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A GAME-THEORETIC MODEL OF SOCIAL ADAPTATION IN AN INFINITE POPULATION

Abstract. The paper deals with the question of existence and properties of equilibrated distributions of individual characteristics in an infinite population. General game-theoretic methods are applied and special attention is focused on the case of fitness functions depending only on the distance of an individual characteristic from a reference point and from the mean characteristics. Iterative procedures leading to equilibrated distributions are also considered.

1. Formulation of the problem. In the present paper we consider an infinite population whose members are classified into n types (animal species, human races, professions, genders, age ranges, etc.). The situation of each individual is characterized by a vector of reals; all possible characteristics of the individuals of type i constitute a nonempty bounded set $W^i \subset \mathbb{R}^{k_i}$. A *distribution* of characteristics of type i is a probability measure $\mathbf{m}^i$ on the Borel subsets of W^i. The *state* of the population is completely determined by a vector $\mathbf{m} = (\mathbf{m}^1, \ldots, \mathbf{m}^n)$ of distributions of characteristics of respective types.

The *mean* of a distribution $\mathbf{m}^i$ is the k_i-dimensional vector

$$m^i = (m^i_1, m^i_2, \ldots, m^i_{k_i}) = \Big(\int_{W^i} x_1 \, d\mathbf{m}^i, \int_{W^i} x_2 \, d\mathbf{m}^i, \ldots, \int_{W^i} x_{k_i} \, d\mathbf{m}^i \Big);$$

the mean is well defined, since all the sets W^i are bounded.

We assume that the *fitness* of every individual of type i is measured by a real function

$$\Psi^i(x; m^1, \ldots, m^n)$$

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depending on the individual characteristic $x \in W^i$ and the mean characteristics $m^1, \ldots, m^n$ of the individuals of respective types $1, \ldots, n$. So, formally, the function Ψ^i is defined over the product $W^i \times \operatorname{co} W^1 \times \ldots \times \operatorname{co} W^n$ ($\operatorname{co} A$ stands for the *convex hull* of A).

The individuals of each type may be choosing their characteristics in a conscious way or else the characteristics may be assigned to them as a result of a genetic process of inheritance (this subject will be considered in Section 2). No matter what is the actual process of the choice of individual characteristics, the meaning of an equilibrated distribution is the same: a vector of distributions $\mathbf{m} = (\mathbf{m}^1, \ldots, \mathbf{m}^n)$ is said to be *equilibrated* whenever, for each type i, the measure $\mathbf{m}^i$ is concentrated on the *set of best replies for the type* i:

$$B^i(m) = \operatorname{Argmax}_{x \in W^i} \Psi^i(x; m^1, \ldots, m^n)$$

(we rather write $B^i(m)$ than $B^i(\mathbf{m})$, since this set depends on $\mathbf{m}$ only via m, the vector of the means of $\mathbf{m}$).

The meaning of this definitions is as follows: a system of distributions is equilibrated whenever, for each type i, the measure of the set of individuals of this type whose situation could be improved upon by an individual deviation from the prevailing characteristic x is zero.

We can state it another way: $\mathbf{m}$ is equilibrated whenever it is concentrated on the set of *population's best replies* to $\mathbf{m}$ itself,

$$B(m) = B^1(m) \times \ldots \times B^n(m).$$

Systems $(W^1, \ldots, W^n; \Psi^1, \ldots, \Psi^n)$ consisting of the sets of available characteristics of respective types and their fitness functions have been considered by Wieczorek in [14], allowing also for more general interpretation. They were called *simple large games*. The cited paper contains the following general existence result:

THEOREM 1. *If all the sets $W^1, \ldots, W^n$ are compact and all the functions $\Psi^1, \ldots, \Psi^n$ are continuous then there exists an equilibrated system of distributions of characteristics. If, moreover, for each m in $\operatorname{co} W^1 \times \ldots \times \operatorname{co} W^n$ and each i in a set of indices $N_0 \subseteq \{1, \ldots, n\}$, the set $B^i(m)$ is acyclic* (e.g. homeomorphic to a nonempty compact convex set) *then there exists an equilibrated system of distributions $\mathbf{m}$ such that for every $i \in N_0$, $\mathbf{m}^i$ is concentrated at a point.* ∎

The present paper includes some general considerations concerning the adaptation systems described above; we present some basic notions, answer most natural questions and consider in more detail some typical examples. Also iterative procedures leading to equilibrated distributions are considered. The paper is intended as an introduction to the subject and a starting point for further investigations.

2. Iterative computational procedures. Iterative procedures are both the means to reach equilibrated distributions and the way to show how a population behaves and how it changes. We may consider a distribution prevailing now as a result of an iterative process in which characteristics are partly inherited and partly chosen by individuals as best replies to the distribution prevailing before the choice. We are interested in the questions of achieving an equilibrated distribution by an iterative procedure.

At first we shall concentrate on the case of a uniform population. We define an iterative process as follows:

A number $0 \le c \le 1$ is used in the construction of consecutive iterations; if each iteration is interpreted as describing the behavior of a new generation of players, c might be understood as a *coefficient of genetic inheritance*. Let $\mathbf{m}_0$ be an initial distribution of characteristics. The distribution prevailing at time t is defined by

$$\mathbf{m}_t = c \cdot \mathbf{m}_{t-1} + (1-c) \cdot b(m_{t-1}), \qquad t = 1, 2, \ldots$$

Here b is a function associating with every distribution $\mathbf{m}$ (or rather its mean m) a distribution $b(m)$ concentrated on the set of best replies to $\mathbf{m}$, $B(m)$. We may refer to b as a *rule of choosing the reply*.

The constant c may change in time. Then we have a sequence of constants $0 \le c_t \le 1$ and the modified process is defined by

$$\mathbf{m}_t = c_t \cdot \mathbf{m}_{t-1} + (1-c_t) \cdot b(m_{t-1}), \qquad t = 1, 2, \ldots \tag{1}$$

We assume that $c_t \to c < 1$ as $t \to \infty$. Then the following theorem holds:

THEOREM 2. *Let W be compact and let Ψ be continuous in x and satisfy the "equi-upper semicontinuity" condition*:

(2) *for every $m \in \operatorname{co} W$ and every positive ε there exists a positive δ such that for all $m' \in \operatorname{co} W$,*

$$\|m - m'\| < \delta \quad \textit{implies} \quad \sup_{x \in W} [\Psi(x, m') - \Psi(x, m)] < \varepsilon.$$

If the weak limit $\lim_{t\to\infty} \mathbf{m}_t = \mathbf{m}$ exists then $\mathbf{m}$ is an equilibrated distribution.*

P r o o f. We define $T : \operatorname{co} W \times \operatorname{Prob}(W) \to \mathbb{R}$ by

$$T(m, \mu) := \int_W \Psi(x, m)\, \mu(dx).$$

To prove that T is upper semicontinuous (w.r.t. the weak* topology in the set $\operatorname{Prob}(W)$ of all probability measures on Borel subsets of W and with ordinary topologies in $\operatorname{co} W$ and $\mathbb{R}$), take a sequence (m_n) of elements of $\operatorname{co} W$ converging to some m and a sequence (μ_n) of probability measures on W weakly* converging to some measure μ. Choose a positive ε; we need to show that, for sufficiently large n,

$$\int_W \Psi(x, m_n)\, \mu_n(dx) - \int_W \Psi(x, m)\, d\mu(x) < \varepsilon. \tag{3}$$

The condition (2) implies that, for sufficiently large n, $\Psi(x, m_n) - \Psi(x, m) < \varepsilon/2$ holds for all $x \in W$, hence

$$\int_W \Psi(x, m_n)\, \mu(dx) - \int_W \Psi(x, m)\, \mu(dx) < \varepsilon/2.$$

By the definition of weak* convergence we also have, because of continuity of Ψ in x,

$$\int_W \Psi(x, m)\, \mu_n(dx) - \int_W \Psi(x, m)\, \mu(dx) < \varepsilon/2$$

for sufficiently large n. The last two inequalities imply (3).

We now apply to T the Maximum Principle of Berge [2], p. 122, to find that the correspondence $B^* : \operatorname{co} W \rightsquigarrow \operatorname{Prob}(W)$ defined by

$$B^*(m) = \operatorname{Argmax}\Big\{ \int_W \Psi(x, m)\mu(dx) \,\Big|\, \mu \in \operatorname{Prob}(W) \Big\}$$

is upper semicontinuous, which, in our case, is equivalent to the closedness of the graph of B^*, as the space of all probability measures on a compact space is compact in the weak* topology.

From (1) we compute

$$b(m_{t-1}) = \frac{\mathbf{m}_t - c_t \cdot \mathbf{m}_{t-1}}{1 - c_t}.$$

This expression makes sense for t large enough and the right hand side converges weakly*:

$$\lim_{t\to\infty} b(m_{t-1}) = \frac{\mathbf{m} - c \cdot \mathbf{m}}{1 - c} = \mathbf{m}.$$

Since $b(m_{t-1}) \in B^*(m_{t-1})$, we have $(m_{t-1}, b(m_{t-1})) \in \operatorname{Gr}(B^*)$.

Because $(m_{t-1}, b(m_{t-1})) \to (m, \mathbf{m})$ and $\operatorname{Gr}(B^*)$, is closed, we have $(m, \mathbf{m}) \in \operatorname{Gr}(B^*)$, which means that $\mathbf{m} \in B^*(m)$, hence $\mathbf{m}$ is an equilibrated distribution. ∎

Now we deal with the case of n types of players. The iterative process is defined in a similar way: c_t are vectors of genetic inheritance of respective types, $c_t = (c_t^1, \ldots, c_t^n)$ with coordinates in $[0, 1]$, and the players of type i behave according to the formula

$$\mathbf{m}_t^i = c_t^i \cdot \mathbf{m}_{t-1}^i + (1 - c_t^i) \cdot b^i(m_{t-1}),$$

where the symbol m_{t-1} stands for the vector $(m_{t-1}^1, \ldots, m_{t-1}^n)$.

As in the case of one type, we shall assume that $\lim_{t\to\infty} c_t = c$ exists and has all coordinates smaller than 1.

THEOREM 3. *Let all the sets W^i be compact, all the functions Ψ^i be continuous in x and satisfy the condition* (2), *jointly w.r.t.* $(m^1,\dots,m^n)$. *If, for $i=1,\dots,n$, the weak* limits $\lim_{t\to\infty}\mathbf{m}_t^i=\mathbf{m}^i$ exist then the system $\mathbf{m}=(\mathbf{m}^1,\dots,\mathbf{m}^n)$ of distributions is equilibrated.*

Proof. As in the proof of Theorem 2, we define, for each $i=1,\dots,n$, a correspondence $B^{i*}:\operatorname{co}W^1\times\ldots\times\operatorname{co}W^n\rightsquigarrow\operatorname{Prob}(W^i)$ by

$$B^{i*}(m^1,\dots,m^n):=\operatorname{Argmax}\Big\{\int\limits_W \Psi(x;m^1,\dots,m^n)\mu^i(dx)\,\Big|\,\mu^i\in\operatorname{Prob}(W^i)\Big\}$$

and we find that its graph is closed. Consequently, the graph of the correspondence $B^*:\operatorname{co}W^1\times\ldots\times\operatorname{co}W^n\rightsquigarrow\operatorname{Prob}(W^1)\times\ldots\times\operatorname{Prob}(W^n)$, defined by $B^*(m^1,\dots,m^n):=B^{1*}(m^1,\dots,m^n)\times\ldots\times B^{n*}(m^1,\dots,m^n)$, is also closed.

We have

$$b^i(m_{t-1})=\frac{\mathbf{m}_t^i-c_t^i\cdot\mathbf{m}_{t-1}^i}{1-c_t^i}.$$

This expression makes sense for t large enough and its right hand side converges weakly*:

$$\lim_{t\to\infty}b^i(m_{t-1})=\frac{\mathbf{m}^i-c^i\cdot\mathbf{m}^i}{1-c^i}=\mathbf{m}^i.$$

As $b^i(m_{t-1})\in B^{i*}(m_{t-1})$, it follows that $(m_{t-1},(b^1(m_{t-1}),\dots,b^n(m_{t-1})))\in\operatorname{Gr}(B^*)$.

Since $(m_{t-1},(b^1(m_{t-1}),\dots,b^n(m_{t-1})))\to(m,\mathbf{m}^1,\dots,\mathbf{m}^n)$ and $\operatorname{Gr}(B^*)$ is closed, we have $(m,\mathbf{m})\in\operatorname{Gr}(B^*)$, i.e. $\mathbf{m}\in B^*(m)$, so the system $\mathbf{m}$ of distributions is equilibrated. ■

3. Mean-reference fitness functions. From now on we assume that the characteristics of the players of all types belong to the same Euclidean space, i.e. $k_1=\ldots=k_n=k$.

In this section we shall be specially interested in the case of games with fitness functions, for a type i, of the form

$$\Psi^i(x;m)=\psi^i(\|x-x_0^i\|,\|x-m^1\|,\dots,\|x-m^n\|) \tag{4}$$

for some $x_0^i\in\mathbb{R}^k$; here and elsewhere in this paper $\|\cdot\|$ denotes the Euclidean norm. Such a function ψ^i, along with x_0^i, will be called a *mean-reference representation* of Ψ^i; x_0^i itself is a *reference point* (for the type i).

For the time being we are not interested in equilibrated distributions but only in properties of mean-reference representations for just one type; the mean values of the remaining $n-1$ distributions may be considered as parameters. So there is not much loss of generality if we restrict our attention to games with just one type of players. If all individuals are of the same type, the population is called *uniform*; in this case we shall skip the superscript "1" in W^1, Ψ^1, m^1 etc.

For every x_0 there is at most one function ψ which, along with x_0, constitutes a mean-reference representation of Ψ; more precisely, such a function ψ may be defined over all of $\mathbb{R}_+ \times \mathbb{R}_+$, but it must be unique (and it is sufficient to have it defined only there) on the set

$$E(x_0) = \{(\zeta,\mu) \in \mathbb{R}_+^2 \mid (\zeta,\mu) = (\|x-x_0\|, \|x-m\|) \text{ for some } x \in W \text{ and } m \in \operatorname{co} W\}$$

The shape of the set $E(x_0)$ strongly depends on the shape of W itself as well as on the geometric position of x_0. In any case, $E(x_0)$ is always included in the set

$$[\inf\{\|x - x_0\| \mid x \in W\}, \sup\{\|x - x_0\| \mid x \in W\}] \times [0, \operatorname{Diam} W].$$

We say that a reference point is *desired* (resp. *strictly desired*, *undesired*, *strictly undesired*) whenever the corresponding function ψ is nonincreasing (resp. decreasing, nondecreasing, increasing) in the first argument for any fixed values of the remaining arguments.

If the fitness function has the form $\Psi(x;m) = \varphi(\|x - m\|)$ then every x_0 can be used as a reference point with the same ψ, namely $\psi(\|x - x_0\|, \|x - m\|) \equiv \varphi(\|x - m\|)$. Also in nontrivial cases there may be many possible reference points:

Example 1. Let W be included in a $(k - 1)$-dimensional hyperplane $H \subset \mathbb{R}^k$ and let (x_0, ψ) be a mean-reference representation of a fitness function Ψ with $x_0 \notin H$. Let x_0' denote the symmetric image of x_0 with respect to H. Then also the pair (x_0', ψ) is a mean-reference representation of Ψ. Every point on the line passing through x_0 and x_0' is also a reference point, along with an appropriately modified function ψ; of course only one of those points, the one situated on H, has a chance to belong to W.

There may be even distinct reference points which do belong to W:

Example 2. Take $W = [0,1] \subset \mathbb{R}$ and let $\Psi(x;m)$ have the form $x + f(\|x - m\|)$. Every $x_0 \le 0$ may serve as a reference point along with the function $\psi_{x_0}(u,v) := u - \|x_0\| + f(v)$; indeed, we have

$$\begin{aligned}\Psi(x;m) = \psi_{x_0}(\|x - x_0\|, \|x - m\|) &= \|x - x_0\| - \|x_0\| + f(\|x - m\|)\\ &= x + f(\|x - m\|).\end{aligned}$$

Similarly, every $x_0 \ge 1$ may also serve as a reference point, this time with the function $\psi_{x_0}'(u,v) := \|x_0\| - u + f(v)$.

Notice that only the reference points $x_0 \ge 1$ are desired; only $x_0 = 0$ and $x_0 = 1$ belong to W.

We have, however, the following:

PROPOSITION 4. *Let x_0 be a strictly desired or strictly undesired reference point and let x'_0 be another point in $\mathbb{R}^k$. If there exist distinct points $y, z \in W$ such that $\|y - x_0\| = \|z - x_0\|$ while $\|y - x'_0\| \neq \|z - x'_0\|$ then x'_0 is not a reference point.* ∎

If $k > 2$ then for the existence of such distinct y and z, it suffices that there exists a hyperplane $H \subset \mathbb{R}^k$ to which x_0 and x'_0 belong, such that the relative interior of $H \cap W$ is nonempty.

Moreover, the hypothesis of Proposition 4 is satisfied for any reference point x_0 and any $x'_0 \neq x_0$, whenever W has a nonempty interior and $k > 1$; then x_0 is the unique reference point.

We now determine under what circumstances a reference point and the mean of an equilibrated distribution coincide. We first consider the uniform case. For any points p, q (in the Euclidean space considered) denote:

- by $L^+(p, q)$ the ray originating at p and passing through q, i.e. $L^+(p, q) = \{\lambda \cdot q + (1 - \lambda) \cdot p \mid \lambda \geq 0\}$;
- by $L^-(p, q)$ the ray originating at p and passing through $2p - q$ (the symmetric image of q w.r.t. p), i.e. $L^-(p, q) = \{\lambda \cdot q + (1 - \lambda) \cdot p \mid \lambda \leq 0\}$.

THEOREM 5. *In the uniform case, for $k > 1$, let the fitness function Ψ have the form* (4), *$\Psi(x; m) = \psi(\|x - x_0\|, \|x - m\|)$ and let $\mathbf{m}$ be any distribution. Then*:

(a) *If x_0 is strictly desired and $m \neq x_0$ then $B(m) \subset \partial W \cup L^+(m, x_0)$.*

(b) *If x_0 is strictly undesired and $m \neq x_0$ then $B(m) \subset \partial W \cup L^-(m, x_0)$.*

(c) *If W is open, x_0 is strictly desired or strictly undesired and the distribution $\mathbf{m}$ is equilibrated then either $m = x_0$ or $\mathbf{m}$ is concentrated at m.*

Proof. To prove (a), suppose that the mean m of $\mathbf{m}$ is not x_0. Take any interior point x in W which is not in $L^+(m, x_0)$. Then there exists $\overline{x} \in W$ such that $\|x - m\| = \|\overline{x} - m\|$ and $\|\overline{x} - x_0\| < \|x - x_0\|$, so, because x_0 is strictly desired, $\psi(\|\overline{x} - x_0\|, \|\overline{x} - m\|) > \psi(\|x - x_0\|, \|x - m\|)$. Hence $x \notin B(m)$, which means that $B(m) \subset \partial W \cup L^+(m, x_0)$.

The proof of (b) is analogous to that of (a).

To prove the statement (c) just note that in this case $\partial W \cap W = \emptyset$. Since $\mathbf{m}$ is equilibrated, we then have $B(m) \subset L^{\pm}(m, x_0)$ (the sign + or − depends on whether x_0 is strictly desired or undesired), hence $\mathbf{m}$ must be concentrated at m. ∎

Due to the geometric nature of the above proof, it only works for $k > 1$; even more, the theorem is false in general for $k = 1$. However, it will be still true under some additional assumptions, for instance if W is an interval centered around x_0 or if W is included and dense in an interval centered around x_0 and Ψ is lower semicontinuous.

In the case (c), an equilibrated distribution $\mathbf{m}$ might be concentrated at m only if $\psi(|\|m-x_0\|+z|,|z|)$, as a function of z, attains its maximum at $z=0$.

Remark 1. Under the assumptions of Theorem 5(c), every equilibrated distribution $\mathbf{m}$ whose mean coincides with x_0 is concentrated on the set of those elements of W whose distance r from x_0 maximizes the value of $\psi(r,r)$. ∎

Some statements of Theorem 5 can be extended to the case of n types of players, for instance we have:

Proposition 6. *If $k>n$ and for some type $i=1,\dots,n$, W^i is open, a reference point x_0^i is either strictly desired or strictly undesired and $\mathbf{m}=(\mathbf{m}^1,\dots,\mathbf{m}^n)$ is an equilibrated system of distributions such that x_0^i is not in the affine subspace M spanned by $m^1,\dots,m^n$ then $\mathbf{m}^i$ is concentrated on M.* ∎

4. Special cases. For a uniform population, the case of our special interest is that of fitness functions of the form

$$\psi(\|x-x_0\|,\|x-m\|)=-\alpha\|x-x_0\|+\beta\|x-m\|-\gamma\|x-m\|^2,$$

where α, β and γ are nonnegative constants.

Such fitness functions correspond to situations where all individuals have their preferred characteristics and they like to differ from the others but only to some extent. In the case where the position of a player is understood as spatial location, interpretation of the terms in the payoff (fitness) function may also involve the natural trends to form a society increasing its reproduction abilities (second term) as well as negative effects of congestion, including shortage of food (the last term).

Let us determine the best reply correspondence for a function of this form in the case of positive α (which means that x_0 is strictly desired) and γ; we also assume that $W\subset\mathbb{R}^k$ is *star-shaped* w.r.t. x_0 (i.e. for every $y\in W$, the interval $\{\lambda y+(1-\lambda)x_0\mid 0\le\lambda\le 1\}$ is included in W) and W contains the sphere with center at x_0 and radius $r=\max\{(\beta-\alpha)/(2\gamma),0\}$, denoted by $S(x_0,r)$ (Figure 1 shows the graph of this correspondence in the case of $k=1$, $x_0=0$ and W being a closed interval).

For the time being assume that $k>1$ and W is open. We first consider the case of distributions $\mathbf{m}$ with $m\ne x_0$. Since x_0 is strictly desired, we have, as stated in Theorem 5(a), $B(m)\subset L^+=\{x\mid x=\lambda\cdot x_0+(1-\lambda)\cdot m,\ \lambda\ge 0\}$. Decompose L^+ into three parts: $\{x_0\}$, $L_1=\{x\mid x=\lambda\cdot x_0+(1-\lambda)\cdot m,\ \lambda>1\}$ and $L_2=\{x\mid x=\lambda\cdot x_0+(1-\lambda)\cdot m,\ 0\le\lambda<1\}$.

If $x\in B(m)\cap L_1$ then $\|x-x_0\|=\|x-m\|-\|m-x_0\|$.

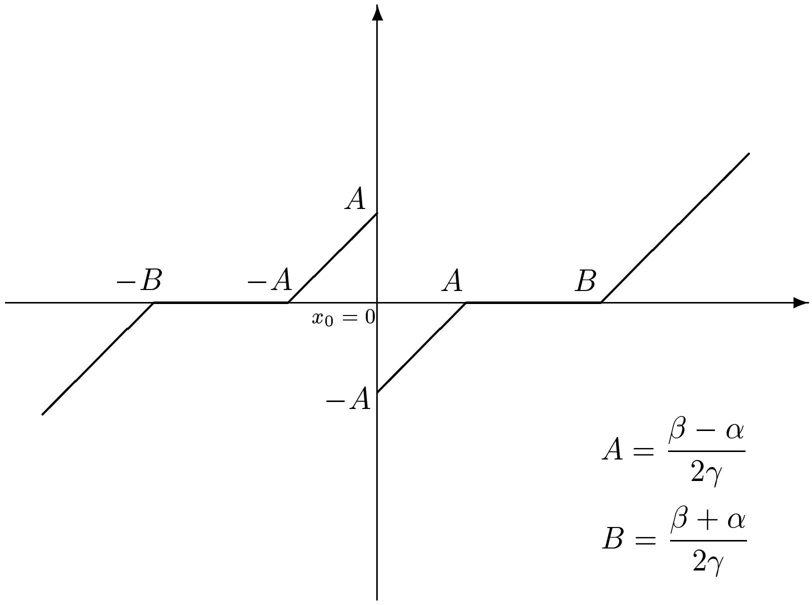

Fig. 1

In this case x is a solution to the maximization problem

$$x = \operatorname{argmax}_{x \in L_1} \{-\alpha\|x - m\| + \alpha\|m - x_0\| + \beta\|x - m\| - \gamma\|x - m\|^2\}.$$

It is easy to calculate that x must satisfy $\|x - m\| = (\beta - \alpha)/(2\gamma)$ and $x \in L_1$. Such an x exists if and only if $\|x_0 - m\| < (\beta - \alpha)/(2\gamma)$. Then

$$x = \frac{\beta - \alpha}{2\gamma\|x_0 - m\|} \cdot x_0 + \left(1 - \frac{\beta - \alpha}{2\gamma\|x_0 - m\|}\right) m.$$

If $x \in B(m) \cap L_2$ then $\|x - x_0\| = \|m - x_0\| - \|x - m\|$, so the corresponding maximization problem has the form

$$x = \operatorname{argmax}\{-\alpha\|x_0 - m\| + \alpha\|m - x\| + \beta\|x - m\| - \gamma\|x - m\|^2\}.$$

It follows that the solution x must satisfy $\|x - m\| = (\beta + \alpha)/(2\gamma)$ and $x \in L_2$. A necessary condition for the existence of such an x is $\|x_0 - m\| > (\beta + \alpha)/(2\gamma)$. Then

$$x = \frac{\beta + \alpha}{2\gamma\|x_0 - m\|} \cdot x_0 + \left(1 - \frac{\beta + \alpha}{2\gamma\|x_0 - m\|}\right) m.$$

If neither of the above conditions holds then we find that $x_0 \in B(m)$, actually $\{x_0\} = B(m)$.

We finally consider the case where $m = x_0$. Then we have $x = \operatorname{argmax}\{(\beta - \alpha)\|x_0 - x\| - \gamma\|x - x_0\|^2\}$. If $\beta < \alpha$ then $x = x_0$, otherwise $\|x - x_0\| = (\beta - \alpha)/(2\gamma)$ and $B(x_0) = \{x \mid \|x - m\| = (\beta - \alpha)/(2\gamma)\}$.

Eventually we have:

$$
B(m) = \begin{cases}
\left\{x \mid \|x-x_0\| = \max\left\{\frac{\beta-\alpha}{2\gamma}, 0\right\}\right\} & \text{if } m = x_0\text{; otherwise} \\
\left\{\frac{\beta-\alpha}{2\gamma\|x_0-m\|}\cdot x_0 + \left(1-\frac{\beta-\alpha}{2\gamma\|x_0-m\|}\right)m\right\} & \text{if } \|x_0-m\| < \frac{\beta-\alpha}{2\gamma}; \\
\left\{\frac{\beta+\alpha}{2\gamma\|x_0-m\|}\cdot x_0 + \left(1-\frac{\beta+\alpha}{2\gamma\|x_0-m\|}\right)m\right\} & \text{if } \|x_0-m\| > \frac{\beta+\alpha}{2\gamma}; \\
\{x_0\} & \text{if none of the above.}
\end{cases}
$$

If $k = 1$, it is easy to show that $B(m) \cap L^-(m, x_0) = \emptyset$, so the same reasoning applies.

Finally, notice that the assumption of openness of W is redundant: indeed, the adopted assumptions imply that the closed ball of radius $r = \max\{(\beta-\alpha)/(2\gamma), 0\}$ around x_0 is included in W; outside this ball, the best reply always tends to x_0, which means that the best reply is never a boundary point unless it is on the sphere $S(x_0, r)$.

The above calculations have led us to the following:

THEOREM 7. *Let α and γ be positive and let $W \subset \mathbb{R}^k$ include x_0. If $\beta \le \alpha$ then the unique equilibrated distribution is concentrated at x_0. If $\beta > \alpha$, W is star-shaped w.r.t. x_0 and it contains $S = S(x_0, (\beta-\alpha)/(2\gamma))$ then every equilibrated distribution is concentrated on S and its mean is equal to x_0.* ∎

A few remarks are in order now. Actually B', the restriction of B to the subdomain

$$D = \left\{x \,\middle|\, \|x_0 - m\| < \frac{\beta-\alpha}{2\gamma}\right\} \setminus \{x_0\},$$

is acting into D, it is one-to-one and $B'^{-1} = B'$.

Let us get back to the iterative process considered in Section 2. Assume that $\beta > \alpha$, W is convex and it contains the sphere with center at x_0 and radius $(\beta-\alpha)/(2\gamma)$. We shall consider two cases.

At first let the coefficient of genetic inheritance c equal 0. Then the process will reach x_0 in a finite number of iterations from any initial distribution with a mean whose distance from x_0 belongs to an interval

$$\left[\frac{\beta-\alpha}{2\gamma} + l\frac{\beta+\alpha}{2\gamma}, \frac{\beta+\alpha}{2\gamma} + l\frac{\beta+\alpha}{2\gamma}\right],$$

for some integer $l \ge 0$. All means with this property constitute a union of rings (intersected with W); this set is an area of possible convergence, since reaching x_0 is a necessary condition for attaining any equilibrated distribution in our game. This condition is not sufficient, because if the mean of the next iterate after reaching x_0 misses x_0, the further process will diverge, since in D the trajectory following the best reply oscillates. If the initial distribution $\mathbf{m}_0$ is not concentrated at x_0 and its mean m_0 does not belong to any of those rings then neither an equilibrated distribution nor the distribution concentrated at x_0 can ever be reached (actually the process will diverge). We can sum up: either an equilibrated distribution may be reached in a finite number of iterations (the first iterate after reaching the mean x_0 is an equilibrated distribution), or the process starts oscillating.

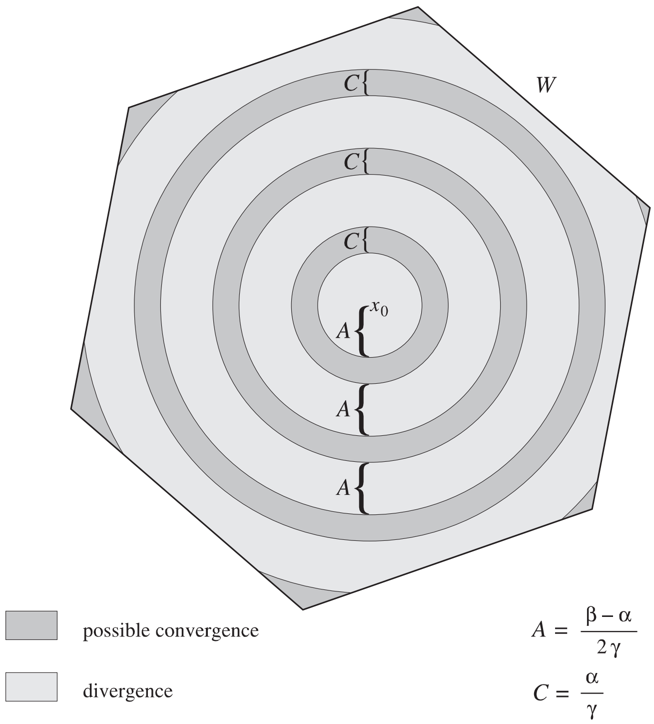

Fig. 2

The areas of divergence and possible convergence are shown in Figure 2.

In the second case let $0 < c < 1$. Then for any initial distribution whose mean differs from x_0, the trajectory not only depends on the mean but also on a particular choice of initial distribution. For an initial distribution concentrated at a point, the areas of possible convergence form a union of isolated spheres, not rings, and an equilibrated distribution will not be reached in a finite number of iterations.

We finally consider the *prey-predator game* which is an extension of the previously considered uniform case to a scheme involving two types of players.

There are two types of species, the prey 1 and the predator 2, with respective fitness functions:

$$\begin{aligned}
&\psi^1(\|x-x_0^1\|,\|x-m^1\|,\|x-m^2\|)\\
&\qquad= -\alpha^1\|x-x_0^1\|+\beta^1\|x-m^1\|-\gamma^1\|x-m^1\|^2+\delta^1\|x-m^2\|^{\sigma^1},\\
&\psi^2(\|x-x_0^2\|,\|x-m^1\|,\|x-m^2\|)\\
&\qquad= -\alpha^2\|x-x_0^2\|+\beta^2\|x-m^2\|-\gamma^2\|x-m^2\|^2-\delta^2\|x-m^1\|^{\sigma^2},
\end{aligned}$$

where α^i, β^i, γ^i, δ^i, σ^i, for $i=1,2$, are nonnegative constants (σ^1 and σ^2 would be typically 1 or 2) while x_0^1 and x_0^2 are the two types' respective reference points.

This game describes, for instance, the situation in a habitat around a water pool, where the two species live. The pool is the ideal area, represented by a point. Both species want to live as close to it as possible. The first three terms in the payoff function can then be interpreted as in the previous case considered in this section. The last term reflects the species' desire or aversion to "enjoy" the company of the antagonist, so the prey (type 1) wants to live as far from the predator's mean as possible, but the predator (type 2) wants to live as close to the prey's mean as possible.

The following theorem holds true:

THEOREM 8. *Let $W^1=W^2=W$ be open, star-shaped w.r.t. x_0 and include the sphere*

$$S\bigg(x_0,\max\bigg\{0,\frac{\beta^1-(\alpha^1-\delta^1)}{2\gamma^1},\frac{\beta^2-(\alpha^2+\delta^2)}{2\gamma^2}\bigg\}\bigg).$$

Assume that $k>1$, $x_0^1=x_0^2=x_0$, $\sigma^1=\sigma^2=1$ and the remaining parameters are positive. A pair $(\mathbf{m}^1,\mathbf{m}^2)$ of distributions is equilibrated if and only if both means m^1 and m^2 equal x_0, $\mathbf{m}^1$ is concentrated on $S(x_0,\max\{0,(\beta^1-(\alpha^1-\delta^1))/(2\gamma^1)\})$ while $\mathbf{m}^2$ is concentrated on $S(x_0,\max\{0,(\beta^2-(\alpha^2+\delta^2))/(2\gamma^2)\})$.

Proof. Take any distribution $\mathbf{m}^2$ with a mean $m^2\neq x_0$ and suppose that there exists $\mathbf{m}^1$ such that the pair $(\mathbf{m}^1,\mathbf{m}^2)$ of distributions is equilibrated. Calculations analogous to those used in the proof of the previous theorem lead to the conclusion that $\mathbf{m}^1$ must be concentrated at one point.

The prey's fitness function $-\alpha^1\|x-x_0\|+\beta^1\|x-m^1\|-\gamma^1\|x-m^1\|^2+\delta^1\|x-m^2\|$ cannot attain its maximum at $x=m^1$ because $\beta^1\|x-m^1\|-\gamma^1\|x-m^1\|^2$ is increasing with the distance of x from m^1 (within a small neighborhood of m^1). This means that no distribution concentrated at one point can form an equilibrated pair of distributions along with $\mathbf{m}^2$, so m^2 must be equal to x_0.

For $m^2=x_0$, the problem of seeking an $\mathbf{m}^1$ that might enter an equilibrated pair of distributions reduces to the former problem considered in this section (the uniform case) with the constants $\alpha = \alpha^1 - \delta^1$, $\beta = \beta^1$, $\gamma = \gamma^1$; we find that $\mathbf{m}^1$ must be concentrated on $S(x_0, \max\{0, (\beta^1 - (\alpha^1 - \delta^1))/(2\gamma^1)\})$ and its mean must be x_0. The same reasoning is then applied to $\mathbf{m}^2$ which must be concentrated on $S(x_0, \max\{0, (\beta^2 - (\alpha^2 + \delta^2))/(2\gamma^2)\})$, its mean being also x_0. ■

Theorem 8 is not true in general in the case of $k = 1$.

5. Concluding remarks. One of the aims of our paper is to provide a mathematical tool to study models of spatial allocation and migration but it was not our intention to apply this tool to any specific model in this paper. Many articles concerning biological aspects of the problems just mentioned can be found, for instance, in the collections edited by Cody and Diamond [3] or Diamond and Case [4] or in the journals like Oecologia, The American Naturalist, Ecology or Science.

The literature of the subject of evolutionary behavior and biological adaptation is enormous and it is senseless to include any exhaustive survey here; we just mention an excellent mathematically oriented monograph written by Hofbauer and Sigmund [5]. A list of rather mathematically oriented papers somehow related to our paper (although the results are of entirely different nature) includes works of Parker and Sutherland [10], Kacelnik, Krebs and Bernstein [6] and two recent papers of Milchtaich [8], [9]. In particular, the last paper contains an optimality condition (p. 763, (1)) similar to our equilibration.

As already mentioned, the adaptation model presented in this paper directly stems from the general scheme of *simple large games* introduced by Wieczorek [14]; a special case with finite sets of individual characteristics W^i is related to the scheme of *elementary large games* of Wieczorek [13]. The roots of the presented model go back, however, to noncooperative games with a continuum of players, first studied by Schmeidler [11], in their "anonymous" version (players not directly represented by elements of a space) by Mas-Colell [7], Balder [1] and others. A topic close to but different from the main subject of the present paper is that of evolutionary stable equilibria (analogous to the classic evolutionary stable strategies, ESS, as considered e.g. by Hofbauer and Sigmund [5] or Van Damme [12]); we do not tackle this topic here as it certainly deserves a separate analysis including an elaboration of its proper definition in the present framework.

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