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APPROXIMATION OF FINITE-DIMENSIONAL DISTRIBUTIONS FOR INTEGRALS DRIVEN BY α-STABLE LÉVY MOTION

Abstract. We present a method of numerical approximation for stochastic integrals involving α-stable Lévy motion as an integrator. Constructions of approximate sums are based on the Poissonian series representation of such random measures. The main result gives an estimate of the rate of convergence of finite-dimensional distributions of finite sums approximating such stochastic integrals.

Stochastic integrals driven by such measures are of interest in constructions of models for various problems arising in science and engineering, often providing a better description of real life phenomena than their Gaussian counterparts.

1. Introduction. Recent studies of various physical and biological problems (see, e.g., Buldyrev *et al.* (1993) and Wang (1992)), signal processing (Shao and Nikias (1993)), various extremal events models (Embrechts *et al.* (1997)) etc. reinforce the need for infinite variance stochastic models, including processes with discontinuous trajectories. Of particular interest are problems involving α-stable processes. Such processes also appear in stochastic models described by stochastic integrals with respect to α-stable random measures.

In this paper we are particularly concerned with the constructive methods of investigation of stochastic integrals driven by α-stable random measures. Such models only begin to find their way into different branches of applied probability and statistics (some examples are presented in Janicki and Weron (1994a), (1994b)).

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The main goal of this work is to prove a convergence result justifying numerical methods proposed, based on discretization of the time parameter t and simulation of α-stable random measures, and providing approximate sums for stochastic integrals with α-stable integrators defined by an α-stable Lévy motion process $\{Z_\alpha(t) : t \geq 0\}$. We are interested in construction of sequences $\{X_n(t)\}_{n=1}^{\infty}$ of real-valued processes which converge to a stochastic integral $\mathbf{X} = \{X(t)\}$ of the form

$$X(t) = \int_0^t f(s-)\, dZ_\alpha(s), \qquad t \in [0,1]. \tag{1.1}$$

A good introduction to α-stable processes is the review article by Weron (1984) with more comprehensive and up-to-date treatment in the monographs of Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994a).

The stochastic process described above can be regarded as a special case of general process driven by semimartingales, i.e. as a process of the following form:

$$X(t) = \int_0^t f(s-)\, dY(s), \qquad t > 0,$$

where $\{Y(t)\}$ stands for a given semimartingale process.

In fact, it is not difficult to notice that an α-stable Lévy motion belongs to the class of semimartingale processes. It is enough to observe that any Lévy process $\{Z(t) : t \geq 0\}$ (defined, e.g., in Protter (1990), Chapter I) can be described by its characteristic function given by the Lévy–Khinchine formula

$$E e^{i\theta Z(t)} = \exp(t\psi(\theta)),$$

where

$$\psi(\theta) = ib\theta - \frac{1}{2}c\theta^2 + \int_{-\infty}^{\infty} \left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2} \right) d\nu(x).$$

Here ν denotes the Lévy measure of the random variable $Z(1)$, i.e. a deterministic measure with the following properties: $\nu \geq 0$, $\int_{|x|\leq 1} x^2\, d\nu(x) < \infty$, $\nu(\{0\}) = 0$, and $\nu(\{x : |x| > \delta\}) < \infty$ for all $\delta > 0$.

Lévy processes are semimartingales (see, e.g., Protter (1990), Chapter I) and an α-stable Lévy motion process $\{Z_\alpha(t) : t \geq 0\}$ can be considered as an example of a Lévy process. Simply, in this case the Lévy measure $d\nu(u) = d\nu_\alpha(u)$ takes the form

$$d\nu_\alpha(u) = \begin{cases} \alpha\{C^+ \boldsymbol{I}_{(0,\infty)}(u) + C^- \boldsymbol{I}_{(-\infty,0)}(u)\}|u|^{-\alpha-1}du, & 0 < \alpha < 2, \\ 0, & \alpha = 2, \end{cases}$$

where C^+ and C^- are nonnegative constants such that $C^+ + C^- > 0$.

So, when studying convergence of sums approximating (1.1) one can lean on some results concerning the stability properties of integrals (1.1) in the space $\mathbb{D}([0,1],\mathbb{R})$ (see, e.g., Kurtz and Protter (1991) or Kasahara and Maejima (1986)). Alternatively, we propose rather elementary convergence results for constructive methods and algorithms specific to the case of α-stable random measures, providing additional information on the speed of convergence.

It is well known that any distribution on the space $\mathbb{D}([0,1],\mathbb{R})$ is completely determined by its finite-dimensional distributions (see Parthasarathy (1967), Chapter VII), so we propose a method of numerical approximation of finite-dimensional distributions of the process (1.1), based on the use of the so-called *series representation* of α-stable random variables and measures (see LePage (1980), (1989)). Our approach is based on Ferguson and Klass (1972), and on further developments by Rosinski (1990). We are concerned with representations of α-stable random variables X as a.s.-limits of sequences of sums $X_n := \sum_{j=1}^{n} \xi_j \tau_j^{-1/\alpha}$, where the τ_j's are the arrival times of a Poisson process and the ξ_j's are appropriately chosen random variables. After evaluating the expectation $\mathsf{E}|X_{n+m} - X_n|^2$ for $n > 0$, $m > 0$, it is possible to establish an upper bound for

$$\mathsf{P}\{\max_{1\le k\le 2^w} |X_n(2^{-w}k) - X(2^{-w}k)| > d\},$$

with any $d > 0$, where the distribution of the vector $\{X(2^{-w}k)\}_{k=1}^{2^w}$ coincides with the appropriate finite-dimensional distribution of the stochastic integral (1.1) and the variables $X_n(2^{-w}k)$, constructed by means of LePage-type sums, converge to the $X(2^{-w}k)$'s.

We provide the appropriate convergence result with rather sharp estimation of the error.

2. Series representations of stable random variables. The most common and convenient way to introduce an α-stable random variable is to define its characteristic function.

The characteristic function $\phi_X = \phi = \phi(\theta)$ of an α-stable random variable X involves four parameters: α—the index of stability, β—the skewness parameter, σ—the scale parameter and μ—the shift. This function is given by

$$\log\phi(\theta) = -\sigma^\alpha|\theta|^\alpha\{1 - i\beta\,\mathrm{sgn}(\theta)\tan(\alpha\pi/2)\} + i\mu\theta$$

when $\alpha \in (0,1)\cup(1,2]$, $\beta\in[-1,1]$, $\sigma\in\mathbb{R}_+$, $\mu\in\mathbb{R}$, and by

$$\log\phi(\theta) = -\sigma|\theta| + i\mu\theta$$

when $\alpha = 1$, which gives the well-known symmetric Cauchy distribution (notice that the case of $\alpha = 1$ with $\beta \ne 0$ is not considered here).

For a random variable X distributed according to the law derived from $\phi = \phi_X$ we use the notation $\mathcal{L}_X = S_\alpha(\sigma,\beta,\mu)$ or $\mathrm{Law}(X) = S_\alpha(\sigma,\beta,\mu)$. When $\mu = \beta = 0$, i.e., X is a symmetric α-stable random variable, we will write $\mathcal{L}_X = S\alpha S$. For convenience we denote by $S_{\alpha,\beta}$ the law $S_\alpha(1,\beta,0)$.

With the use of the Central Limit Theorem it is possible to describe the asymptotic behavior of α-stable variables. Namely, if $\mathcal{L}_X = S_\alpha(\sigma,\beta,\mu)$ and $\alpha \in (0,2)$, then

$$\begin{aligned}\lim_{x\to\infty} x^\alpha P\{X > x\} &= C_\alpha \frac{1+\beta}{2}\sigma^\alpha,\\ \lim_{x\to\infty} x^\alpha P\{X < -x\} &= C_\alpha \frac{1-\beta}{2}\sigma^\alpha,\end{aligned} \tag{2.1}$$

where

$$C_\alpha = \Big(\int_0^\infty x^{-\alpha}\sin(x)\,dx\Big)^{-1}. \tag{2.2}$$

Notice also that if $\mathcal{L}_X = S\alpha S$, then X belongs to $\boldsymbol{L}^{\alpha'}$ for $\alpha' \in (0,\alpha)$, and

$$\|X\|_{\alpha,\infty} := (\sup_{x>0}[x^\alpha P\{|X| > x\}])^{1/\alpha} < \infty.$$

To introduce series representations of α-stable random variables we need the sequence $\{\tau_1,\tau_2,\ldots\}$ composed of the arrival times or successive jump times of a right continuous Poisson process with unit rate; e.g., for $j \ge 1$, $\tau_j = \sum_{i=1}^j \lambda_i$, where $\{\lambda_1,\lambda_2,\ldots\}$ is a sequence of independent random variables with common exponential distribution

$$P\{\lambda_i > x\} = e^{-x}, \qquad x \ge 0.$$

Thus,

$$P\{\tau_j \le x\} = \int_0^y \frac{y^{j-1}}{(j-1)!} e^{-y}\,dy, \qquad x \ge 0,$$

and the random variable τ_j has the density

$$f_j(x) = x^{j-1}e^{-x}\boldsymbol{I}_{[0,\infty)}(x)/\Gamma(j). \tag{2.3}$$

Further on by a *series representation* of a given α-stable random variable X we mean a series $\sum_{j=1}^\infty \tau_j^{-1/\alpha}\xi_j$ such that

$$\lim_{J\to\infty}\sum_{j=1}^J \tau_j^{-1/\alpha}\xi_j = X \qquad \text{a.s.},$$

where $\{\xi_1,\xi_2,\ldots\}$ stands for an appropriately chosen sequence of i.i.d. random variables which is assumed to be independent of the sequence $\{\tau_j\}$.

LePage (1980) remarked that the series representations of the kind discussed here provide a fine insight into the structure of stable distributions.

In the symmetric case we have a stronger result (see Theorem 5.1 in Ledoux and Talagrand (1991)).

THEOREM 2.1. *Let* $\alpha \in (0,2)$ *and* η *be a symmetric real-valued random variable such that* $\mathsf{E}|\eta|^\alpha < \infty$. *Denote by* $\{\eta_j\}$ *a sequence of independent copies of* η *assumed to be independent of* $\{\tau_j\}$. *Then*

$$\lim_{M\to\infty} \sup_{N\geq M} \Big\| \sum_{j=M}^{N} \tau_j^{-1/\alpha}\eta_j \Big\|_{\alpha,\infty} = 0,$$

and the almost surely convergent series

$$X = \sum_{j=1}^{\infty} \tau_j^{-1/\alpha}\eta_j$$

defines an α*-stable random variable* $\mathcal{L}_X = S_\alpha(\sigma,0,0)$ *with* $\sigma = C_\alpha^{-1/\alpha}\|\eta\|_\alpha$ *and* C_α *from* (2.1)–(2.2).

In that case it is enough to take the *Rademacher sequence* for $\{\eta_i\}$, that is, a sequence of independent copies of η defined by

$$\mathsf{P}\{\eta = 1\} = 1/2 = \mathsf{P}\{\eta = -1\}.$$

Generally, by Theorem 5.1.2 of Samorodnitsky and Taqqu (1994), we have

THEOREM 2.2. *Let* $\alpha \in (0,1)\cup(1,2)$ *and* ξ *be a real-valued random variable such that* $\mathsf{E}|\xi|^\alpha < \infty$. *Denote by* $\{\xi_j\}$ *a sequence of independent copies of* ξ *assumed to be independent of* $\{\tau_j\}$. *Then the almost surely convergent series*

$$X = \begin{cases} \sum_{j=1}^{\infty} \tau_j^{-1/\alpha}\xi_j & \text{for } \alpha \in (0,1), \\ \sum_{j=1}^{\infty} (\tau_j^{-1/\alpha}\xi_j - k_j^{(\alpha)}) & \text{for } \alpha \in (1,2), \end{cases}$$

where

$$k_j^{(\alpha)} = \frac{\alpha}{\alpha-1}\big(j^{(\alpha-1)/\alpha} - (j-1)^{(\alpha-1)/\alpha}\big)\mathsf{E}\xi,$$

defines an α*-stable random variable* $\mathcal{L}_X = S_\alpha(\sigma,\beta,0)$ *with* $\sigma = C_\alpha^{-1/\alpha}\|\eta\|_\alpha$ *and* C_α *from* (2.2) *and* $\beta = \mathsf{E}(|\xi|^\alpha \operatorname{sgn}(\xi))(\mathsf{E}|\xi|^\alpha)^{-1}$.

In order to remove the centering constants $k_j^{(\alpha)}$ from the above series representation we propose the following choice of ξ. Let $\mathsf{P}\{\xi = t_1\} = p_1$ and $\mathsf{P}\{\xi = t_2\} = p_2$, where

$$t_1 = (1+\beta)^{1/(\alpha-1)}, \qquad t_2 = -(1-\beta)^{1/(\alpha-1)},$$
$$p_1 = -t_2/(t_1 - t_2), \qquad p_2 = 1 - p_1.$$

Then we have

Corollary 2.1. *Let $\alpha \in (1,2)$ and $\beta \in (-1,1)$, and let $\{\xi_j\}$ denote a sequence of independent copies of ξ, independent of $\{\tau_j\}$. Let*

$$Y := \sum_{j=1}^{\infty} \xi_j \tau_j^{-1/\alpha}. \tag{2.4}$$

Then the series defining Y converges a.s. to a stable random variable Y with characteristic function

$$\phi(\theta) = \exp\{-\sigma^\alpha |\theta|^\alpha (1 - i\beta \operatorname{sgn}(\theta) \tan(\alpha\pi/2))\}, \tag{2.5}$$

where the parameters α, β, σ satisfy

$$\begin{aligned} \sigma^\alpha &= C(\alpha,\beta), \\ C(\alpha,\beta) &= 2K(\alpha) p_1 t_1, \\ K(\alpha) &= -\alpha\Gamma(-\alpha)\cos(\alpha\pi/2), \end{aligned}$$

which means that $\mathcal{L}_Y = S_\alpha(\sigma, \beta, 0)$.

Proof. Let λ be the law of ξ. Then F defined by

$$F(A) = \int_0^\infty \int_{\{t_1,t_2\}} \boldsymbol{I}_A(svu^{-1/\alpha})\, d\lambda(v)\, du = p_1 F_{st_1}(A) + p_2 F_{st_2}(A),$$

for A such that $0 \notin A$, is the Lévy measure of a stable law. Therefore,

$$\int_{\{|x|>1\}} |x|^p \, dF(x) < \infty \qquad \text{whenever } p \in (1,\alpha),$$

and we are in a position to apply Theorem 3.1 of Rosinski (1990). First of all note that

$$\begin{aligned} p_1 t_1 + p_2 t_2 &= 0, \\ p_1 t_1^{\langle\alpha\rangle} + p_2 t_2^{\langle\alpha\rangle} &= \beta(p_1|t_1|^\alpha + p_2|t_2|^\alpha), \end{aligned} \tag{2.6}$$

where $t^{\langle\alpha\rangle} = |t|^\alpha \operatorname{sgn}(t)$. It follows from (2.6) that the centering constant appearing in Theorem 3.1 of Rosinski (1990) vanishes. Consequently, series (2.4) converges a.s. to a random variable X with

$$\widehat{\mathcal{L}}(X)(\theta) = \exp\Big\{ \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x)\, dF(x) \Big\}.$$

Now, noticing that for $c \in \mathbb{R}$ the formula

$$F_c(A) = \int_0^\infty \boldsymbol{I}_A(cu^{-1/\alpha})\, du, \qquad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

defines the Lévy measure of a stable law on $\mathbb{R}$, which is concentrated on

$\mathbb{R}\setminus\{0\}$ and

$$\int_{-\infty}^{\infty}(e^{i\theta x}-1-i\theta x)\,dF_c(x)=\alpha\Gamma(-\alpha)|c\theta|^{\alpha}e^{-i\,\mathrm{sgn}(c\theta)\alpha\pi/2},$$

we get

$$\begin{aligned}\int_{-\infty}^{\infty}&(e^{i\theta x}-1-i\theta x)\,dF(x)\\&=p_1\alpha\Gamma(-\alpha)|\theta|^{\alpha}|t_1|^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right)-i\sin\left(\frac{\alpha\pi}{2}\right)\mathrm{sgn}(\theta)\right)\\&\quad+p_2\alpha\Gamma(-\alpha)|\theta|^{\alpha}|t_2|^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\mathrm{sgn}(\theta)\right)\\&=-K(\alpha)|\theta|^{\alpha}(p_1|t_1|^{\alpha}+p_2|t_2|^{\alpha})\left(1-i\beta\,\mathrm{sgn}(\theta)\tan\left(\frac{\alpha\pi}{2}\right)\right)\end{aligned}$$

and (2.5) follows. ∎

For $\alpha\in(0,1)$ it is enough to take for $\{\xi_j\}$ a sequence of independent copies of a random variable ξ defined by

$$P\{\xi=1\}=\frac{1+\beta}{2},\qquad P\{\xi=-1\}=\frac{1-\beta}{2},$$

and notice that

$$C_\alpha=\frac{1}{C(\alpha,0)}=\frac{1-\alpha}{\Gamma(2-\alpha)\cos(\alpha\pi/2)}.$$

A notable shortcoming of the above result is the exclusion of the case $|\beta|=1$. However, as Corollary 2.2 below demonstrates, this cannot be remedied as long as we insist that $E\xi=0$. Before formulating the next proposition we state a simple lemma in which the τ_j's and ξ_j's are as above.

LEMMA 2.1. *For $\alpha>0$ set $T_n=\sum_{j=1}^{n}\xi_j\tau_j^{-1/\alpha}$. If $E|\xi|^{\alpha}=\infty$, then the sequence $\{T_1,T_2,\ldots\}$ diverges a.s.*

Proof. The event $\Omega_0=\{\omega:\lim_{n\to\infty}\tau_n(\omega)/n=1\}$ has probability one. Therefore, to prove that $\{T_n\}$ diverges a.s., it suffices to show that it diverges a.s. for sequences $\{\tau_n(\omega)\}$ with ω belonging to Ω_0. Fix such a sequence. Then the summands of T_n are independent and $\tau_j^{-1/\alpha}>2^{-1/\alpha}j^{-1/\alpha}$ eventually. Consequently,

$$P\{|\xi_j\tau_j^{-1/\alpha}|>2^{-1/\alpha}\}\geq P\{|\xi_j|^{\alpha}>j\}.$$

Now the assertion follows from the three series theorem and the fact that $E\zeta<\infty$ if and only if $\sum_{j=1}^{\infty}P\{\zeta>j\}<\infty$ for any positive random variable ζ. ∎

COROLLARY 2.2. *Let* $\alpha \in (1,2)$ *and* $\mathsf{E}\xi = 0$. *Set* $T_n = \sum_{j=1}^{n} \xi_j \tau_j^{-1/\alpha}$.

(i) *If* $\{T_n\}$ *converges a.s., then its limit* T_∞ *is a strictly stable random variable.*

(ii) *If* T_∞ *is nondegenerate, then its skewness parameter* β *belongs to* $(-1,1)$.

Proof. Let λ denote the law of ξ and $D = \operatorname{supp}(\lambda)$.

(i) By Lemma 2.1, $\mathsf{E}|\xi|^\alpha < \infty$, so the series $\sum_{j=1}^{\infty} \varepsilon_j \xi_j \tau_j^{-1/\alpha}$, with $\{\varepsilon_1, \varepsilon_2, \ldots\}$ denoting a sequence of i.i.d. Rademacher random variables independent of all the other sequences introduced so far, converges a.s. to a symmetric stable random variable (Theorem 1.5.1 of Samorodnitsky and Taqqu (1993)).

By Corollary 3.6 of Rosinski (1990), G defined by

$$G(A) = \int_0^\infty \int_D \boldsymbol{I}_A(vu^{-1/\alpha})\, d\lambda(v)\, du, \qquad 0 \notin A,$$

is a Lévy measure and thus the symmetrization of G is the Lévy measure of a stable law. Consequently, G is the Lévy measure of a stable law. (See Corollaries 6.3.1 and 6.3.2 of Linde (1986).) Thus,

$$\int_{\{|x|>1\}} |x|^p\, dG(x) < \infty \qquad \text{for all } p \in (1,\alpha),$$

and it remains to apply Theorem 3.1 of Rosinski (1990).

(ii) By applying the same argument as in the proof of Corollary 2.1 one gets

$$\begin{aligned}
\log \widehat{\mathcal{L}}(T_\infty)(\theta) &= \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x)\, dG(x) \\
&= \int_D \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x)\, dG_v(x)\, d\lambda(v) \\
&= \int_D \alpha\Gamma(-\alpha)|v\theta|^\alpha e^{i \operatorname{sgn}(v\theta)\alpha\pi/2}\, d\lambda(v) \\
&= -K(\alpha)|\theta|^\alpha \Big\{ \int_D |v|^\alpha\, d\lambda(v) - i \operatorname{sgn}(\theta)\tan(\alpha\pi/2) \int_D v^{\langle\alpha\rangle}\, d\lambda(v) \Big\}.
\end{aligned}$$

Recall that G_v is the Lévy measure defined by $G_v(A) = \int_0^\infty \boldsymbol{I}_A(vu^{-1/\alpha})\, du$. If T_∞ is nondegenerate, then $\int_D |v|^\alpha\, d\lambda(v) > 0$ and T_∞ has the skewness parameter

$$\beta = \frac{\int_D v^{\langle\alpha\rangle}\, d\lambda(v)}{\int_D |v|^\alpha\, d\lambda(v)}.$$

Thus, the requirements $|\beta| = 1$ and $\int_D v\, d\lambda(v) = 0$ are incompatible. ∎

Now some comment is in order. If $\alpha < 1$, then there is a clear difference between totally skewed ($|\beta| = 1$) and remaining stable random variables. If $\mathcal{L}_X = S_\alpha(\sigma, 1, 0)$ and $\mathcal{L}_Y = S_\alpha(\sigma, \beta, 0)$ with $0 < \alpha < 1$ and $|\beta| < 1$, then $\mathrm{supp}(\mathcal{L}_X) = [0, \infty)$ and $\mathrm{supp}(\mathcal{L}_Y) = (-\infty, \infty)$. By contrast, if $\alpha > 1$, each α-stable random variable has positive density on the whole line. In the light of this, Corollaries 2.1 and 2.2 exhibit a qualitative distinction between totally skewed and remaining stable random variables in the case of $\alpha > 1$.

Corollary 2.1 also raises the question whether the series representation of stable vectors taking values in Banach spaces can be so modified that no centering is needed.

3. α-Stable integrals. Denote by $(\Omega, \mathcal{F}, P)$ the underlying probability space and by $\boldsymbol{L}^0(\Omega, \mathcal{F}, P)$ the set of all real random variables defined on it. Let $(E, \mathcal{E}, m)$ be a measure space, and let

$$\mathcal{E}_{\mathrm{f}} = \{A \in \mathcal{E} : m(A) < \infty\}$$

be the subset of $\mathcal{E}$ of sets of finite m-measure.

A stochastic process $\{L_{\alpha,\beta}(t) : t \geq 0\}$ is called an *α-stable Lévy motion* if

1. $L_{\alpha,\beta}(0) = 0$ a.s.,
2. $\{L_{\alpha,\beta}(t) : t \geq 0\}$ has independent increments,
3. the stationary increments $L_{\alpha,\beta}(t) - L_{\alpha,\beta}(s)$ are distributed according to the law $S_\alpha((t - s)^{1/\alpha}, \beta, 0)$ for all $0 \leq s < t < \infty$.

Since each Lévy process admits a unique modification which has cadlag trajectories (see Protter (1990), Section I.4), we can assume that the trajectories of an α-stable Lévy motion are cadlag functions.

Observe that each α-stable Lévy motion has stationary increments. It is a Brownian motion when $\alpha = 2$. The α-stable Lévy motions are $S\alpha S$ when $\beta = 0$.

An independently scattered σ-additive set function

$$\boldsymbol{M} : \mathcal{E}_{\mathrm{f}} \ni A \mapsto M(A) \in \boldsymbol{L}^0(\Omega, \mathcal{F}, P)$$

such that for each $A \in \mathcal{E}_{\mathrm{f}}$,

$$\mathrm{Law}(M(A)) = \mathcal{L}_{M(A)} = S_\alpha((m(A))^{1/\alpha}, \beta, 0)$$

is called an *α-stable random measure* on $(E, \mathcal{E})$ with control measure m and skewness intensity β. The measure $\boldsymbol{M}$ is called an *$S\alpha S$ random measure* if the skewness intensity β is zero.

The definition of the *α-stable stochastic integral*

$$I(f) = \int_E f(x)\, dM(x)$$

with respect to an α-stable random measure on $(E,\mathcal{E})$ for a measurable function $f \in \boldsymbol{L}^\alpha(E,\mathcal{E},m)$ is quite well known (see, e.g., Samorodnitsky and Taqqu (1994)). Let us briefly recall this definition in the case of $(E,\mathcal{E},m) = ([0,1],\mathcal{B},\mathrm{Leb})$.

In this case, for any $0 \le a < b \le 1$, we just have

$$M([a,b)) \stackrel{\mathrm{d}}{=} L_{\alpha,\beta}(b) - L_{\alpha,\beta}(a) \stackrel{\mathrm{d}}{=} L_{\alpha,\beta}(b-a),$$

with

$$\mathrm{Law}(M([a,b))) = S_\alpha((b-a)^{1/\alpha},\beta,0).$$

So, further on instead of $dM(s)$ we write $dL_{\alpha,\beta}(s)$.

Let us formulate a basic property of α-stable random variables in a form suitable for our purposes: if $\mathcal{L}_X = S_{\alpha,\beta}$ and X_i are independent copies of X, then

$$\sum_{i=1}^{I} f_i h^{1/\alpha} X_i \stackrel{\mathrm{d}}{=} \Big(\sum_{i=1}^{I} |f_i|^\alpha h\Big)^{1/\alpha} X \tag{3.1}$$

for any sequence $\{f_i\}$ of real numbers and $h > 0$.

This explains immediately that if $f^I \in \boldsymbol{L}^\alpha([0,1],\mathcal{B},\mathrm{Leb})$ is a step function of the form

$$f^I(t) = \sum_{i=1}^{I} f_i^I \boldsymbol{I}_{[t_i,t_{i+1})}(t), \tag{3.2}$$

for $t_i = (i-1)h$, $h = 1/I$, then, with X_i as above, we get

$$\begin{aligned} I(f^I) = \int_0^1 f^I(s)\, dL_{\alpha,\beta} &:= \sum_{i=1}^{I} f_i^I h^{1/\alpha} X_i \\ &\stackrel{\mathrm{d}}{=} \sum_{i=1}^{I} f_i^I (L_{\alpha,\beta}(t_i) - L_{\alpha,\beta}(t_{i-1})), \end{aligned}$$

where

$$\mathrm{Law}(I(f^I)) = S_\alpha(\|f\|_\alpha,\beta,0).$$

We will call $I(f^I)$ a *stochastic integral* of f (with respect to an α-stable Lévy motion process).

Further on we assume for any $f \in \boldsymbol{L}^\alpha([0,1],\mathcal{B},\mathrm{Leb})$ to be given a sequence $\{f^I\}$ of step functions from $\boldsymbol{L}^\alpha([0,1],\mathcal{B},\mathrm{Leb})$ such that

$$\lim_{I\to\infty} \|f - f^I\|_\alpha = 0.$$

The sequence of integrals $\{I(f^I)\}_{I=1,2,\ldots}$ is well defined. It is a Cauchy sequence in the complete space $S_{\alpha,\beta}$ of α-stable random variables with the metric induced in $\boldsymbol{L}^0(\Omega,\mathcal{F},P)$ by convergence in probability. So, there exists a random variable $I(f)$ which is the limit of $\{I(f^I)\}$ in this space. Therefore the *α-stable stochastic integral* of any function $f \in \boldsymbol{L}^\alpha([0,1], \mathcal{B}, \mathrm{Leb})$ is defined by

$$I(f) := \lim_{I\to\infty} I(f^I) \quad \text{ in probability.}$$

Notice also that, by Breiman (1992), Chapter VIII, one can derive the following estimate:

$$(3.3)\quad P\Big\{\Big|\int_0^1 f(s)\,dL_{\alpha,\beta}(s) - \int_0^1 f^I(s)\,dL_{\alpha,\beta}(s)\Big| \ge \delta\Big\} \le K\delta \int_0^{1/\delta} (1-\psi_I(v))\,dv,$$

where $\psi_I = \psi_I(v)$ stands for the characteristic function of the difference of $I(f)$ and $I(f^I)$.

Our main goal is to propose an algorithm for approximate construction of the stochastic process

$$(3.4)\qquad X(t) = I(f;t) = \int_0^t f(s)\,dL_{\alpha,\beta}(s) \quad \text{ for } t \in [0,1].$$

It is well known that the process $\{X(t)\}$ defined by (3.4), being an infinitely divisible process with stationary increments, is a cadlag process, i.e., its trajectories belong to the space $\mathbb{D}([0,1],\mathbb{R})$ (see Protter (1990), Chapter I). Let us recall that such processes can be characterized by the following theorem (Parthasarathy (1967), Chapter VII).

THEOREM 3.1. *The class $\mathcal{B}_D$ of the Borel subsets of $\mathbb{D}([0,1],\mathbb{R})$ coincides with the smallest σ-algebra of subsets of $\mathbb{D}([0,1],\mathbb{R})$ with respect to which the maps $\pi^t : x \mapsto x(t)$ are measurable for all $t \in [0,1]$. If μ and ν are two measures on $\mathbb{D}([0,1],\mathbb{R})$ then a necessary and sufficient condition for $\mu=\nu$ to hold is that $\mu^{t_1,\dots,t_k} = \nu^{t_1,\dots,t_k}$ for all k and $t_1,\dots,t_k$ from $[0,1]$, where $\mu^{t_1,\dots,t_k}$ and $\nu^{t_1,\dots,t_k}$ are the measures in $\mathbb{R}^k$ induced by μ and ν, respectively, through the map $\pi^{t_1,\dots,t_k} : x \mapsto (x(t_1),\dots,x(t_k))$.*

This theorem leads to our idea of approximating $X(t) = I(f;t)$ defined by (3.4) by means of finite-dimensional random vectors $\{X(t_i)\}_{i=1}^I$ in the following way.

Having a step function $f^I = f^I(t)$ on $[0,1]$ of the form (3.2), and such that

$$P\Big\{\Big|\int_0^1 f(s)\,dL_{\alpha,\beta} - \int_0^1 f^I(s)\,dL_{\alpha,\beta}\Big| \ge \delta\Big\} \le \varepsilon$$

for any fixed $\delta > 0$ and given $\varepsilon > 0$ (the choice of I can be controlled by (3.3)), we construct inductively the $(I+1)$-dimensional random vector $\{X_i^{I,J}\}_{i=0}^I$ in the following way:

$$(3.5)\quad X_0^{I,J} = 0 \quad \text{a.s.}, \qquad X_i^{I,J} := X_{i-1}^{I,J} + f_i^I\, h^{1/\alpha} \Delta L_i^J, \qquad i = 1,\dots,I,$$

where

$$\Delta L_i^J := C(\alpha,\beta)^{-1/\alpha}\sum_{j=1}^{J}\xi_{i,j}\tau_{i,j}^{1/\alpha}, \tag{3.6}$$

and $\{\xi_{i,j}\}_{i=1,j=1}^{I,J}$, $\{\tau_{i,j}\}_{i=1,j=1}^{I,J}$ are independent copies of the random variables ξ and τ_j which were defined in the previous section.

Notice that $h^{-1/\alpha}\Delta L_i^J$ can be regarded as a good approximation for the α-stable random measure $M([t_{i-1},t_i))$ of the interval $[t_{i-1},t_i)$, so one can expect that $\{X_i^{I,J}\}$ is a good approximation for $\{I(f^I;t_i)\}$ and thus also for $\{I(f;t_i)\}$ for a function f from $\boldsymbol{L}^\alpha([0,1],\mathcal{B},\mathrm{Leb})$, in the sense of convergence in probability.

Notice also that

$$I(f^I;t_i)-X_i^{I,J}=\sum_{l=1}^{i}f_l^I\,h^{1/\alpha}(X_l-\Delta L_l^J). \tag{3.7}$$

The theorem providing an estimate allowing one to control the parameter J is proved in the next section.

4. Convergence of approximations for stable integrals. What we now need is an estimation of the probability of generating an approximate trajectory of $I(f^I;t)$ which deviates too far away from a real trajectory. Let δ and ε be arbitrary, small enough, positive numbers. It seems reasonable to require that

$$\mathsf{P}\{\exists_{i\in\{1,\ldots,I\}}:|I(f^I;t_i)-X_i^{I,J}|>\delta\}<\varepsilon. \tag{4.1}$$

It will follow from our main theorem that, given positive δ, ε and natural number I, (4.1) holds if J satisfies

$$R_J(\alpha,\beta)<\|f^I\|_\alpha^{-2}\delta^2 I^{2/\alpha-1}\varepsilon,$$

where

$$R_J(\alpha,\beta)=C(\alpha,\beta)^{-2/\alpha}(1-\beta)^{1/(\alpha-1)}\sum_{j=J+1}^{\infty}(j-2/\alpha)^{-2/\alpha}. \tag{4.2}$$

THEOREM 4.1. *Let $\alpha\in(1,2)$ and $|\beta|\neq 1$, or $\alpha\in(0,1)$ and $\beta\in[-1,1]$, or $\alpha\in(0,2)$ and $\beta=0$. Let $X_i^{I,J}$ be the random variables defined by* (3.5). *Then*

$$I(f^I;t_i)=\lim_{J\to\infty}X_i^{I,J}\qquad a.s.$$

Moreover, for any positive δ and $J>2/\alpha$ we have

$$\mathsf{P}\{\exists_{i\in\{1,\ldots,I\}}:|I(f^I;t_i)-X_i^{I,J}|>\delta\}<\|f^I\|_\alpha^2\delta^{-2}I^{1-2/\alpha}R_J(\alpha,\beta),$$

where $R_J(\alpha,\beta)$ is given by (4.2).

P r o o f. I. First we prove some technical results concerning the case of $\alpha \in (1,2)$ and $|\beta| \neq 1$.

For $C(\alpha,\beta)$ and the ξ_j's, τ_j's as in Corollary 2.1, and $h > 0$, define

$$L^J_{\alpha,\beta}(h) := h^{1/\alpha} C(\alpha,\beta)^{-1/\alpha} \sum_{j=1}^{J} \xi_j \tau_j^{-1/\alpha}.$$

Now we prove that for any $J > 2/\alpha$ and $m > 0$,

$$E|L^{J+m}_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)|^2 < h^{2/\alpha} R_J(\alpha,\beta). \tag{4.3}$$

Write

$$\begin{aligned} E|L^{J+m}_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)|^2 &= E\Big| h^{1/\alpha} C(\alpha,\beta)^{-1/\alpha} \sum_{j=J+1}^{J+m} \xi_j \tau_j^{-1/\alpha}\Big|^2 \\ &= h^{2/\alpha} C(\alpha,\beta)^{-2/\alpha} E|\xi_j|^2 E\tau_j^{-2/\alpha}. \end{aligned}$$

The last equality is justified by the fact that for $j \neq k$ we have

$$E(\xi_j\xi_k(\tau_j\tau_k)^{-1/\alpha}) = E(\xi_j\xi_k)E((\tau_j\tau_k)^{-1/\alpha}) = E\xi_j E\xi_k E((\tau_j\tau_k)^{-1/\alpha}) = 0.$$

Since $E|\xi_j|^2 = E|\xi|^2 = t_1^2 p_1 + t_2^2 p_2 = (1-\beta^2)^{1/(\alpha-1)}$, we get

$$\begin{aligned} &E|L^{J+m}_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)|^2 \\ &\quad = h^{2/\alpha} C(\alpha,\beta)^{-2/\alpha} (1-\beta^2)^{1/(\alpha-1)} \sum_{j=J+1}^{\infty} E\tau_j^{-2/\alpha}. \end{aligned} \tag{4.4}$$

By (2.3), $E\tau_j^{-2/\alpha} = \Gamma(j-2/\alpha)/\Gamma(j)$ whenever $j > 2/\alpha$. It is obvious that $\Gamma(j-2/\alpha)/\Gamma(j) = (j-2/\alpha)^{-2/\alpha}$ for $\alpha = 2$, so, after some calculations, we derive the inequality

$$\Gamma(j-2/\alpha)/\Gamma(j) < (j-2/\alpha)^{-2/\alpha} \qquad \text{for all } \alpha \in (1,2)$$

and thus for $j > 2/\alpha$ we have

$$E\tau_j^{-2/\alpha} \le (j-2/\alpha)^{-2/\alpha}. \tag{4.5}$$

Combining (4.4) and (4.5) we get (4.3).

Noticing that

$$L_{\alpha,\beta}(h) \stackrel{\mathrm{d}}{=} h^{1/\alpha} C(\alpha,\beta)^{-1/\alpha} \sum_{j=1}^{\infty} \xi_j \tau_j^{-1/\alpha},$$

we derive the a.s.-convergence of $\{L^J_{\alpha,\beta}(h)\}$ to $L_{\alpha,\beta}(h)$ from Corollary 2.1. Using Theorem 3.1 of Rosinski (1990), one can easily check that the convergence is also in $\boldsymbol{L}^{\alpha'}$ for each $\alpha' \in (1,\alpha)$. Namely, for any $J > 2/\alpha$, $h > 0$ and $\alpha' \in (1,\alpha)$, we get

$$P\{|L_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)| > \eta\} \le \eta^{-2} h^{2/\alpha} R_J(\alpha,\beta), \tag{4.6}$$

$$E\{(|L_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)|^{\alpha'})^{1/\alpha'}\} \le h^{1/\alpha}(R_J(\alpha,\beta))^{1/2}.$$

To get (4.6) it is enough to notice that for any $\delta \in (0,\eta)$ and $m > 0$,

$$P\{|L_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)| > \eta\} \le P\{|L_{\alpha,\beta}(h) - L^{J+m}_{\alpha,\beta}(h)| > \delta\} + (\eta-\delta)^{-2}h^{2/\alpha}R_J(\alpha,\beta),$$

and let first $m \to \infty$ and then $\delta \to 0$.

Taking now into account (3.1) and (3.7), by the Kolmogorov inequality and the above argument we can write, for any $J > 2/\alpha$,

$$P\{\exists_{i\in\{1,\dots,I\}} : |I(f^I;t_i) - X^{I,J}_i| \ge \delta\} = P\Big\{\max_{1\le H\le I} : \Big|\sum_{i=1}^{H} f^I_i h^{1/\alpha}(L^{(i)}_{\alpha,\beta}(1) - \Delta L^J_i)\Big| \ge \delta\Big\} \le \delta^{-2}\|f^I\|^2_\alpha h^{2/\alpha-1}R_J(\alpha,\beta).$$

This completes the proof in the case of $\alpha \in (1,2)$ and $|\beta| \ne 1$.

II. Now we deal with the case of $\alpha \in (0,1)$ and β arbitrary from $[-1,1]$. This means that we admit here totally skewed α-stable stochastic integrals.

Since now $E\xi = \beta$, the above procedure cannot be applied. However, we can proceed as follows. First note that

$$E|L^{J+m}_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)| \le h^{1/\alpha}Q_J(\alpha)$$

for $J > 1/\alpha$, where

$$Q_J(\alpha) = C_\alpha \sum_{j=J+1}^{\infty} (j - 1/\alpha)^{-1/\alpha}.$$

As in the previous case, our objective is to determine the values of J for which

$$P\{\exists_{i\in\{1,\dots,I\}} : |I(f^I;t_i) - X^{I,J}_i| > \delta\} < \varepsilon. \tag{4.7}$$

The above inequality will be satisfied if we have

$$P\{\forall_{i\in\{1,\dots,I\}} : |I(f^I;t_i) - X^{I,J}_i| \le \delta h\} > 1 - \varepsilon.$$

It is enough to have

$$(1 - P\{|L_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)| \ge \delta h\})^I > 1 - \varepsilon.$$

As in the previous case we see that

$$P\{|L_{\alpha,\beta}(h) - L^J_{\alpha,\beta}(h)| > \eta\} \le \eta^{-1}h^{1/\alpha}Q_J(\alpha).$$

Consequently, all the J satisfying the condition

$$Q_J(\alpha) > \delta I^{(1-1/\alpha)}(1 - (1-\varepsilon)^2)$$

also satisfy (4.7).

III. The proof in the case of $\alpha \in (0,2)$ and $\beta = 0$ now seems quite obvious. ■

Notice that approximating sums $X_i^{I,J}$ defined by (3.5) are well suited for computer simulations. In particular, it is possible to apply some of statistical estimation methods providing more information on the approximate stochastic integrals constructed (some of those techniques are widely utilised in Janicki and Weron (1994a)).

REMARK 4.1. Making use of the sets $\{X_i^{I,J}\}_{i=1}^I$ of random variables defined by (3.5), (3.6) for given natural numbers I, J, and applying obvious interpolation techniques it is possible to get a sequence $\{I^J(f^I;t) : t \in [0,1]\}$ of processes approximating the stochastic integral $\{I(f^I;t) : t \in [0,1]\}$ as $J \to \infty$. The problem of estimation of the rate of convergence of these approximations in the Skorokhod topology of the space $\mathbb{D}([0,1],\mathbb{R})$ seems to be an open question.

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