \hat{\theta}_{\mu_0, \sigma_1} < \hat{\theta}_{\mu_0, \sigma_2} \). Solving the inequalities
\[
\hat{\theta}_{\mu_0, \sigma_1} \leq \hat{\theta}_{\mu_0} \leq \hat{\theta}_{\mu_0, \sigma_2}
\]
we obtain the condition
\[
(*) \quad \exp[(\lambda_1 - \lambda_2)(a^2/2 + av_n)] + av_n(\lambda_2 - \lambda_1) \geq 1.
\]

For \( v_n + a/2 < 0 \) we have \( \hat{\theta}_{\mu_0, \sigma_1} > \hat{\theta}_{\mu_0, \sigma_2} \). Solving the inequalities
\[
\hat{\theta}_{\mu_0, \sigma_1} \geq \hat{\theta}_{\mu_0} \geq \hat{\theta}_{\mu_0, \sigma_2}
\]
we also obtain \((*)\). Thus if \( v_n(v_n + a/2) > 0 \) and \((*)\) is true then \( \hat{\theta}_{\mu_0} = \hat{\theta}_{\mu_0} \).

If \( v_n + a/2 > 0 \) and \( v_n > 0 \) and \((*)\) is not true then
\[
\hat{\theta}_{\mu_0, \sigma_1} < \hat{\theta}_{\mu_0, \sigma_2} < \hat{\theta}_{\mu_0}
\]
and
\[
\sup_{\lambda \in [\lambda_1, \lambda_2]} q_{\mu_0}(\lambda, \hat{\theta}) = \begin{cases} 
q_{\mu_0}(\lambda_2, \hat{\theta}) & \text{if } \hat{\theta} \leq \hat{\theta}_{\mu_0}, \\
q_{\mu_0}(\lambda_1, \hat{\theta}) & \text{if } \hat{\theta} \geq \hat{\theta}_{\mu_0},
\end{cases}
\]
and the oscillation \( r_{\mu_0}(\hat{\theta}) \) is a decreasing function for \( \hat{\theta} < \hat{\theta}_{\mu_0} \).
If \( v_n + a/2 < 0 \) and \( v_n < 0 \) and \((*)\) is not true then
\[
\hat{\theta}_{\mu_0, \sigma_2}^{\text{Bay}} > \hat{\theta}_{\mu_0, \sigma_2}^{\text{Bay}} > \hat{\theta}_{\mu_0}
\]
and
\[
\sup_{\lambda \in [\lambda_1, \lambda_2]} \varrho_{\mu_0}(\lambda, \hat{\theta}) = \begin{cases} 
\varrho_{\mu_0}(\lambda_1, \hat{\theta}) & \text{if } \hat{\theta} \leq \hat{\theta}_{\mu_0}, \\
\varrho_{\mu_0}(\lambda_2, \hat{\theta}) & \text{if } \hat{\theta} \geq \hat{\theta}_{\mu_0},
\end{cases}
\]
and the oscillation \( r_{\mu_0}(\hat{\theta}) \) is an increasing function for \( \hat{\theta} \geq \hat{\theta}_{\mu_0} \).

Thus if \( v_n(v_n + a/2) > 0 \) and \((*)\) is not true then \( \hat{\theta}_{\mu_0} = \hat{\theta}_{\mu_0, \sigma_2}^{\text{Bay}} \) and \( \hat{\theta}_{\mu_0, \sigma_2}^{\text{Bay}} \) is the most stable estimator in the class \( \mathcal{L} \).

The monotonicity of the function \( r_{\mu_0} \) shows that \( \hat{\theta}_{\mu_0, \sigma_2}^{\text{Bay}} \) is also the most stable estimator in the class \( \mathcal{L} \) for \( v_n(v_n + a/2) \leq 0 \). □

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