APPLICATIONES MATHEMATICAE
25,4 (1999), pp. 411–415

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ON A MULTI-OBJECTIVE OPTIMIZATION PROBLEM ARISING FROM PRODUCTION THEORY

Abstract. The paper presents a natural application of multi-objective programming to household production and consumption theory. A contribution to multi-objective programming theory is also included.

1. A multi-objective optimization problem. The following multi-objective optimization problem will be considered and applied in the present paper:

Given a $k\times n$ matrix A, an $m\times n$ matrix C and an m-vector b, we consider the set of all k-vectors of the form Ax for some $x \in \mathbb{R}^n_+$, satisfying the constraint $Cx = b$; such x's are called *solutions* of the *problem* $P = (A, C, b)$. The set of all solutions of P will be denoted by $X(P)$; it is a convex set. We are specially interested in characterizing the existence of efficient and weakly efficient solutions: a solution x^* is called *efficient* whenever there exists no $x \in X(P)$ for which $Ax > Ax^*$ (by $y > z$ we always mean that the respective coordinates of the vector z do not exceed those of y and the vectors are different; $y \gg z$ will mean that all respective coordinates of y are larger than those of z); a solution x^* is called *weakly efficient* whenever there exists no $x \in X(P)$ for which $Ax \gg Ax^*$. Obviously, *every efficient solution is also weakly efficient.*

Isermann has proven in [2] the following result, actually reducing the problem of finding all efficient solutions to solving a class of linear maximization problems:

THEOREM 1. *A vector x^* is an efficient solution of a problem $P = (A, C, b)$ if and only if there exists a vector $w \in \mathbb{R}^k_+$, $w \gg \mathbf{0}$, such that*

1991 *Mathematics Subject Classification*: 90A11, 90A12, 90A15.

Journal of Economic Literature Classification: C62.

Key words and phrases: multi-objective optimization, (weakly) efficient solution, household production, (weak) Pareto optimality.

$w^\top Ax^* \geq w^\top Ax$ for all $x \in X(P)$ (i.e. x^* maximizes $w^\top Ax$ subject to the constraint $Cx = b$). ■

An analogous result holds for weakly efficient solutions:

For any solution x, we denote by $J(x)$ the set of those indices j among $1, \ldots, k$ for which there exists no solution y such that $Ay > Ax$ and $(Ay)_j > (Ax)_j$.

THEOREM 2. *A vector x^* is a weakly efficient solution of a problem $P = (A, C, b)$ if and only if there exists a vector $w \in \mathbb{R}^k_+$, $w \neq \mathbf{0}$, such that $w^\top Ax^* \geq w^\top Ax$ for all solutions x while $w_j > 0$ if and only if $j \in J(x^*)$.*

Proof. To prove the implication $\Rightarrow$ suppose first that $J(x^*)$ is empty; then x^* is not a weakly efficient solution (by the convexity of $X(P)$). Now suppose that $J(x^*)$ is not empty and $w^\top Ax^* \geq w^\top Ax$ for all solutions x but x^* is not weakly efficient. Then there exists a solution x such that $Ax \gg Ax^*$. But now we have $w^\top Ax > w^\top Ax^*$ (w has at least one positive coordinate), which contradicts the hypothesis.

To prove $\Leftarrow$, we apply Theorem 1 to the matrix $A_{J(x^*)}$ composed of those rows of A whose indices belong to $J(x^*)$. We first show that x^* is an efficient solution of the problem $(A_{J(x^*)}, C, b)$.

If x^* is efficient for P then $J(x^*) = \{1, \ldots, k\}$, $A_{J(x^*)} = A$ and the above statement is obviously true. Now let x^* be weakly efficient but not efficient for P. For each $l \notin J(x^*)$ there exists a solution y^l such that $(Ay^l)_j = (Ax^*)_j$ for $j \in J(x^*)$, $(Ay^l)_j \geq (Ax^*)_j$ for $j \notin J(x^*)$ and $(Ay^l)_l > (Ax^*)_l$. The vector $\widetilde{y}$ defined by $\widetilde{y} := (k-m)^{-1} \sum_{l \notin J(x^*)} y^l$, where m denotes the cardinality of $J(x^*)$, is a solution such that $(A\widetilde{y})_j = (Ax^*)_j$ for $j \in J(x^*)$ and $(A\widetilde{y})_j > (Ax^*)_j$ otherwise.

Suppose that x^* is not an efficient solution of the problem $(A_{J(x^*)}, C, b)$, i.e. there exists a solution x such that $A_{J(x^*)}x > A_{J(x^*)}x^*$. This means that $(Ax)_j > (Ax^*)_j$ for at least one $j \in J(x^*)$ and $(Ax)_j \geq (Ax^*)_j$ for all remaining $j \in J(x^*)$. Let L denote the set of those $j \notin J(x^*)$ for which $(Ax)_j < (Ax^*)_j$. If L is empty then x is a solution of P such that $Ax > Ax^*$ and $(Ax)_j > (Ax^*)_j$ for some $j \in J(x^*)$, which contradicts the definition of $J(x^*)$. If L is not empty, we have for $j \in L$, $(Ax)_j < (Ax^*)_j < (A\widetilde{y})_j$. Choose any number μ so that

$$1 > \mu \geq \max_{j \in L} \left\{ \frac{(Ax^*)_j - (Ax)_j}{(A\widetilde{y})_j - (Ax)_j} \right\}.$$

Then $z = \mu\widetilde{y} + (1-\mu)x$ will be a solution of P such that $Az > Ax^*$ and $(Az)_j > (Ax^*)_j$ for some $j \in J(x^*)$, which again contradicts the definition of $J(x^*)$.

According to Theorem 1, there exists a positive vector $v \in \mathbb{R}^m_+$ such that $v^\top A_{J(x^*)}x^* \geq v^\top A_{J(x^*)}x$ for all solutions x. We define a vector $w \in \mathbb{R}^k_+$ as follows: $w_i := v_i$ whenever $i \in J(x^*)$ and $w_i := 0$ otherwise. We have

$$w^\top Ax^* = v^\top A_{J(x^*)}x^* \geq v^\top A_{J(x^*)}x = w^\top Ax \quad \text{for all } x \in X(P),$$

which completes the proof. ∎

One can also easily prove something more:

PROPOSITION 3. *If x^* is a weakly efficient solution of the problem $P = (A, C, b)$ and $w \in \mathbb{R}^k_+$ is such that $w^\top Ax^* \geq w^\top Ax$ for all solutions x then $w_j = 0$ for all $j \notin J(x^*)$.* ∎

2. A model of household production and consumption. The model describes the behavior of infinitely many households classified into n types. Each household of each type can choose among k kinds of activity. The choice of a jth activity by a household of type i results in producing r^i_j units of the jth good, $j = 1, \ldots, k$. Hence, we can characterize the production possibilities of a household of type i by a vector $r^i = (r^i_1, r^i_2, \ldots, r^i_k)$ of nonnegative numbers; r^i_j's are interpreted here as *coefficients of efficiency.* (For instance, r^i_1 might be the output of an individual of type i if she decides to produce shoes, r^i_2 would be her output if driving a lorry, r^i_k can be her output if acting as a businesswoman.)

The model is completely determined by the vectors r^i, $i = 1, \ldots, n$, and a distribution $d = (d_1, \ldots, d_n)$, in the $(n-1)$-dimensional standard simplex Δ_n, of the respective types in the population.

Once the agents decide which activity to undertake, a distribution of households of type i, producing respective goods, is created: it is a vector $p^i = (p^i_1, p^i_2, \ldots, p^i_k)$ in the standard simplex Δ_k. Set $\mathbf{p} := (p^1, p^2, \ldots, p^n)$. The volume of the production of the jth good is then equal to $S_j(\mathbf{p}) = \sum_{i=1}^n d_i r^i_j p^i_j$; the vector of *total supply* is

$$S(\mathbf{p}) = (S_1(\mathbf{p}), \ldots, S_k(\mathbf{p})).$$

This model has been constructed by Wieczorek in [3], where the author was interested mainly in the existence and properties of *competitive equilibria.* In the present paper we are rather interested in the distributions $\mathbf{p}^*$ leading to supply vectors which are *efficient* [or *weakly efficient*] *in the sense of Pareto*, i.e. $\mathbf{p}^*$ such that there exists no other distribution $\mathbf{p}$ such that $S(\mathbf{p}) > S(\mathbf{p}^*)$ [respectively, $S(\mathbf{p}) \gg S(\mathbf{p}^*)$]. Usually such efficiency concepts are regarded as measuring efficiency of the organization of a society.

We shall prove that $\mathbf{p}$ is efficient if and only if there exists a system $\pi = (\pi_1, \ldots, \pi_k)$ of positive prices at which p^i maximizes the *total profit* (of all individuals) of type i, for each i (we speak of the total profit, but it is achieved as a result of decentralized action of the players acting independently and having only their own profit in mind). More precisely, we have:

THEOREM 4. *A distribution vector* $\mathbf{p} = (p^1, \ldots, p^n)$ *is Pareto efficient if and only if there exists a system* $\pi = (\pi_1, \ldots, \pi_k) \in \Delta_k$ *of positive prices at which* p^i *maximizes the total profit of type* i, $d_i \sum_{j=1}^{k} r_j^i \pi_j p_j^i$, *for each* i.

Proof. Define

$$\begin{aligned} Q &:= \{\mathbf{p} \in \mathbb{R}_+^{nk} \mid \mathbf{p} = (p^1, \ldots, p^n),\ p^i \in \Delta_k \text{ for } i = 1, \ldots, n\} \\ &= \{\mathbf{p} \in \mathbb{R}_+^{nk} \mid C\mathbf{p} = \mathbf{1}\}, \end{aligned}$$

where $\mathbf{1}$ stands for the n-vector with all entries 1 while C is the $n \times nk$ matrix defined by

$$c_{ij} = \begin{cases} 1 & \text{for } i = 1, \ldots, n \text{ and } (i-1)k + 1 \le j \le ik; \\ 0 & \text{for the remaining pairs } (i, j). \end{cases}$$

Let A be the $k \times nk$ matrix defined by

$$a_{ij} = \begin{cases} d_l r_i^l & \text{for } i = 1, \ldots, k \text{ and } j = (l-1)k + i \ (l = 1, \ldots, n); \\ 0 & \text{for the remaining } (i, j). \end{cases}$$

So we have $A\mathbf{p} = S(\mathbf{p})$ for each $\mathbf{p}$; we have constructed the optimization problem $(A, C, \mathbf{1})$. According to Theorem 1, a distribution $\mathbf{p}$ is an efficient solution of this problem if and only if there exists a vector $w \in \mathbb{R}_+^k$, $w \gg \mathbf{0}$, such that $w^\top A\mathbf{p} \ge w^\top A\mathbf{q}$ holds for all $\mathbf{q} \in Q$. We can normalize w and get a vector $\pi = (\pi_1, \ldots, \pi_k) \in \Delta_k$ for which the above inequality also holds. The proof will be complete if we show that π is a system of prices we are looking for. This will be formulated as a separate lemma. ∎

LEMMA 5. *A price vector* π *satisfies*

$$\pi^\top A\mathbf{p} = \sum_{j=1}^{k} \sum_{i=1}^{n} d_i r_j^i \pi_j p_j^i = \max_{\mathbf{q} \in Q} \pi^\top A\mathbf{q}$$

if and only if p^i *maximizes the total profit of the type* i, $d_i \sum_{j=1}^{k} r_j^i \pi_j p_j^i$, *for each* $i = 1, \ldots, n$.

Proof. The implication $\Rightarrow$ is obvious. If every element of the sum is maximal then the sum is also maximal.

To prove $\Leftarrow$, assume that $\pi^\top A\mathbf{p} = \max_{\mathbf{q} \in Q} \pi^\top A\mathbf{q}$ and that there is a type i whose total profit is not maximal. Then there exists a vector $s \in \Delta_k$ such that $\sum_{j=1}^{k} d_i r_j^i \pi_j p_j^i < \sum_{j=1}^{k} d_i r_j^i \pi_j s_j^i$. For the vector $\mathbf{q} = (p^1, \ldots, p^{i-1}, s, p^{i+1}, \ldots, p^n) \in Q$ we have $\pi^\top A\mathbf{q} > \pi^\top A\mathbf{p}$, which contradicts the hypothesis. ∎

An analogue to Theorem 4 for weak efficiency is the following:

THEOREM 6. *A distribution vector* $\mathbf{p} = (p^1, \ldots, p^n)$ *is weakly Pareto efficient if and only if there exists a system* $(\pi_1, \ldots, \pi_k) \in \Delta_k$ *of prices at which* p^i *maximizes the total profit of type* i, $d_i \sum_{j=1}^{k} r_j^i \pi_j p_j^i$, *for each* i, *and* *such that*, *for* $j = 1, \ldots, k$, $\pi_j > 0$ *if and only if there exists no distribution* $\mathbf{q}$ *such that* $S(\mathbf{q}) > S(\mathbf{p})$ *and* $(S(\mathbf{q}))_j > (S(\mathbf{p}))_j$.

P r o o f. The proof is analogous to that of Theorem 4 except that we use Theorem 2 instead of Theorem 1. ■

A consequence of Proposition 3 is the following:

PROPOSITION 7. *If a distribution vector* $\mathbf{p} = (p^1, \ldots, p^n)$ *is weakly Pareto efficient and* $(\pi_1, \ldots, \pi_k) \in \Delta_k$ *is any system of prices at which* p^i *maximizes the total profit of type* i *for each* i *then*, *for* $j = 1, \ldots, k$, $\pi_j = 0$ *whenever there exists a distribution* $\mathbf{q}$ *such that* $S(\mathbf{q}) > S(\mathbf{p})$ *and* $(S(\mathbf{q}))_j > (S(\mathbf{p}))_j$. ■

The results in this section actually describe the process of decentralizing economic behavior of a society: efficient (or weakly efficient) states of an economy are rather obtained in a cooperative manner, an efficient state is jointly elaborated by all agents; in contrast, states at which individuals are maximizing their income have, *a fortiori*, noncooperative decentralized character. Such "decentralizing" results are known in many economic models for a long time (see e.g. Hildenbrand [1], p. 232) although the mathematical tools to get them may be entirely different from ours.

References

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Received on 5.11.1997;
revised version on 6.4.1998