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REGULARITY OF THE MULTIDIMENSIONAL SCALING FUNCTIONS: ESTIMATION OF THE L^p-SOBOLEV EXPONENT

Abstract. The relationship between the spectral properties of the transfer operator corresponding to a wavelet refinement equation and the L^p-Sobolev regularity of solution for the equation is established.

1. Introduction. Let us consider the d-dimensional refinement equation

$$f(x) = 2^d \sum_{k\in\mathbb{Z}^d} c_k f(2x-k), \tag{1}$$

where $x \in \mathbb{R}^d$, and

$$\sum_{k\in\mathbb{Z}^d} c_k = 1. \tag{2}$$

Any solution φ of (1) is called a *scaling function* or *refinable function.*

One of the fundamental problems for the scaling function is to estimate its regularity. For the one-dimensional case with a finite number of nonzero coefficients c_k, $k \in \mathbb{Z}$, the estimations of Hölder exponent were derived in [13], [4, 5], [14], and the Sobolev and L^p regularity was studied in [7], [16], [2], [8], [10], [12], [9]. But only [10] and [2] concern the case with an infinite number of nonzero coefficients in (1).

For $d = 2$ the L^p regularity for compactly supported scaling functions was studied in [11]. In this article we adopt the methods of [2] for deriving the estimation for the coefficient of L^p-Sobolev regularity in the case $d = 2$. We establish a connection between the L^p-Sobolev exponent s_p and the spectral radius of the so called transfer operator corresponding to the equation (1).

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Beginning from Lemma 2.7, for clarity, we confine ourselves to the case $d = 2$.

2. The transfer operator. The following notations are used: $\Lambda = \{(j_1,\ldots,j_d) : j_k \in \{0,1\},\ k = 1,\ldots,d\}$. For any function $f \in L^1(\mathbb{R}^d)$ we consider the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{i\langle x,\xi\rangle}\,dx$$

and for any function from $L^2([-\pi,\pi]^d)$ we consider the nth Fourier coefficient

$$f_n = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} f(x)e^{i\langle n,x\rangle}\,dx, \qquad n \in \mathbb{Z}^d.$$

The L^p-*Sobolev exponent* s_p is defined by

$$s_p = \sup\Big\{s : \int_{\mathbb{R}^d} |\widehat{f}(x)|^p(1+\|x\|^p)^s\,dx < \infty\Big\}.$$

Let $\mathcal{P}$ denote the set of all continuous functions $f : \mathbb{R}^d \to \mathbb{C}$, 2π-periodic with respect to each variable. Let $\omega \in \mathcal{P}$. Then the *transfer operator* $\mathcal{L}_\omega : \mathcal{P} \to \mathcal{P}$ associated with ω is defined by

$$(\mathcal{L}_\omega f)(x) = \sum_{e\in\Lambda} \omega(2^{-1}x+\pi e)f(2^{-1}x+\pi e). \tag{3}$$

It is called the *Perron–Frobenius operator*.

The following lemmas concerning $\mathcal{L}_\omega$ will be important in our further considerations:

Lemma 2.1. *Let $f, g \in \mathcal{P}$ and $k \in \mathbb{N}$. Then*

$$\int_{[-\pi,\pi]^d} f(x)(\mathcal{L}_\omega^k g)(x)\,dx = \int_{[-2^k\pi,2^k\pi]^d} f(x)\Big[\prod_{n=1}^{k}\omega(2^{-n}x)\Big]g(2^{-k}x)\,dx$$
$$= 2^{dk}\int_{[-\pi,\pi]^d} f(2^k x)\Big[\prod_{n=0}^{k-1}\omega(2^n x)\Big]g(x)\,dx.$$

The proof is a straightforward generalization of the one-dimensional case (see [2]).

Lemma 2.2. *Let $f \in \mathcal{P}$ and $n \in \mathbb{N}$. Then*

$$(\mathcal{L}_\omega^n f)(x) = \sum_{m\in I_n}\Big[\prod_{j=1}^{n}\omega(2^{-j}(x+2\pi m))\Big]f(2^{-n}(x+2\pi m)), \tag{4}$$

where $I_n = \{m \in \mathbb{Z}^d : m_i \in \{-2^{n-1}+1,\ldots,2^{n-1}\},\ i = 1,\ldots,d\}$.

Proof (by induction). The first step is obvious. Suppose that (4) holds for any $k \le n$ and let

$$I_n = \{m \in \mathbb{Z}^d : m_i \in \{-2^{n-1}+1, \ldots, 2^{n-1}\},\ i = 1, \ldots, d\}.$$

Then

$$\begin{aligned}(5)\quad & (\mathcal{L}_\omega^{n+1} f)(x) \\ &= \sum_{e\in\Lambda} \omega(2^{-1}x + \pi e)(\mathcal{L}_\omega^n f)(2^{-1}x + \pi e) \\ &= \sum_{e\in\Lambda} \omega(2^{-1}(x + 2\pi e)) \sum_{m\in I_n} \Big[\prod_{j=2}^{n+1} \omega(2^{-j}(x + 2\pi(e+2m)))\Big] \\ &\quad \times f(2^{-(n+1)}(x + 2\pi(e+2m))) \\ &= \sum_{e\in\Lambda}\sum_{m\in I_n} \omega(2^{-1}(x + 2\pi(e+2m)))\Big[\prod_{j=2}^{n+1} \omega(2^{-j}(x + 2\pi(e+2m)))\Big] \\ &\quad \times f(2^{-(n+1)}(x + 2\pi(e+2m))) \\ &= \sum_{m\in I'_{n+1}} \Big[\prod_{j=1}^{n+1} \omega(2^{-j}(x + 2\pi m))\Big] f(2^{-(n+1)}(x + 2\pi m)),\end{aligned}$$

where

$$(6)\qquad I'_{n+1} = \{m \in \mathbb{Z}^d : m_i \in \{-2^n+2, \ldots, 2^n+1\},\ i = 1, \ldots, d\}.$$

Now consider the set

$$I = \{m \in I'_{n+1} : \text{there exists } i \in \{1, \ldots, d\} \text{ such that } m_i = 2^n+1\}.$$

Then for each $m \in I$ such that

$$m = (m_1, \ldots, m_{i-1}, 2^n+1, m_{i+1}, \ldots, m_d),$$

by periodicity we have

$$\omega(2^{-j}(x + 2\pi m)) = \omega(2^{-j}(x + 2\pi(m_1, \ldots, m_{i-1}, -2^n+1, m_{i+1}, \ldots, m_d))),$$

and similarly

$$\begin{aligned}& f(2^{-(n+1)}(x + 2\pi m)) \\ &\quad = f(2^{-(n+1)}(x + 2\pi(m_1, \ldots, m_{i-1}, -2^n+1, m_{i+1}, \ldots, m_d))).\end{aligned}$$

Hence from (5), (6) we obtain our inductive claim.

REMARK 2.1. *For any function $f \in \mathcal{P}$ and $n \in \mathbb{Z}^d$,*

$$(\mathcal{L}_\omega f)_n = 2^d \sum_{k\in\mathbb{Z}^d} \omega_{2n-k} f_k.$$

For $\mathbb{R} \ni \alpha > 0$ the function space

$$E_\alpha = \Big\{ f \in \mathcal{P} : f(x) = \sum_{n\in\mathbb{Z}^d} f_n e^{-i\langle n,x\rangle},\ \|f\|_\alpha^2 = \sum_{n\in\mathbb{Z}^d} |f_n|^2 e^{2\|n\|\alpha} < \infty \Big\},$$

is a Hilbert space of analytic functions (see Theorem A.4) with the inner product

$$\langle f, g\rangle_\alpha = \sum_{n\in\mathbb{Z}^d} f_n \overline{g}_n e^{2\alpha\|n\|}.$$

For each function f from E_α we estimate

$$\begin{aligned} |f(x)| &\le \sum_{n\in\mathbb{Z}^d} |f_n| = \sum_{n\in\mathbb{Z}^d} e^{-\|n\|\alpha} |f_n| e^{\|n\|\alpha} \\ &\le \Big(\sum_{n\in\mathbb{Z}^d} e^{-2\|n\|\alpha}\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}^d} |f_n|^2 e^{2\|n\|\alpha}\Big)^{1/2}. \end{aligned}$$

Hence we have proved:

REMARK 2.2. *We have* $\|f\|_{L^\infty} \le C_\alpha \|f\|_\alpha$ *for* $f \in E_\alpha$, *where* $C_\alpha = (\sum_{n\in\mathbb{Z}^d} e^{-2\alpha\|n\|})^{1/2}$ *is a universal constant.*

REMARK 2.3. *Let* $e_{n,\alpha}(x) = e^{-i\langle n,x\rangle} e^{-\alpha\|n\|}$, *where* $n \in \mathbb{Z}^d$. *Then* $\{e_{n,\alpha}\}$ *is an orthonormal basis of* E_α.

LEMMA 2.3. *Let* $\omega \in \mathcal{P}$ *and suppose that* $\alpha \in (\gamma, 2\gamma)$ *and* $|\omega_n| \le Ce^{-\gamma\|n\|}$ *for some* $C, \gamma > 0$. *Then*:

(i) $\mathcal{L}_\omega$ *maps* E_α *to* E_α.
(ii) $\mathcal{L}_\omega$ *is compact.*
(iii) $\mathcal{L}_\omega$ *is a trace-class operator.*

Proof. (i) $\|\mathcal{L}_\omega f\|_\alpha^2$ can be estimated as follows:

$$\begin{aligned} \|\mathcal{L}_\omega f\|_\alpha^2 &= 2^{2d} \sum_{n\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} \omega_{2n-k} f_k \Big|^2 e^{2\|n\|\alpha} \\ &\le 2^{2d} \sum_{n\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |\omega_{2n-k} e^{-\|k\|\alpha} f_k e^{\|k\|\alpha}|^2 e^{2\|n\|\alpha} \\ &\le 2^{2d} \|f\|_\alpha^2 \sum_{n\in\mathbb{Z}^d} \Big[\sum_{k\in\mathbb{Z}^d} |\omega_{2n-k}|^2 e^{-2\|k\|\alpha} \Big] e^{2\|n\|\alpha} \\ &\le 2^{2d} \|f\|_\alpha^2 C^2 \sum_{n\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} e^{-2\|k\|\alpha} e^{2\|n\|\alpha} e^{2\gamma\|2n-k\|\alpha} \\ &\le 2^{2d} \|f\|_\alpha^2 C^2 \Big[\sum_{n\in\mathbb{Z}^d} e^{-2\|n\|(2\gamma-\alpha)} \Big] \Big[\sum_{k\in\mathbb{Z}^d} e^{-2\|k\|(\alpha-\gamma)} \Big] < \infty. \end{aligned}$$

(ii) We must prove that $\mathcal{L}_\omega(K)$ is relatively compact, where $K = \{f \in E_\alpha : \|f\|_\alpha \le 1\}$. One can immediately see that $\mathcal{L}_\omega(K)$ is a bounded subset in E_α.

Now let $(\varepsilon_k)_{k=1}^\infty$ be a basis of E_α such that:

(a) for each n from $\mathbb{Z}^d$ there exists exactly one $k \in \mathbb{N}$ such that $e_{n,\alpha} = \varepsilon_k$,

(b) for each k from $\mathbb{N}$ there exists exactly one $n \in \mathbb{Z}^d$ such that $e_{n,\alpha} = \varepsilon_k$,

(c) for each $n \in \mathbb{Z}^d$ and $k \in \mathbb{N}$ such that $e_{n,\alpha} = \varepsilon_k$, $\sum_{i=1}^d |n_i| \le k$,

(d) for each $n, m \in \mathbb{Z}^d$ and $k, l \in \mathbb{N}$ such that $e_{n,\alpha} = \varepsilon_k$, $e_{m,\alpha} = \varepsilon_l$ the following condition holds: if $\sum_{i=1}^d |n_i| = \sum_{i=1}^d |m_i|$ then $k \le l$; if $\sum_{i=1}^d |n_i| < \sum_{i=1}^d |m_i|$ then $k < l$.

Let $R_k : E_\alpha \to \operatorname{span}\{\varepsilon_{k+1}, \ldots\}$ and $f \in K$. Consider $n^0 \in \mathbb{Z}^d$, $k_1 \in \mathbb{N}$ such that $e_{n^0,\alpha} = \varepsilon_{k+1}$ and $k_1 = \sum_{i=1}^d |n_i^0|$ and set $I(k_1) = \{m \in \mathbb{Z}^d : \sum_{i=1}^d |m_i| = k_1$ and for all $l \in \mathbb{N}$ if $\varepsilon_l = e_{m,\alpha}$ then $l < k+1\}$. Then

$$
\begin{aligned}
(7)\quad & \|R_k(\mathcal{L}_\omega f)\|_\alpha \\
&\le \Big\| \sum_{\|n\| \ge k_1} (\mathcal{L}_\omega f)_n e^{-i\langle n,\cdot\rangle} \Big\|_\alpha + \Big\| \sum_{n \in I(k_1)} (\mathcal{L}_\omega f)_n e^{-i\langle n,\cdot\rangle} \Big\|_\alpha \\
&= \Big(\sum_{\|n\| \ge k_1} |(\mathcal{L}_\omega f)_n|^2 e^{2\|n\|\alpha} \Big)^{1/2} + \Big(\sum_{n \in I(k_1)} |(\mathcal{L}_\omega f)_n|^2 e^{2\|n\|\alpha} \Big)^{1/2} \\
&\le \overline{C} \|f\|_\alpha \Big(\Big(\sum_{\|n\| \ge k_1} e^{-2\|n\|(2\gamma-\alpha)} \Big)^{1/2} + \Big(\sum_{n \in I(k_1)} e^{-2\|n\|(2\gamma-\alpha)} \Big)^{1/2} \Big),
\end{aligned}
$$

where the last inequality is obtained as in the proof of (i).

From (7) we see that

$$
\begin{aligned}
&\sup_{f \in K} \|R_k(\mathcal{L}_\omega f)\|_\alpha \\
&\qquad \le \overline{C} \Big(\Big(\sum_{\|n\| \ge k_1} e^{-2\|n\|(2\gamma-\alpha)} \Big)^{1/2} + \Big(\sum_{n \in I(k_1)} e^{-2\|n\|(2\gamma-\alpha)} \Big)^{1/2} \Big) \to 0
\end{aligned}
$$

as $k \to \infty$. Hence applying Theorem A.1 we get the assertion.

(iii) Let us estimate the expression $|\langle \mathcal{L}_\omega e_{n,\alpha}, e_{n,\alpha} \rangle_\alpha|$. After some calculations we see that

$$|\langle \mathcal{L}_\omega e_{k,\alpha}, e_{n,\alpha} \rangle_\alpha| \le 2^d C e^{-(\alpha-\gamma)\|k\|} e^{-(2\gamma-\alpha)\|k\|}.$$

Then

$$\sum_{n,k \in \mathbb{Z}^d} |\langle \mathcal{L}_\omega e_{k,\alpha}, e_{n,\alpha} \rangle_\alpha| < \infty.$$

For any orthonormal basis $(\varphi_i)_{i\in\mathbb{Z}^d}$ of E_α we see that

$$\sum_{i\in\mathbb{Z}^d} |\langle \mathcal{L}_\omega \varphi_i, \varphi_i\rangle_\alpha| \le \sum_{k,l\in\mathbb{Z}^d} |\langle \mathcal{L}_\omega e_{k,\alpha}, e_{l,\alpha}\rangle_\alpha| \Big[\sum_{i\in\mathbb{Z}^d} |\langle \varphi_i, e_{k,\alpha}\rangle_\alpha||\langle e_{l,\alpha}, \varphi_i\rangle_\alpha|\Big]$$
$$\le \sum_{k,l\in\mathbb{Z}^d} |\langle \mathcal{L}_\omega e_{k,\alpha}, e_{l,\alpha}\rangle_\alpha|,$$

which yields that $\mathcal{L}_\omega$ is a trace-class operator in E_α because of the following theorem:

THEOREM (see [15, p. 219]). *Let H be a separable complex Hilbert space and $T \in L(H)$ a bounded operator. Suppose that for any orthonormal base $\{\varphi_i\}_{i\ge 1}$ the series $\sum_{i=1}^{\infty}\langle T\varphi_i, \varphi_i\rangle$ is absolutely convergent. Then T is a trace-class operator.*

This concludes the proof of Lemma 2.3.

The fact that $\mathcal{L}_\omega$ is a trace-class operator allows us to control the error of the spectral radius of $\mathcal{L}_\omega$ in E_α in numerical calculations (see [2]).

LEMMA 2.4. *Let $\omega \in \mathcal{P}$ and suppose that there exist $C, \gamma > 0$ such that for each $n \in \mathbb{Z}^d$, $|\omega_n| \le Ce^{-\gamma\|n\|}$. Then there exist $\gamma_\varepsilon \in (0,\gamma)$ and $C_2 > 0$ such that for the Fourier coefficients $(|\omega|^2)_n$, $n \in \mathbb{Z}^d$ we have*

$$(|\omega|^2)_n \le C_2 e^{-\gamma_\varepsilon\|n\|}.$$

Proof. One can see that

$$(|\omega|^2)(x) = \sum_{n,m\in\mathbb{Z}^d} \omega_n\overline{\omega}_m e^{-i\langle n-m,x\rangle},$$

and hence

$$(|\omega|^2)_k = \frac{1}{(2\pi)^d}\sum_{n,m\in\mathbb{Z}^d}\omega_n\overline{\omega}_m \int_{[-\pi,\pi]^d} e^{-i\langle n-m+k,x\rangle}\,dx$$
$$= \sum_{n,m\in\mathbb{Z}^d,\, m-n=k}\omega_n\overline{\omega}_m = \sum_{n\in\mathbb{Z}^d}\omega_n\overline{\omega}_{n+k}.$$

Then for any ε, γ_1 such that $0 < 2\varepsilon < \gamma$ and $\gamma_1 = \gamma - \varepsilon$,

$$|(|\omega|^2)_k| \le \sum_{n\in\mathbb{Z}^d} |\omega_n||\overline{\omega}_{n+k}| \le C^2\sum_{n\in\mathbb{Z}^d} e^{-\gamma\|n\|}e^{-\gamma\|n+k\|}$$
$$= C^2\sum_{n\in\mathbb{Z}^d} e^{-\gamma_1(\|n\|+\|n+k\|)}e^{-\varepsilon(\|n\|+\|n+k\|)}$$
$$\le C^2\sum_{n\in\mathbb{Z}^d} e^{-\gamma_1\|k\|}e^{-\varepsilon(2\|n\|-\|k\|)} \le C_2 e^{-(\gamma-2\varepsilon)\|k\|},$$

where we used the inequalities $\|k\| \le \|n\| + \|n+k\|$, $\|n\| - \|k\| \le \|n+k\|$.

Iterating Lemma 2.4 l times we obtain:

LEMMA 2.5. *Let $\omega\in\mathcal{P}$ and suppose that $|\omega_n|\le Ce^{-\gamma\|n\|}$ for each $n\in\mathbb{Z}^d$ and some $C,\gamma>0$. Then for any $l\in\mathbb{N}$ there exist $C_{2l}>0$ and $\gamma'\in(0,\gamma)$ such that for each $n\in\mathbb{Z}^d$ the Fourier coefficients $(|\omega|^{2l})_n$ satisfy the estimate*

$$(|\omega|^{2l})_n\le C_{2l}e^{-\gamma'\|n\|}.$$

LEMMA 2.6. *Let ω satisfy the assumptions of Lemma 2.5 and $\omega\neq 0$ on $[0,2\pi]^d$. Then for any $p\in\mathbb{N}$ there exist γ_1 in $(0,\gamma)$ and $C_p>0$ such that the Fourier coefficients of $|\omega|^p$ satisfy the estimate*

$$(|\omega|^p)_n\le C_pe^{-\gamma_1\|n\|},\qquad n\in\mathbb{Z}^d.$$

Proof. $|\omega|^2$ is an analytic function. We can extend $|\omega|^2$ to a function of a complex variable for $|\operatorname{Im} z|<\gamma$. Then there exists $\gamma_1\in(0,\gamma)$ such that $|\omega|^2\neq 0$ on $R_{\gamma_1}=\{z\in\mathbb{C}^d: e^{-\gamma_1}\le|z_k|\le e^{\gamma_1}\text{ for }k=1,\dots,d\}$ and we can define on R_{γ_1} an analytic function

$$|\omega|^p=\exp\left(\frac{p}{2}\log|\omega|^2\right).$$

From the analyticity of $|\omega|^p$ and the form of R_{γ_1} we get the assertion.

To proceed with our considerations we recall the Cohen condition (see [3]).

A set K is called *congruent* to $[-\pi,\pi]^d$ (modulo $2\pi\mathbb{Z}^d$) if $|K|=(2\pi)^d$ and for all $x\in[-\pi,\pi]^d$ there exists $x'\in K$ such that $x-x'\in 2\pi\mathbb{Z}^d$. We say that a function ω satisfies the *Cohen condition* if there exists a compact set K congruent to $[-\pi,\pi]^d$ (modulo $2\pi\mathbb{Z}^d$) such that it contains a neighbourhood of 0 and

$$\inf_{j\ge 1,\,x\in K}|\omega(2^{-j}x)|>0.$$

We finish our preparatory considerations. From now on we assume $d=2$.

LEMMA 2.7. *Let $\omega\in\mathcal{P}$ be real-valued and satisfy the following conditions*:

(i) *there exist $C>0$, $\gamma>0$ such that for each $n\in\mathbb{Z}^2$, $|\omega_n|\le Ce^{-\gamma\|n\|}$*,
(ii) $\omega\ge 0$, $\omega(0)=1$,
(iii) ω *satisfies the Cohen condition*,
(iv) $\omega(s,r)>0$ *when $r\in[0,2\pi], s=0$ or $s=\pi$, and $\omega(2^{-n}\pi,r)>0$ for $r\in[0,\pi/2]$ and $n\in\mathbb{N}$.*

If $f\in E_\alpha\setminus\{0\}$ ($\alpha\in(\gamma,2\gamma)$) is a real-valued function such that $f\ge 0$, then for each $x\in[-\pi,\pi]^2$ there exists $n\in\mathbb{N}$ such that $(\mathcal{L}_\omega^n f)(x)>0$.

Proof. Assume, on the contrary, that there exist a function $0\le f\in E_\alpha$ and $x^0\in\mathbb{R}^2$ such that $(\mathcal{L}_\omega^n f)(x^0)=0$ for any $n\ge 1$. We can assume that $x^0=0$, because if $x^0\neq 0$ then by Lemma 2.2 for any $p\ge 0$ we can write

$$
\begin{aligned}
(8)\qquad 0 &= (\mathcal{L}_\omega^{n+p} f)(x^0)\\
&= \sum_{m\in I_n} \Big[\prod_{j=1}^{n} \omega(2^{-j}(x^0+2m\pi))\Big](\mathcal{L}_\omega^p f)(2^{-n}(x^0+2m\pi)),
\end{aligned}
$$

where $I_n = \{(m_1, m_2) : m_j \in [-2^{n-1}+1, 2^{n-1}]\cap\mathbb{Z},\ j = 1,2\}$.

By the Cohen condition there exist $c > 0$ and a set K congruent to $[-\pi,\pi]^2$ such that $\omega(2^{-j}x) \ge c\chi_K(x)$ for any $x \in \mathbb{R}^2$, $j \ge 1$. By (8),

$$
(9)\qquad 0 = (\mathcal{L}_\omega^{n+p} f)(x^0) \ge c^n \sum_{m\in I_n} \chi_K(x^0+2m\pi)(\mathcal{L}_\omega^p f)(2^{-n}(x^0+2m\pi)).
$$

There exist $m^0 \in \mathbb{Z}^2$ and $\overline{x} \in K$ such that $x^0 + 2\pi m^0 = \overline{x}$. Now if $2^{n-1} > |m_i^0|$, $i = 1,2$, then by (9),

$$
0 = (\mathcal{L}_\omega^{n+p} f)(x^0) \ge c^n (\mathcal{L}_\omega^p f)(2^{-n}\overline{x}).
$$

Hence by analyticity $\mathcal{L}_\omega^p f$ vanishes on the line $\{y = t\overline{x}\} \subseteq \mathbb{R}^2$, $p \ge 0$.

The next steps of the proof are as follows. First we show that

$$
(10)\qquad (\mathcal{L}_\omega^p f)\left(\pi, \frac{l}{2^j}\pi\right) = 0
$$

for any $p \ge 0$ and $l \in \{0, 1, \ldots, 2^j\}$, $j \ge 1$. Then we deduce that

$$
(11)\qquad f\left(r\left(\pi, \frac{l}{2^j}\pi\right)\right) = 0
$$

for each $r \in \mathbb{R}$. Hence we conclude that $f \equiv 0$ by analyticity.

To prove (10) let us take into account (8). For $x^0 = 0$ and $n \ge j-1$ we derive

$$
(12)\qquad 0 = (\mathcal{L}_\omega^{n+p} f)(0) \ge \Big[\prod_{k=1}^{n} \omega(2^{-k}2m\pi)\Big](\mathcal{L}_\omega^p f)\left(\pi, \frac{l}{2^j}\pi\right) \ge 0,
$$

where $m = (2^{n-1}, 2^{n-j-1}l)$, $l \in \{0, 1, \ldots, 2^j\}$. If $k \in \{1, \ldots, n-1\}$ then

$$
\omega(2^{-k}2\pi(2^{n-1}, 2^{n-j-1}l)) = \omega(2\pi, 2^{n-k-j}l\pi) = \omega(0, 2^{n-k-j}l\pi) > 0,
$$

and for $k = n$,

$$
\omega(2^{-k}2\pi(2^{n-1}, 2^{n-j-1}l)) = \omega\left(\pi, \frac{l}{2^j}\pi\right) > 0.
$$

Hence by (12) we obtain (10).

To prove (11) it is necessary to show that the function f vanishes on the line $r\big(\pi, \frac{l}{2^j}\pi\big)$, $r \in \mathbb{R}$, or equivalently that it vanishes at infinitely many points having a point of accumulation. Once more rewrite (8) for $p = 0$ and

$x = \left(\pi, \frac{l}{2^j}\pi\right)$:

$$\begin{aligned}0 &= (\mathcal{L}_\omega^n f)\left(\pi, \frac{l}{2^j}\pi\right)\\ &= \sum_{m\in I_n}\left[\prod_{p=1}^{n}\omega\left(2^{-p}\left(\left(\pi, \frac{l}{2^j}\pi\right) + 2m\pi\right)\right)\right]\\ &\quad\times f\left(2^{-n}\left(\left(\pi, \frac{l}{2^j}\pi\right) + 2m\pi\right)\right).\end{aligned}$$

Then inserting $m = 0$ we observe that

$$0 = (\mathcal{L}_\omega^n f)\left(\pi, \frac{l}{2^j}\pi\right) \ge \prod_{p=1}^{n}\omega\left(2^{-p}\left(\pi, \frac{l}{2^j}\pi\right)\right)f\left(2^{-n}\left(\pi, \frac{l}{2^j}\pi\right)\right) \ge 0, \tag{13}$$

where the last inequality follows from (iv). Then by (13),

$$f\left(2^{-n}\left(\pi, \frac{l}{2^j}\pi\right)\right) = 0$$

for any n and hence we obtain (11). The set $\left\{\frac{l}{2^j}\pi : j \ge 1,\ l = 0, 1, \ldots, 2^j\right\}$ is dense in $[0, \pi]$ hence f vanishes on the triangle with vertices $(0, 0)$, $(\pi, 0)$, (π, π). So $f \equiv 0$ and we obtain a contradiction.

From the proof it is clear that (iv) can be replaced by another condition given in the following:

REMARK 2.4. $\omega(s, r) > 0$ whenever $s \in [0, 2\pi]$, $r = 0$ or $r = \pi$, and $\omega(r, 2^{-n}\pi) > 0$ for $r \in [0, \pi/2]$ and $n \in \mathbb{N}$.

Let us remark that the second part of (iv) (i.e. $\omega(2^{-n}\pi, r) > 0$ for $r \in [0, \pi/2]$, $n \in \mathbb{N}$) concerns only a finite number of $n \in \mathbb{N}$, $n \in \{1, \ldots, k_0\}$, where $k_0 \ge 1$ is such that the square $[0, 2^{-k_0}\pi]^2 \subseteq 2^{-1}K$, K being the compact set from the Cohen condition. We recall that for $x \in 2^{-1}K$, $w(x) > 0$.

It seems that the assumption (iv) in Lemma 2.7 is excessively strong, and it is an open problem how to relax it.

In the case $d = 1$ assumptions (i)–(iii) suffice for proving the assertion of Lemma 2.7 (see [2]).

3. Regularity of the refinable function. An operator $T \in L(X)$, where X is a Banach space, is called *positive* with respect to the cone $K \subset X$ if $T(K) \subset K$. If $\operatorname{Int} K \ne \emptyset$ we say that T is *strictly positive* when $T(K \setminus \{0\}) \subseteq \operatorname{Int} K$. We use $r(T)$ for the spectral radius of T and $B(x, r)$ for the ball with center at x and radius r.

Define

$$E_{\alpha,\mathbb{R}} = \{f \in E_\alpha : f(x) \in \mathbb{R} \text{ for all } x \in \mathbb{R}^d\}.$$

Then

$$E_\alpha = E_{\alpha,\mathbb{R}} + iE_{\alpha,\mathbb{R}}.$$

For $E_{\alpha,\mathbb{R}}$ and E_α the sets

$$E^+_{\alpha,\mathbb{R}} = \{f \in E_{\alpha,\mathbb{R}} : f \geq 0\} \quad \text{and} \quad E^+_\alpha = E^+_{\alpha,\mathbb{R}} + iE^+_{\alpha,\mathbb{R}}$$

are cones.

LEMMA 3.1. *Let $f \in E_{\alpha,\mathbb{R}}$. Suppose that $f > 0$. Then*

(i) *$B(f, a_f/(2C_\alpha)) \subset E^+_{\alpha,\mathbb{R}}$, where $\min\{f(x) : x \in [-\pi,\pi]^d\} > a_f > 0$ and C_α is as in Remark 2.2.*

(ii) *For each $g \in E_{\alpha,\mathbb{R}}$ we have $g > 0$ whenever $g \in B(f, a_f/(2C_\alpha))$.*

Proof. Let $a_f > 0$ be such that $f > a_f$ and assume $g \in B(f, a_f/(2C_\alpha))$. Then

$$g(x) \geq f(x) - |f(x) - g(x)| \geq a_f - \|f - g\|_{L^\infty} \geq a_f - C_\alpha \|f - g\|_\alpha > 0.$$

As a direct consequence of this lemma we get the following

REMARK 3.1. *$E^+_{\alpha,\mathbb{R}}$ and E^+_α are cones with nonempty interior.*

Let f be an integrable and normalized solution of the equation (1), i.e. $\int_{\mathbb{R}^d} f(x)\,dx = 1$. Applying the Fourier transform to (1) one obtains

$$\widehat{f}(x) = m(2^{-1}x)\widehat{f}(2^{-1}x), \tag{14}$$

where $m(x) = \sum_{n\in\mathbb{Z}^2} c_n e^{i\langle n,x\rangle}$.

From now on we assume that the function m can be factored as

$$m(x) = \left(\frac{1+e^{ix_1}}{2}\right)^N \left(\frac{1+e^{ix_2}}{2}\right)^M q(x), \tag{15}$$

where $N, M \in \mathbb{N} \cup \{0\}$ and q is a $2\pi\mathbb{Z}^2$-periodic function such that the Fourier coefficients q_n satisfy the estimate

$$|q_n| \leq Ce^{-\gamma\|n\|} \tag{16}$$

for some $C, \gamma > 0$.

We can rewrite (15) as

$$m(x) = q(x) \sum_{k\in I} 2^{-(N+M)} \binom{N}{k_1} \binom{M}{k_2} e^{i\langle k,x\rangle},$$

where $I = \{k \in \mathbb{Z}^2 : k_1 = 0, 1, \ldots, N,\ k_2 = 0, 1, \ldots, M\}$. Then the Fourier coefficients of m can be estimated as follows:

$$c_n = m_n = \sum_{k\in I} 2^{-(N+M)} \binom{N}{k_1} \binom{M}{k_2} q_{n+k},$$

and applying (16) we obtain

$$|m_n| \le 2^{-(N+M)} C e^{-\gamma\|n\|} \sum_{k\in I} \binom{N}{k_1}\binom{M}{k_2} e^{\gamma\|k\|},$$

hence

$$|m_n| \le \overline{C} e^{-\gamma\|n\|} \quad \text{for any } n \in \mathbb{Z}^2. \tag{17}$$

One sees that $\prod_{j=1}^{\infty} m(2^{-j}x)$ and $\prod_{j=1}^{\infty} q(2^{-j}x)$ are uniformly convergent on each compact subset K of $\mathbb{R}^2$, since using (2) and (17) it follows that

$$\begin{aligned} |q(x)| = |m(x)| &\le 1 + |m(x) - 1| = 1 + \Big|\sum_{n\in\mathbb{Z}^2} c_n e^{i\langle n,x\rangle} - \sum_{n\in\mathbb{Z}^2} c_n\Big| \\ &\le 1 + 2\sum_{n\in\mathbb{Z}^2} |c_n| \Big|\sin\frac{1}{2}\langle n,x\rangle\Big| \le 1 + 2\sum_{n\in\mathbb{Z}^2} \overline{C} e^{-2\gamma\|n\|} \Big|\frac{1}{2}\langle n,x\rangle\Big| \\ &\le 1 + \overline{C}\sum_{n\in\mathbb{Z}^2} e^{-2\gamma\|n\|}\|n\|\cdot\|x\| \le 1 + C_1\|x\|. \end{aligned} \tag{18}$$

LEMMA 3.2. *Assume that* m, q *satisfy* (15), (16) *and one of the following conditions*:

1. $p > 0$ *and* $q \ne 0$ *on* $[-\pi,\pi]^2$,
2. $p \in 2\mathbb{N}$, m *satisfy the Cohen condition and* $|q|$ *satisfies the condition* (iv) *of Lemma* 2.7.

Let $\mathcal{L}_{|q|^p}$ *be the transfer operator associated with the function* $|q|^p$ *and* r_p *be the spectral radius of this operator on* E_α *for any* $\alpha \in (\gamma, 2\gamma)$. *Then*:

(i) r_p *is an eigenvalue of* $\mathcal{L}_{|q|^p}$,
(ii) *the eigenfunction corresponding to* r_p *is strictly positive* (*i.e. is in* E_α^+),
(iii) $r_p > 1$.

P r o o f. For $\lambda > \|\mathcal{L}_{|q|^p}\|$ consider the operator

$$T = \sum_{k=1}^{\infty} \lambda^{-k} \mathcal{L}_{|q|^p}^k \quad \text{acting on } E_\alpha. \tag{19}$$

T is compact and by Lemma 2.7 it is strongly positive. Then by Theorem A.3 its spectral radius $r(T) > 0$ is an eigenvalue of T. Moreover, the corresponding eigenfunction F is in $\operatorname{Int} E_\alpha^+$. Recall that for $\lambda > \|\mathcal{L}_{|q|^p}\|$ the resolvent $R(\lambda, \mathcal{L}_{|q|^p})$ equals $\sum_{k=0}^{\infty} \lambda^{-(k+1)} \mathcal{L}_{|q|^p}^k$. So we can write

$$I + T = \lambda R(\lambda, \mathcal{L}_{|q|^p}). \tag{20}$$

Because also F is an eigenfunction of $I + T$ corresponding to the eigenvalue $1 + r(T)$, from (20) we derive

$$\lambda R(\lambda, \mathcal{L}_{|q|^p}) F = (1 + r(T)) F.$$

This immediately gives $\lambda F = (1+r(T))(\lambda I - \mathcal{L}_{|q|^p})F$ and therefore $\mathcal{L}_{|q|^p}F = \kappa F$, $\kappa \equiv \frac{\lambda}{1+r(T)}r(T) > 0$. So $r_p \ge \kappa > 0$ where r_p is the spectral radius of $\mathcal{L}_{|q|^p}$. Now the Krein–Rutman Theorem (see Theorem A.2) applied to $\mathcal{L}_{|q|^p}$ shows that r_p is an eigenvalue of $\mathcal{L}_{|q|^p}$ and the corresponding eigenfunction G is in E_α^+. By (19), G is also an eigenfunction for T and

$$TG = \bigg(\sum_{k=1}^{\infty}\bigg(\frac{r_p}{\lambda}\bigg)^k\bigg)G \in \operatorname{Int} E_\alpha^+.$$

Hence we obtain (i), (ii).

Now write

$$r_pF(0) = (\mathcal{L}_{|q|^p}F)(0) = F(0) + \sum_{e\in\Lambda'}|q(\pi e)|^pF(\pi e), \qquad \Lambda' = \Lambda\setminus\{(0,0)\}.$$

The assumption imposed on q guarantees that $|q(0,\pi)| > 0$. Hence the sum on the right hand side of the latter formula is positive. Thus $r_p > 1$ and the proof is finished.

Let

$$E'_\alpha = \{g : g(x) = |\sin(2^{-1}x_1)|^{Np}|\sin(2^{-1}x_2)|^{Mp}f(x) \text{ and } f\in E_\alpha\},$$

and for any $g\in E'_\alpha$ the norm of g is identified with the norm of the corresponding f in E_α.

LEMMA 3.3. *Let $\mathcal{L}_{|q|^p}$ (resp. $\mathcal{L}'_{|m|^p}$) be the transfer operator associated with $|q|^p$ (resp. $|m|^p$). For any $\alpha\in(\gamma,2\gamma)$, $\mathcal{L}'_{|m|^p}$ is a trace-class operator on the space E'_α. Moreover, if f is a continuous eigenfunction of $\mathcal{L}_{|q|^p}$ with eigenvalue λ then $g(x) = |\sin(2^{-1}x_1)|^{Np}|\sin(2^{-1}x_2)|^{Mp}f(x)$ is a continuous eigenfunction of $\mathcal{L}'_{|m|^p}$ with eigenvalue $2^{-(N+M)p}\lambda$.*

Proof. As in the one-dimensional case (see [2]), it is enough to show

$$\begin{aligned}(\mathcal{L}'_{|m|^p}g)(2x) &= \sum_{e\in\Lambda}|m(x+\pi e)|^pg(x+\pi e)\\ &= \bigg|\sin\bigg(\frac{x_1}{2}\bigg)\cos\bigg(\frac{x_1}{2}\bigg)\bigg|^{Np}\bigg|\sin\bigg(\frac{x_2}{2}\bigg)\cos\bigg(\frac{x_2}{2}\bigg)\bigg|^{Mp}\\ &\quad\times\sum_{e\in\Lambda}|q(x+\pi e)|^pf(x+\pi e)\\ &= 2^{-(N+M)p}|\sin x_1|^{Np}|\sin x_2|^{Mp}(\mathcal{L}_{|q|^p}f)(2x).\end{aligned}$$

THEOREM 1. *Assume that m, q satisfy* (15), (16) *and one of the conditions of Lemma* 3.2. *Let $\mathcal{L}_{|q|^p}$ be the transfer operator associated with the function $|q|^p$ and r_p be the spectral radius of this operator on E_α for any*

$\alpha \in (\gamma, 2\gamma)$. *Then the L^p-Sobolev exponent of the scaling function f satisfies*

$$s_p = N + M - \frac{1}{p}\log_2 r_p. \tag{21}$$

Proof. Applying (14) and (15) we see that

$$\begin{aligned} |\widehat{f}(x)| &= \Big[\prod_{k=1}^{\infty} |\cos^N(2^{-k-1}x_1)|\Big]\Big[\prod_{k=1}^{\infty} |\cos^M(2^{-k-1}x_2)|\Big]\prod_{k=1}^{\infty}|q(2^{-k}x)| \\ &= \left|\frac{2\sin(2^{-1}x_1)}{x_1}\right|^N \left|\frac{2\sin(2^{-1}x_2)}{x_2}\right|^M \prod_{k=1}^{\infty}|q(2^{-k}x)|. \end{aligned} \tag{22}$$

For all $x \in [-2^n\pi, 2^n\pi]^2$ we obtain

$$\Big|\prod_{k=1}^{\infty} q(2^{-k}x)\Big|^p \le C_p \prod_{k=1}^{n}|q(2^{-k}x)|^p, \tag{23}$$

where $C_p = \sup\{|\prod_{k=1}^{\infty} q(2^{-k}x)|^p : x \in [-\pi,\pi]^2\}$ and C_p is finite by (18). Using (23) we obtain

$$\begin{aligned} \int\limits_{[-2^n\pi,2^n\pi]^2} \Big|\prod_{k=1}^{\infty} q(2^{-k}x)\Big|^p\,dx &\le C_p \int\limits_{[-2^n\pi,2^n\pi]^2} \prod_{k=1}^{n}|q(2^{-k}x)|^p\,dx \\ &\le C_p \int\limits_{[-\pi,\pi]^2} (\mathcal{L}_{|q|^p})^n 1(x)\,dx \qquad \text{by Lemma 2.1} \\ &\le (2\pi)^2 C_p \langle (\mathcal{L}_{|q|^p})^n 1, 1\rangle_\alpha \le (2\pi)^2 C_p \|\mathcal{L}^n_{|q|^p}\|. \end{aligned} \tag{24}$$

For each $\varepsilon > 0$ and $n \ge n_0(\varepsilon) \ge 1$ we have

$$|\|\mathcal{L}^n_{|q|^p}\|^{1/n} - r_p| < \varepsilon.$$

Hence applying (24) we see that

$$\int\limits_{[-2^n\pi,2^n\pi]^2} \Big|\prod_{k=1}^{\infty} q(2^{-k}x)\Big|^p\,dx \le \begin{cases} (2\pi)^2 C_p (r_p+\varepsilon)^n & \text{for } n \ge n_0(\varepsilon) \ge 1, \\ (2\pi)^2 C_p \|\mathcal{L}_{|q|^p}\|^n & \text{for } 1 \le n < n_0(\varepsilon). \end{cases} \tag{25}$$

Consider the family of sets $A_0 = [-\pi,\pi]^2$, $A_j = [-2^j\pi, 2^j\pi]^2 \setminus [-2^{j-1}\pi, 2^{j-1}\pi]^2$ for $j \ge 1$. Then using (22) and (25) we estimate

$$\begin{aligned} &\int\limits_{\mathbb{R}^2} |\widehat{f}(x)|^p (1+\|x\|^p)^s\,dx \\ &\quad= \int\limits_{[-\pi,\pi]^2} |\widehat{f}(x)|^p (1+\|x\|^p)^s\,dx + \sum_{j=1}^{\infty} \int\limits_{x\in A_j} |\widehat{f}(x)|^p (1+\|x\|^p)^s\,dx \end{aligned}$$

$$\le C_1 + C\sum_{j=1}^{\infty} 2^{jp(s-N-M)} C_p \int_{x\in A_j} \prod_{k=1}^{j} |q(2^{-k}x)|^p\,dx \qquad \text{by (22), (23)}$$

$$\le C_1 + C_2\Big(\sum_{j=1}^{n_0-1} 2^{jp(s-N-M)} \|\mathcal{L}_{|q|^p}\|^j + \sum_{j=n_0}^{\infty} 2^{jp(s-N-M)}(r_p+\varepsilon)^j\Big)$$

$$\le C_3 + C_2\sum_{j=n_0}^{\infty} 2^{j(p(s-N-M)+\log_2(r_p+\varepsilon))}.$$

Then for any s such that $j(p(s-N-M)+\log_2 r_p) < 0$ the series is convergent and hence the L^p-Sobolev exponent s_p is greater than or equal to $N+M-\frac{1}{p}\log_2 r_p$.

Let $K \subseteq \mathbb{R}^2$ be a compact set congruent to $[-\pi,\pi]^2$ modulo $2\pi\mathbb{Z}^2$ from the Cohen condition. Define

$$I_n = \int_{x\in 2^n K} \|x\|^{(N+M)p} |\widehat{f}(x)|^p\,dx$$

and

$$\varrho = \inf\Big\{\Big|\prod_{k=1}^{\infty} m(2^{-k}x)\Big|^p : x\in K\Big\}.$$

Then $\varrho > 0$ by the Cohen condition.

Let F be a strictly positive eigenfunction of $\mathcal{L}_{|q|^p}$ (see Lemma 3.2) corresponding to r_p. Define

$$S = \sup\{|F(x)| : x\in[-\pi,\pi]^2\},$$

$$g(x) = |\sin(2^{-1}x_1)|^{Np}|\sin(2^{-1}x_2)|^{Mp}F(x), \qquad G = \int_{[-\pi,\pi]^2} g(x)\,dx.$$

We can estimate I_n as follows:

$$I_n = \int_{x\in 2^n K} \|x\|^{(N+M)p}\Big[\prod_{k=1}^{n}|m(2^{-k}x)|\Big]^p\Big[\prod_{k=1}^{\infty}|m(2^{-(k+n)}x)|\Big]^p\,dx$$

$$\ge \varrho\int_{x\in 2^n K} |x_1|^{Np}|x_2|^{Mp}\Big[\prod_{k=1}^{n}|m(2^{-k}x)|\Big]^p\,dx$$

$$\ge \varrho 2^{(N+M)p(n+1)}\int_{x\in 2^n K} |\sin(2^{-(n+1)}x_1)|^{Np}|\sin(2^{-(n+1)}x_2)|^{Mp} \times\Big[\prod_{k=1}^{n}|m(2^{-k}x)|\Big]^p\,dx$$

$$\ge S^{-1}\varrho 2^{(N+M)p(n+1)}\Big|\int_{x\in 2^n K} g(2^{-n}x)\Big[\prod_{k=1}^{n}|m(2^{-k}x)|\Big]^p\,dx\Big|$$

$$\geq S^{-1}\varrho 2^{(N+M)p(n+1)}\Big|\int_{x\in[-2^n\pi,2^n\pi]^2} g(2^{-n}x)\Big[\prod_{k=1}^{n}|m(2^{-k}x)|\Big]^p\,dx\Big|$$
$$= S^{-1}\varrho 2^{(N+M)p(n+1)}\Big|\int_{x\in[-\pi,\pi]^2}(\mathcal{L}'_{|m|^p})^n g(x)\,dx\Big| \quad \text{by Lemma 2.1}$$
$$= |G|\varrho S^{-1}2^{(N+M)p}(r_p)^n \quad \text{by Lemma 3.3}$$
$$= C(r_p)^n.$$

Since K is compact there exists a finite L such that $K\subseteq[-2^L\pi,2^L\pi]^2$. Hence

$$\bar{I}_n = \int_{[-2^n\pi,2^n\pi]^2}\|x\|^{(N+M)p}|\widehat{f}(x)|^p\,dx \geq I_{n-L}\geq \overline{C}(r_p)^n. \tag{26}$$

Put

$$J_n = \bar{I}_n - \bar{I}_{n-1} = \int_{A_n}\|x\|^{(N+M)p}|\widehat{f}(x)|^p\,dx.$$

Now we prove that $r_p>0$ and (26) gives

(27) for each $C>0$ and $\varepsilon>0$ we have $J_n\geq C(r_p/2^{\varepsilon})^n$ for infinitely many $n\geq 1$.

In fact, suppose not. Then there exist $n_0\geq 1, C_0>0$, and $\varepsilon_0>0$ such that $J_n<C_0(r_p/2^{\varepsilon_0})^n$ for each $n\geq n_0$. For $n>n_0$ this yields

$$0<\overline{C}\leq (r_p)^{-n}\bar{I}_n = (r_p)^{-n}\Big(\bar{I}_{n_0}+\sum_{k=n_0+1}^{n}J_k\Big) < (r_p)^{-n}\bar{I}_{n_0} + C_0(r_p)^{-n}\sum_{k=n_0+1}^{n}\Big(\frac{r_p}{2^{\varepsilon_0}}\Big)^n. \tag{28}$$

It is clear that for $r_p/2^{\varepsilon_0}\leq 1$ the right hand side tends to zero as n tends to infinity. Now we show that the same holds for $r_p/2^{\varepsilon_0}>1$. Actually, in this case we have

$$(r_p)^{-n}\sum_{k=n_0+1}^{n}\Big(\frac{r_p}{2^{\varepsilon_0}}\Big)^n \leq (r_p)^{-n}\int_{n_0}^{n}\Big(\frac{r_p}{2^{\varepsilon_0}}\Big)^x dx = \frac{(r_p)^{-n}}{\ln\frac{r_p}{2^{\varepsilon_0}}}\Big[\Big(\frac{r_p}{2^{\varepsilon_0}}\Big)^n - \Big(\frac{r_p}{2^{\varepsilon_0}}\Big)^{n_0}\Big],$$

which gives the claim.

We thus get a contradiction, and therefore (27) is valid.

Let us write (27) in the form

$$\int_{A_n}\|x\|^{(N+M)p-\log_2 r_p+\varepsilon}|\widehat{f}(x)|^p\,dx\geq C_1>0 \tag{29}$$

for infinitely many $n\geq 1$. Now for

$$\int_{\mathbb{R}^2} (1+\|x\|^p)^s |\widehat{f}(x)|^p\,dx \ge \sum_{n=0}^{\infty} \int_{A_n} \|x\|^{ps} |\widehat{f}(x)|^p\,dx,$$

using (29) we see that when $s > N = M - \frac{1}{p}\log_2 r_p + \frac{\varepsilon}{p}$, the integral $\int_{\mathbb{R}^2}(1+\|x\|^p)^s|\widehat{f}(x)|^p\,dx$ is divergent. Since $\varepsilon > 0$ can be chosen arbitrarily small we infer $s_p \le N+M-\frac{1}{p}\log_2 r_p$. This concludes the proof of Theorem 1.

From the first part of the proof we get

Remark 3.2. If we impose on m, q only (15), (16), and the spectral radius r_p of $\mathcal{L}_{|q|^p}$ is greater than zero then

$$s_p \ge N + M - \frac{1}{p}\log_2 r_p \qquad \text{for } p \in 2\mathbb{N}.$$

4. Appendix. Let us recall three theorems which were used in the article:

Theorem A.1 (Proposition 7.4 of [6]). *Let X be a Banach space with a basis. Then $B \subseteq X$ is relatively compact if and only if B is bounded and* $\sup\{|R_n x| : x \in B\} \to 0$ *as $n \to \infty$, where $R_n : X \to \operatorname{span}\{\varepsilon_{n+1}, \ldots\}$ are projections and $(\varepsilon_i)_{i=1}^{\infty}$ is a basis of X.*

Theorem A.2 (Theorem 19.2 of [6]). *Let X be a Banach space, $K \subset X$ a total cone, and $T \in L(X)$ compact positive with $r(T) > 0$. Then $r(T)$ is an eigenvalue of T with positive eigenvector.*

Theorem A.3 (Theorem 19.3 of [6]). *Let X be a Banach space, $K \subset X$ a cone with* $\operatorname{Int} K \ne \emptyset$, *and $T \in L(X)$ compact and strongly positive* (*i.e.* $T(K \setminus \{0\}) \subseteq \operatorname{Int} K$). *Then*:

(a) $r(T) > 0$, *$r(T)$ is a simple eigenvalue with an eigenvector $v \in \operatorname{Int} K$ and there is no other eigenvalue with a positive eigenvector.*

(b) $|\lambda| < r(T)$ *for all eigenvalues $\lambda \ne r(T)$.*

(c) *For $y > 0$, the equation $\lambda x - Tx = y$ has a unique solution $x \in \operatorname{Int} K$ if $\lambda > r(T)$ and no solution in K if $\lambda \le r(T)$. The equation $r(T)x - Tx = -y$ also has no solution in K.*

(d) *If $S \in L(X)$ and $Sx \ge Tx$ on K then $r(S) \ge r(T)$, while $r(S) > r(T)$ if $Sx - Tx \in \operatorname{Int} K$ for $x > 0$.*

The next theorem is a generalization of a well-known theorem for functions of one variable (see [1]):

Theorem A.4. *Let $f \in \mathcal{P}$, and suppose that $f(x) = \sum_{n\in\mathbb{Z}^d} f_n e^{-i\langle n,x\rangle}$ for each $x \in \mathbb{R}^d$. Then the following conditions are equivalent*:

(i) *for some $C, \gamma > 0$ and each $n \in \mathbb{Z}^d$ we have $|f_n| \le Ce^{-\gamma\|n\|}$,*

(ii) *f is an analytic function.*

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