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ON NONSTATIONARY MOTION OF A FIXED MASS OF A GENERAL VISCOUS COMPRESSIBLE HEAT CONDUCTING CAPILLARY FLUID BOUNDED BY A FREE BOUNDARY

Abstract. The motion of a fixed mass of a viscous compressible heat conducting capillary fluid is examined. Assuming that the initial data are sufficiently close to a constant state and the external force vanishes we prove the existence of a global-in-time solution which is close to the constant state for any moment of time. Moreover, we present an analogous result for the case of a barotropic viscous compressible fluid.

1. Introduction. The aim of this paper is to prove the global existence theorem for a free boundary problem for equations of a viscous compressible heat conducting capillary fluid in the general case, i.e. without assuming any conditions on the form of the internal energy.

In papers [13], [18], [19] the global existence theorem was proved under the assumption of a special form of the internal energy $\varepsilon = \varepsilon(\varrho, \theta)$, where ϱ is the density of the fluid and θ is the temperature. More precisely, we assumed

$$\varepsilon(\varrho, \theta) = a_0 \varrho^\alpha + h(\varrho, \theta),$$

where $a_0 > 0$, $\alpha > 0$, $h(\varrho, \theta) \ge h_* \ge 0$ for $\varrho \in [\varrho_*, \varrho^*]$, $\theta \in [\theta_*, \theta^*]$; a_0, α, h are constants, and

$$\varrho_* = \min_{t\in[0,T]} \min_{\overline{\Omega}_t} \varrho(x,t), \quad \varrho^* = \max_{t\in[0,T]} \max_{\overline{\Omega}_t} \varrho(x,t),$$

$$\theta_* = \min_{t\in[0,T]} \min_{\overline{\Omega}_t} \theta(x,t), \quad \theta^* = \max_{t\in[0,T]} \max_{\overline{\Omega}_t} \theta(x,t),$$

T is the time of local existence of a solution.

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In this paper we consider the motion of a fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ which depends on time $t \in \mathbb{R}_+$. The shape of the free boundary S_t of Ω_t is governed by surface tension. Let $v = v(x,t)$ be the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature, $f = f(x,t)$ the external force per unit mass, $r = r(x,t)$ the heat sources per unit mass, $\overline{\theta} = \overline{\theta}(x,t)$ the heat flow per unit surface, $p = p(\varrho,\theta)$ the pressure, μ and ν the viscosity coefficients, $\varkappa$ the coefficient of heat conductivity, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume and p_0 the external (constant) pressure. Then the motion of the fluid is described by the following system (see [1], Chs. 2 and 5):

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] + \nabla p - \mu\Delta v - \nu\nabla\operatorname{div} v = \varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta \operatorname{div} v - \varkappa\Delta\theta \\
&\quad - \frac{\mu}{2}\sum_{i,j=1}^{3}(v_{ix_j} + v_{jx_i})^2 - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\varphi_t/|\nabla\varphi| && \text{on } \widetilde{S}^T,\\
&\partial\theta/\partial n = \overline{\theta} && \text{on } \widetilde{S}^T,\\
&v|_{t=0} = v_0, \quad \varrho|_{t=0} = \varrho_0, \quad \theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned} \tag{1.1}
$$

where $\varphi(x,t) = 0$ describes S_t, $\overline{n}$ is the unit outward normal vector to the boundary, $\widetilde{\Omega}^T = \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$, $\Omega_0 = \Omega$ is the initial domain, and $\widetilde{S}^T = \bigcup_{t\in(0,T)} S_t \times \{t\}$. Moreover, $\mathbb{T} = \mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,2,3} = \{-p\delta_{ij} + \mu(v_{ix_j} + v_{jx_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}_{i,j=1,2,3}$ is the stress tensor and H is the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\overline{n} = \Delta(t)x, \quad x = (x_1,x_2,x_3),$$

where $\Delta(t)$ is the Laplace–Beltrami operator on S_t.

Let S_t be determined by

$$x = x(s_1,s_2,t), \quad (s_1,s_2) \in U \subset \mathbb{R}^2,$$

where U is an open set. Then

$$\Delta(t) = g^{-1/2}\frac{\partial}{\partial s_\alpha}\left(g^{-1/2}\widehat{g}_{\alpha\beta}\frac{\partial}{\partial s_\beta}\right) = g^{-1/2}\frac{\partial}{\partial s_\alpha}\left(g^{1/2}g^{\alpha\beta}\frac{\partial}{\partial s_\beta}\right), \quad \alpha,\beta = 1,2,$$

where the summation convention over the repeated indices is assumed, $g = \det\{g_{\alpha,\beta}\}_{\alpha\beta=1,2}$, $g_{\alpha\beta} = x_\alpha\cdot x_\beta$ ($x_\alpha = \partial x/\partial s_\alpha$), $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\widehat{g}_{\alpha\beta}\}$ is the matrix of algebraic complements of $\{g_{\alpha\beta}\}$.

Assume that the domain Ω is given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi,t),\ \xi \in \Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1,\xi_2,\xi_3).$$

Hence, we obtain the following relation between the Eulerian x and the Lagrangian ξ coordinates of the same fluid particle:

$$x = \xi + \int_0^t u(\xi,t')\,dt' \equiv X_u(\xi,t),$$

where $u(\xi,t) = v(X_u(\xi,t),t)$. Moreover, by $(1.1)_5$, $S_t = \{x : x = x(\xi,t),\ \xi \in S = \partial\Omega\}$.

By the continuity equation $(1.1)_2$ and the kinematic condition $(1.1)_5$ the total mass is conserved, i.e.

$$\int_{\Omega_t} \varrho(x,t)\,dx = \int_{\Omega} \varrho_0(\xi)\,d\xi = M, \tag{1.2}$$

where M is a given constant.

Moreover, in view of thermodynamic considerations assume

$$c_v > 0, \quad \varkappa > 0, \quad \nu > \tfrac{1}{3}\mu > 0.$$

In this paper we prove the existence of a global-in-time solution of problem (1.1) near a constant state.

Assume that $p_\varrho > 0$, $p_\theta > 0$ for $\varrho,\theta \in \mathbb{R}_+$ and consider the equation

$$p\left(\frac{M}{\frac{4}{3}\pi R_e^3},\theta_e\right) = p_0 + \frac{2\sigma}{R_e}. \tag{1.3}$$

We assume that there exist $R_e > 0$ and $\theta_e > 0$ satisfying (1.3). Then we introduce the following definition.

Definition 1.1. Let $f = r = \overline{\theta} = 0$. By a *constant* (*equilibrium*) *state* we mean a solution $(v,\theta,\varrho,\Omega_t)$ of problem (1.1) such that $v = 0$, $\theta = \theta_e$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \ge 0$, where $\varrho_e = M/\big(\frac{4}{3}\pi R_e^3\big)$, Ω_e is a ball of radius R_e, and $R_e > 0$ and $\theta_e > 0$ satisfy equation (1.3).

The methods used to prove the main result of the paper, Theorem 3.5, are similar to those applied in [11], [14]–[19], [21]–[23] and [2]–[6]. To prove the global existence theorem (Theorem 3.5) we use the local existence theorem of [12], the differential inequality (3.5) which is similar to the differential inequalities derived in [11], [16], [17] and [21]–[23] and the conservation laws for energy, mass and momentum which are presented together with their consequences in Section 2. Theorem 3.5 is proved without assuming any conditions on the form of the internal energy $\varepsilon = \varepsilon(\varrho,\theta)$.

Theorem 3.7 is the global existence theorem for the case $p_0 = 0$.

In Section 4 we present the global existence theorem for the case of a viscous compressible barotropic capillary fluid (Theorem 4.3).

In contrast to [22]–[25] we do not assume any conditions on the form of the pressure $p = p(\varrho)$. The case of a general $p = p(\varrho)$ and $\sigma = 0$ was examined in [21], where the global existence theorem was proved. On the other hand papers [7]–[8] are devoted to the global motion of a viscous compressible barotropic fluid in the case of a general $p = p(\varrho)$, $\sigma > 0$ and $p_0 = 0$.

Papers [10]–[11] are concerned with the global motion of a viscous compressible barotropic self-gravitating fluid in the case when $p = A\varrho^{\varkappa}$, where $A > 0$ and $\varkappa > 1$ are constants.

Finally, in [9], [12], [20] and [24] local existence theorems are proved, while [2]–[6] are devoted to the motion of a viscous compressible heat-conducting fluid both in the space $\mathbb{R}^3$ and in a fixed domain.

Now, we present the notation used in the paper. We denote by $W_2^{l,1/2}(Q_T)$ the anisotropic Sobolev–Slobodetskiĭ spaces of functions defined in Q_T, where $Q_T = \Omega^T = \Omega \times (0,T)$ ($\Omega \subset \mathbb{R}^3$ is a domain, $T < \infty$ or $T = \infty$) or $Q_T = S^T = S \times (0,T)$, $S = \partial\Omega$. We define $W_2^{l,l/2}(\Omega^T)$ as the space of functions u such that

$$\begin{aligned}\|u\|_{W_2^{l,l/2}(\Omega^T)} = \Bigg[& \sum_{|\alpha|+2i\le[l]} \|D_\xi^\alpha \partial_t^i u\|^2_{L_2(\Omega^T)} \\ & + \sum_{|\alpha|+2i=[l]} \bigg(\int_0^T \int_\Omega \int_\Omega \frac{|D_\xi^\alpha \partial_t^i u(\xi,t) - D_{\xi'}^\alpha \partial_t^i u(\xi',t)|^2}{|\xi-\xi'|^{3+2(l-[l])}} \, d\xi\, d\xi'\, dt \\ & + \int_\Omega \int_0^T \int_0^T \frac{|D_\xi^\alpha \partial_t^i u(\xi,t) - D_\xi^\alpha \partial_{t'}^i u(\xi,t')|^2}{|t-t'|^{1+2(l/2-[l/2])}} \, dt\, dt'\, d\xi \bigg) \Bigg]^{1/2} < \infty,\end{aligned}$$

where we use generalized derivatives, $D_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3}$, $\partial_{\xi_j}^{\alpha_j} = \partial^{\alpha_j}/\partial\xi_j^{\alpha_j}$ ($j = 1,2,3$), $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\partial_t^i = \partial^i/\partial t^i$ and $[l]$ is the integer part of l. In the case when l is an integer the second term in the above formula must be omitted, and in the case of $l/2$ integer also the last term is omitted.

The space $W_2^{l,l/2}(S^T)$ is defined similarly by using local charts and a partition of unity.

By $W_2^l(Q)$, where $l \in \mathbb{R}_+$, $Q = \Omega, S, S^1$ ($\Omega \subset \mathbb{R}^3$ is a bounded domain; $S = \partial\Omega$, S^1 is the unit sphere), we denote the usual Sobolev–Slobodetskiĭ spaces. To simplify notation we write

$$\|u\|_{l,Q} = \|u\|_{W_2^{l,l/2}(Q)} \quad \text{if } Q = \Omega^T \text{ or } Q = S^T,$$

$$\|u\|_{l,Q} = \|u\|_{W_2^l(Q)} \quad \text{if } Q = \Omega \text{ or } Q = S \text{ or } Q = S^1.$$

Next, we introduce the spaces $\Gamma_k^l(Q)$ and $\Gamma_k^{l,l/2}(Q)$ of functions u defined on $Q\times(0,T)$ ($T<\infty$ or $T=\infty$, $Q=\Omega,S$) such that

$$|u|_{l,k,Q}\equiv\|u\|_{\Gamma_k^l(Q)}=\sum_{i\le l-k}\|\partial_t^i u\|_{l-i,Q}<\infty$$

and

$$\|u\|_{\Gamma_k^{l,l/2}(Q)}=\sum_{2i\le l-k}\|\partial_t^i u\|_{l-2i,Q}<\infty,$$

where $l\in\mathbb{R}_+$, $k\ge 0$.

By $|u|_{l,0,p,\Omega^T}$ we denote the norm in the space $L_p(0,T;\Gamma_0^{l,l/2}(\Omega))$ and by $C_B^{2,1}(Q)$ ($Q\subset\mathbb{R}^3\times[0,\infty)$) the space of functions such that $D_x^\alpha\partial_t^i u\in C_B^0(Q)$ for $|\alpha|+2i\le 2$ (where $C_B^0(Q)$ is the space of continuous bounded functions on Q).

Finally, we introduce the seminorm

$$|u|_{\varkappa,S^T}=\bigg(\int_0^T\frac{\|u\|_{0,S}^2}{t^{2\varkappa}}dt\bigg)^{1/2}.$$

2. Conservation laws and their consequences. The following lemma is proved in [15].

Lemma 2.1. *For sufficiently regular solutions* (v,θ,ϱ) *of problem* (1.1) *we have*

$$\frac{d}{dt}\bigg[\int_{\Omega_t}\varrho\bigg(\frac{v^2}{2}+\varepsilon\bigg)dx+p_0|\Omega_t|+\sigma|S_t|\bigg]-\varkappa\int_{S_t}\overline{\theta}\,ds=\int_{\Omega_t}\varrho f\cdot v\,dx\quad (\textit{conservation of energy}),\tag{2.1}$$

where $|\Omega_t|=\operatorname{vol}\Omega_t$, $|S_t|$ *is the surface area of* S_t, *and* $\varepsilon=\varepsilon(\varrho,\theta)$ *is the internal energy per unit mass. Moreover*,

$$\frac{d}{dt}\int_{\Omega_t}\varrho x\,dx=\int_{\Omega_t}\varrho v\,dx$$

and

$$\frac{d}{dt}\int_{\Omega_t}\varrho v\cdot\eta\,dx=\int_{\Omega_t}\varrho f\cdot\eta\,dx,$$

where $\eta=a+b\times x$ *and* a,b *are arbitrary constant vectors.*

Now, assume:

$$f=0,\quad \overline{\theta}\ge 0,\tag{2.2}$$

$$\varrho_1<\varrho(x,t)<\varrho_2,\quad \theta_1<\theta(x,t)<\theta_2\quad\text{for all } x\in\overline{\Omega}_t \text{ and } t\in[0,T],\tag{2.3}$$

where T is the time of existence of a solution of problem (1.1); $0 < \varrho_1 < \varrho_2$ and $0 < \theta_1 < \theta_2$ are constants; and

$$\varepsilon_1 < \varepsilon(\varrho,\theta) < \varepsilon_2 \quad \text{for all } \varrho \in (\varrho_1,\varrho_2) \text{ and } \theta \in (\theta_1,\theta_2). \tag{2.4}$$

Integrating (2.1) with respect to t in an interval $(0,t)$ $(t \le T)$ and using (2.2)–(2.4) we get

$$\begin{aligned} &\frac{\varepsilon_1}{\varrho_2^\alpha}\int_{\Omega_t}\varrho^\beta\,dx + \int_{\Omega_t}\frac{\varrho v^2}{2}\,dx + p_0|\Omega_t| + \sigma|S_t| \\ &\le \int_\Omega \varrho_0\left(\frac{v_0^2}{2}+\varepsilon_0\right)d\xi + p_0|\Omega| + \sigma|S| + \varkappa\sup_t\int_0^t dt'\int_{S_{t'}}\overline{\theta}(s,t')\,ds \equiv d, \end{aligned} \tag{2.5}$$

where $\beta = \alpha+1$, $\alpha > 0$ is a constant, $\varepsilon_0 = \varepsilon(\varrho_0,\theta_0)$. Hence in the same way as in Lemma 2 of [15] we obtain

Lemma 2.2. *Under assumptions* (2.2)–(2.4) *the following estimate holds*:

$$\left(\frac{M^\beta\varepsilon_1}{d\varrho_2^\alpha}\right)^{1/(\beta-1)} \le |\Omega_t| \le \frac{d}{p_0}.$$

Let R_t be the radius of a ball of volume $|\Omega_t|$. Then inequality (2.5) yields

$$\frac{\varepsilon_1}{\varrho_2^\alpha}\int_{\Omega_t}\varrho^\beta\,dx + p_0|\Omega_t| + \sigma\widetilde{c}|\Omega_t|^{2/3} - d + \int_{\Omega_t}\varrho\frac{v^2}{2}\,dx + \sigma(|S_t| - 4\pi R_t^2) \le 0, \tag{2.6}$$

where $\widetilde{c} = (36\pi)^{1/3}$. Multiplying (2.6) by $|\Omega_t|^{\beta-1}$ and using (1.2) we have

$$\begin{aligned} &y(|\Omega_t|) + \frac{\varepsilon_1}{\varrho_2^\alpha}\Big[|\Omega_t|^{\beta-1}\int_{\Omega_t}\varrho^\beta\,dx - \Big(\int_{\Omega_t}\varrho\,dx\Big)^\beta\Big] \\ &\qquad + |\Omega_t|^{\beta-1}\int_{\Omega_t}\varrho\frac{v^2}{2}\,dx + \sigma|\Omega_t|^{\beta-1}(|S_t| - 4\pi R_t^2) \le 0, \end{aligned} \tag{2.7}$$

where

$$y(x) = p_0x^\beta + \sigma\widetilde{c}x^{\beta-1/3} - dx^{\beta-1} + \frac{\varepsilon_1}{\varrho_2^\alpha}M^\beta.$$

Since the last three terms in (2.7) are non-negative we have $y(|\Omega_t|) \le 0$, so we have to consider $y = y(x)$ for $x > 0$ only.

To do this introduce (as in [15])

$$D = \nu_0(\nu_0 - 2\mu_0^3),$$

where

$$\mu_0 = \frac{\widetilde{c}\sigma(\beta - 1/3)}{3p_0\beta}, \qquad \nu_0 = \frac{d(\beta-1)}{2p_0\beta}.$$

We have the following possibilities:

$$\text{if } \nu_0 \in (2\mu_0^3, \infty) \equiv I_1, \quad \text{then } D > 0, \tag{2.8}$$

$$\text{if } \nu_0 \in (\mu_0^3, 2\mu_0^3] \equiv I_2, \quad \text{then } D \le 0, \tag{2.9}$$

$$\text{if } \nu_0 \in (0, \mu_0^3] \equiv I_3, \quad \text{then } D < 0. \tag{2.10}$$

For $\nu_0 \in I_i$, we define φ_i, $i = 1, 2, 3$, by

$$\cosh \varphi_1 \equiv \frac{\nu_0}{\mu_0^3} - 1, \quad \text{where } \nu_0 \in I_1; \tag{2.11}$$

$$\cos \varphi_2 \equiv \frac{\nu_0}{\mu_0^3} - 1, \quad \text{where } \nu_0 \in I_2; \tag{2.12}$$

$$\cos \varphi_3 \equiv 1 - \frac{\nu_0}{\mu_0^3}, \quad \text{where } \nu_0 \in I_3. \tag{2.13}$$

Next, set

$$\begin{aligned} \Phi_1(\mu_0, \varphi_1, p_0, \beta, \varepsilon_1, \varrho_2, M) &= \frac{p_0 \mu_0^{3\beta}}{\beta - 1}\left(2\cosh\frac{\varphi_1}{3} - 1\right)^{3(\beta-1)} \\ &\cdot \left[2\left(\cosh\varphi_1 + 1\right) - \frac{\beta-1}{\beta-1/3}\left(2\cosh\frac{\varphi_1}{3} - 1\right)^2\right] - \frac{\varepsilon_1}{\varrho_2^\alpha} M^\beta, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \Phi_2(\mu_0, \varphi_2, p_0, \beta, \varepsilon_1, \varrho_2, M) &= \frac{p_0 \mu_0^{3\beta}}{\beta - 1}\left(2\cos\frac{\varphi_1}{3} - 1\right)^{3(\beta-1)} \\ &\cdot \left[2(\cos\varphi_2 + 1) - \frac{\beta-1}{\beta-1/3}\left(2\cos\frac{\varphi_2}{3} - 1\right)^2\right] - \frac{\varepsilon_1}{\varrho_2^\alpha} M^\beta, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \Phi_3(\mu_0, \varphi_3, p_0, \beta, \varepsilon_1, \varrho_2, M) &= \frac{p_0 \mu_0^{3\beta}}{\beta - 1}\left[2\cos\left(\frac{\pi}{3} - \frac{\varphi_3}{3}\right) - 1\right]^{3(\beta-1)} \\ &\cdot \left\{2\left(1 - \cos\varphi_2\right) - \frac{\beta-1}{\beta-1/3}\left[2\cos\left(\frac{\pi}{3} - \frac{\varphi_3}{3}\right) - 1\right]^2\right\} - \frac{\varepsilon_1}{\varrho_2^\alpha} M^\beta. \end{aligned} \tag{2.16}$$

In the same way as Theorem 1 of [15] the following theorem can be proved.

THEOREM 2.3. *Let conditions* (2.2)–(2.4) *be satisfied. Let* $\delta_0 \in (0,1)$ *be given. Assume that the parameters* μ_0, ν_0, p_0, β, ε_1, ϱ_2, M *satisfy one of the relations*

$$\nu_0 \in I_i, \quad 0 < \Phi_i(\mu_0, \varphi_i, p_0, \beta, \varepsilon_1, \varrho_2, M) \le \delta_0, \tag{2.17$_i$}$$

$i = 1, 2, 3$, *where* I_i *are defined in* (2.8)–(2.10), *and* Φ_i *are given by* (2.14)–(2.16). *Then there exists a constant* c_1 *independent of* δ_0 (*it can depend on the parameters*) *such that*

$$\operatorname*{var}_{0 \le t \le T} |\Omega_t| \le c_1 \delta, \tag{2.18}$$

where $\delta^2 = c\delta_0$, $c > 0$ *is a constant.*

Moreover, in the case $(2.17)_i$ *we have*

$$||\Omega_t| - Q_i| \le c_2\delta, \qquad t \in [0,T], \tag{2.19}$$

where $Q_1 = \mu_0^3(2\cosh(\varphi_1/3) - 1)^3$, $Q_2 = \mu_0^3(2\cos(\varphi_2/3) - 1)^3$ *and* $Q_3 = \mu_0^3[2\cos(\pi/3 - \varphi_3/3) - 1]^3$, *and* $c_2 > 0$ *is a constant independent of* δ_0.

Remark 2.4. It can be proved in the same way as in Lemma 4 of [15] that for any δ_0 sufficiently small and for any $1 \le i \le 3$ there exist parameters p_0, μ_0, ν_0, β, ε_1, ϱ_2, M such that relation $(2.17)_i$ is satisfied.

Now, consider the case $p_0 = 0$. Instead of (2.7) we have in this case

$$y_0(|\Omega_t|) + \frac{\varepsilon_1}{\varrho_2^\alpha}\Big[|\Omega_t|^{\beta-1}\int_{\Omega_t}\varrho^\beta\,dx - \Big(\int_{\Omega_t}\varrho\,dx\Big)^\beta\Big] + |\Omega_t|^{\beta-1}\int_{\Omega_t}\varrho\frac{v^2}{2}\,dx + \sigma|\Omega_t|^{\beta-1}(|S_t| - 4\pi R_t^2) \le 0,$$

where

$$y_0(x) = \sigma\widetilde{c}x^{\beta-1/3} - d_0x^{\beta-1} + \frac{\varepsilon_1}{\varrho_2^\alpha}M^\beta,$$

$$d_0 = \int_\Omega \varrho_0\Big(\frac{v_0^2}{2} + \varepsilon_0\Big)\,d\xi + \sigma|S| + \varkappa\sup_t\int_0^t dt'\int_{S_{t'}}\overline{\theta}(s,t')\,ds.$$

In this case the following theorem analogous to Theorem 2 of [15] holds:

Theorem 2.5. *Let* $p_0 = 0$ *and let assumptions* (2.2)–(2.4) *be satisfied. Moreover, assume that*

$$\Big|\int_\Omega\varrho_0\frac{v_0^2}{2}\,d\xi + \int_\Omega\varrho_0(\varepsilon(\varrho_0,\theta_0) - \varepsilon(\varrho_e,\theta_e))\,d\xi + \varkappa\sup_t\int_0^t dt'\int_{S_{t'}}\overline{\theta}(s,t')\,ds\Big| \le \delta_0,$$

$$\int_\Omega|\varrho_0 - \varrho_e|\,d\xi \le \delta_0,$$

$$||S| - |S_e|| \le \delta_0,$$

$$0 < \Big[\frac{2}{3}(\beta-1)^{3(\beta-1)/2}(\beta - 1/3)^{-(3\beta-1)/2}(\widetilde{c}\sigma)^{-(3\beta-1)/2} \cdot (\widetilde{c}\sigma|\Omega_e|^{1/3} + \varrho_e\varepsilon(\varrho_e,\theta_e))^{(3\beta-1)/2}|\Omega_e|^{(\beta-1)/2} - \frac{\varepsilon_1}{\varrho_2^\alpha}\varrho_e^\beta\Big]|\Omega_e|^\beta \le \delta_0, \tag{2.20}$$

where $\delta_0 > 0$ *is a sufficiently small constant*, $|S_e| = 4\pi R_e^2$, *and* ϱ_e, R_e, Ω_e *are introduced in Definition* 1.1. *Then*

$$\operatorname*{var}_{0\le t\le T}|\Omega_t| \le c_2\delta,$$

where $c_2 > 0$ *is a constant independent of* δ_0, $\delta^2 = c\delta_0$ *and* $c > 0$ *is a constant.*

Remark 2.6. There exist β, δ, ε_1, ϱ_2, ϱ_e, θ_e, $|\Omega_e|$ such that condition (2.20) is satisfied. In fact, assuming

$$\frac{\widetilde{c}\sigma|\Omega_e|^{-1/3}}{\beta-1}=\varrho_e\varepsilon(\varrho_e,\theta_e) \tag{2.21}$$

we have

$$\begin{aligned}&\bigg[\frac{2}{3}(\beta-1)^{3(\beta-1)/2}(\beta-1/3)^{-(3\beta-1)/2}(\widetilde{c}\sigma)^{-3(\beta-1)/2}\\&\qquad\cdot\bigg(\frac{\beta}{\beta-1}\widetilde{c}\sigma|\Omega_e|^{-1/3}\bigg)^{(3\beta-1)/2}|\Omega_e|^{(\beta-1)/2}-\frac{\varepsilon_1}{\varrho_2^\alpha}\varrho_e^\beta\bigg]|\Omega_e|^\beta\\&=\bigg[\frac{2}{3}\widetilde{c}\sigma\frac{\beta^{(3\beta-1)/2}}{(\beta-1)(\beta-1/3)^{(3\beta-1)/2}|\Omega_e|^{1/3}}-\frac{\varepsilon_1}{\varrho_2^\alpha}\varrho_e^\beta\bigg]|\Omega_e|^\beta\\&=\bigg[\frac{2}{3}\bigg(\frac{\beta}{\beta-1/3}\bigg)^{(3\beta-1)/2}\varrho_e\varepsilon(\varrho_e,\theta_e)-\frac{\varepsilon_1}{\varrho_2^\alpha}\varrho_e^\beta\bigg]|\Omega_e|^\beta.\end{aligned} \tag{2.22}$$

Taking β sufficiently close to 1 and choosing σ, ϱ_e, θ_e, $|\Omega_e|$, ε_1, ϱ_2 satisfying (2.21) and (2.22) we see that condition (2.20) also holds.

3. Global existence of solutions of problem (1.2). In [12] (see also [19]) we proved the existence of a sufficiently smooth local solution of problem (1.1). In order to show the global existence we assume the following condition:

(A) Ω_t is diffeomorphic to a ball, so S_t can be described by

$$|x|=r=R(\omega,t),\qquad \omega\in S^1, \tag{3.1}$$

where S^1 is the unit sphere and we consider the motion near the constant state (see Definition 1.1). Define

$$p_\sigma=p-p_0-\frac{2\sigma}{R},\qquad \varrho_\sigma=\varrho-\varrho_e,\qquad \vartheta_0=\theta-\theta_e.$$

Using the Taylor formula p_σ can be written as (see [17], formula (3.2))

$$p_\sigma=p_1\varrho_\sigma+p_2\vartheta_0, \tag{3.2}$$

where p_i $(i=1,2)$ are positive functions. Formula (3.2) yields

$$\|\vartheta_0\|_{0,\Omega_t}^2\le c_3(\|p_\sigma\|_{0,\Omega_t}^2+\|\varrho_\sigma\|_{0,\Omega_t}^2). \tag{3.3}$$

Next, by the Poincaré inequality we have

$$\begin{aligned}\|\varrho_\sigma\|_{0,\Omega_t}^2&\le\|\varrho-\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}-\varrho_e\|_{0,\Omega_t}^2\\&\le c_4\|\varrho_{\sigma x}\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}-\varrho_e\|_{0,\Omega_t}^2,\end{aligned} \tag{3.4}$$

where $\overline{\varrho}_{\Omega_t}=\frac{1}{|\Omega_t|}\int_{\Omega_t}\varrho\,dx$ and $c_4>0$ is a constant depending on Ω_t.

In the same way as the differential inequality (3.46) of [17] and by using (3.3)–(3.4), the following inequality can be proved:

$$
\begin{aligned}
(3.5)\qquad \frac{d\varphi}{dt} + c_0\Phi \le{}& c_5 P(X)X(1+X^3)\Big(X+\Phi+\int_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'\Big)\\
&+ c_6F + c_7\psi + c_8\|H(\cdot,0)+2/R_e\|^4_{2,S^1}\\
&+ \varepsilon c_9(\|H(\cdot,0)+2/R_e\|^2_{2,S^1} + \|R(\cdot,t)-R(\cdot,0)\|^2_{4,S^1})\\
&+ c_{10}\Big(\|R(\cdot,t)-R(\cdot,0)\|^2_{4+1/2,S^1}\Big\|\int_0^t v\,dt'\Big\|^2_{3,S_t}\\
&+ \|R(\cdot,t)-R(\cdot,0)\|^2_{3,S^1}\Big\|\int_0^t v\,dt'\Big\|^2_{4,S_t}\Big),
\end{aligned}
$$

where

$$c_{11}\varphi_0(t) \le \varphi(t) \le c_{12}\varphi_0(t)$$

and

$$
\begin{aligned}
\varphi_0(t) &= |v|^2_{3,0,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{3,0,\Omega_t} + \Big\|\int_0^t v\,dt'\Big\|^2_{4,S_t}\\
&\quad - \Big\|\int_0^t v\,dt'\Big\|^2_{0,S_t} + |v|^2_{3,1,S_t} + \|H(\cdot,0)+2/R_e\|^2_{2,S^1},\\
\Phi(t) &= |v|^2_{4,1,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{4,1,\Omega_t},\\
X(t) &= |v|^2_{3,0,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{3,0,\Omega_t} + \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt',\\
\psi(t) &= \|v\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t} + \|R(\cdot,t)-R(\cdot,0)\|^2_{0,S^1} + \|\overline{\varrho}_{\Omega_t} - \varrho_e\|^2_{0,\Omega_t},\\
F(t) &= \|r_{ttt}\|^2_{0,\mathbb{R}^3} + |r|^2_{2,0,\mathbb{R}^3} + \|r\|_{0,\mathbb{R}^3} + \|\overline{\theta}\|^2_{4,1,\mathbb{R}^3} + \|\overline{\theta}\|_{1,\mathbb{R}^3},
\end{aligned}
$$

$t \in [0,T]$ (T is the time of local existence); $0 < c_0 < 1$ is a constant depending on ϱ_1, ϱ_2, θ_1, θ_2, μ, ν and $\varkappa$; $c_i > 0$ ($i = 5,\dots,12$) are constants depending on ϱ_1, ϱ_2, θ_1, θ_2, T, $\int_0^T \|v\|^2_{3,\Omega_{t'}}\,dt'$, $\|S\|_{4+1/2}$ and on the constants from the imbedding lemmas and the Korn inequalities; $\varepsilon > 0$ is a small parameter and P is a positive continuous increasing function.

In order to prove the global existence assume also

$$(3.6)\qquad \sup_{t\in[0,T]} F(t) \le \overline{\delta},$$

where $\overline{\delta} > 0$ is sufficiently small.

Next, introduce the spaces

$$\mathfrak{N}(t) = (v,\vartheta_0,\varrho_\sigma) :\ \varphi_0(t) < \infty\},$$

$$\mathfrak{M}(t)=\Big\{(v,\vartheta_0,\varrho_\sigma):\ \varphi_0(t)+\int\limits_0^t\Phi(t')\,dt'<\infty\Big\}.$$

In [19] the following lemma is proved:

LEMMA 3.1 (see [19], Lemma 5.1). *Let the assumptions of Theorem* 4.2 *of* [12] *be satisfied. Let the initial data* v_0, ϱ_0, θ_0, S *of problem* (1.1) *be such that* $(v,\vartheta_0,\varrho_\sigma)\in\mathfrak{M}(0)$ *and* $S\in W_2^{4+1/2}$. *Let*

$$\int\limits_\Omega\varrho_0v_0(a+b\times\xi)\,d\xi=0,\qquad\int\limits_\Omega\varrho_0\xi\,d\xi=0,$$

for all constant vectors a, b. *Let condition* (A) *be satisfied and let the initial data* v_0, ϱ_0, θ_0, S *and the parameters* p_0, σ, d, β, $\varkappa$, M, ε_1, ϱ_2 (d, β, ε_1 *and* ϱ_2 *are defined in Section* 2) *be such that*

$$\varphi(0)\le\alpha_1,\qquad\omega(t)=\sup_{t'\le t}\|R(\cdot,t')-R_e\|^2_{0,S^1}\le\alpha_2\qquad\textit{for } t\le T,$$
$$\chi(0)=\|H(\cdot,0)+2/R_e\|^2_{2+1/2,S^1}\le\alpha_3,$$

where $\alpha_1,\alpha_2,\alpha_3>0$ *are sufficiently small constants*, *and* T *is the time of local existence. Then the local solution of problem* (1.1) *is such that* $(v,\vartheta_0,\varrho_\sigma)\in\mathfrak{M}(t)$ *for* $t\le T$ *and*

$$\varphi(t)+\int\limits_0^t\Phi(t')\,dt'\le c_{14}(\varphi(0)+\chi(0)+\omega(t)+\sup_{t\in[0,T]}F(t))$$
$$\le c_{14}(\alpha_1+\alpha_2+\alpha_3+\overline{\delta}).$$

Now, we prove

LEMMA 3.2. *Assume that there exists a local solution to problem* (1.1) *which belongs to* $\mathfrak{M}(t)$ *for* $t\le T$. *Let the assumptions of Lemma* 3.1 *and Theorem* 2.3 *be satisfied. Moreover*, *if* $(2.17)_i$ *holds then assume that*

$$|Q_i-|\Omega_e||\le\delta_1,\tag{3.7}$$

where $\delta_1>0$ *is sufficiently small and*

$$\int\limits_\Omega\varrho_0\frac{v_0^2}{2}\,d\xi+\int\limits_\Omega\varrho_0(\varepsilon(\varrho_0,\theta_0)-\varepsilon_1)\,d\xi+p_0(|\Omega|-Q_i+\delta_2)+\sigma[|S|-\widetilde{c}(Q_i-\delta_2)^{2/3}]+\varkappa\sup_t\int\limits_0^t dt'\int\limits_{S_{t'}}\overline{\theta}(s,t')\,ds\le\delta_3,\tag{3.8}$$

where $t\le T$, $\delta_2\in(0,1/2]$ *is a constant so small that* $Q_i-\delta_2>0$ *and assume that* δ *from* (2.19) *is so small that* $c_2\delta\le\delta_2$. *Then*

$$\psi(t)\le\delta_4\qquad\textit{for } t\le T,\tag{3.9}$$

where $\delta_4 = c_{15}(\delta + \delta_1 + \delta_3 + \delta'\alpha_1)$, $c_{15} > 0$ *is a constant depending on* ϱ_1, ϱ_2, θ_1, θ_2, β, d; δ *is the constant from estimate* (2.18), α_1 *is the constant from Lemma* 3.1 *and* $\delta' \in (0,1)$ *is a sufficiently small constant.*

Proof. First, assumption (3.7) and estimate (2.19) yield

$$||\Omega_t| - |\Omega_e|| \le c_2\delta + \delta_1.$$

Hence, by Lemma 2.2,

$$\|\overline{\varrho}_{\Omega_t} - \varrho_e\|_{0,\Omega_t}^2 \le \frac{\varrho_e^2}{|\Omega_t|}(c_2\delta + \delta_1)^2 \le \varrho_e^2\left(\frac{d\varrho_2^\alpha}{M^\beta\varepsilon_1}\right)^{1/(\beta-1)}(c_2\delta + \delta_1)^2. \tag{3.10}$$

Next, integrating (2.1) with respect to t in an interval $(0,t)$ $(t \le T)$ we get

$$\begin{aligned}\int_{\Omega_t} \varrho\left(\frac{v^2}{2} + \varepsilon\right) dx + p_0|\Omega_t| + \sigma|S_t| \\ \le \int_{\Omega} \varrho_0\left(\frac{v^2}{2} + \varepsilon(\varrho_0,\theta_0)\right) d\xi + p_0|\Omega| + \sigma|S| + \varkappa \sup_t \int_0^t dt' \int_{S_{t'}} \overline{\theta}(s,t')\, ds.\end{aligned} \tag{3.11}$$

Now, let R_t be the radius of a ball of volume $|\Omega_t|$. Then by (2.19),

$$\widetilde{c}(Q_i - \delta_2)^{2/3} \le 4\pi R_t^2 \le \widetilde{c}(Q_i + \delta_2)^{2/3},$$

where $\widetilde{c} = (36\pi)^{1/3}$. Hence

$$\begin{aligned}|S_t| - \widetilde{c}(Q_i - \delta_2)^{2/3} = |S_t| - 4\pi R_t^2 + 4\pi R_t^2 \\ - \widetilde{c}(Q_i - \delta_2)^{2/3} \ge 0 \quad \text{for } t \le T.\end{aligned} \tag{3.12}$$

Using (2.19), (2.3)–(2.4), (3.12) and assumption (3.8) in (3.11) we obtain

$$\int_{\Omega_t} \varrho\frac{v^2}{2}\, dx + \int_{\Omega_t} \varrho(\varepsilon - \varepsilon_1)\, dx + p_0(|\Omega_t| - Q_i + \delta_2) + \sigma[|S_t| - \widetilde{c}(Q_i - \delta_2)^{2/3}] \le \delta_3.$$

Hence

$$\|v\|_{0,\Omega_t}^2 \le \frac{2}{\varrho_1}\delta_3. \tag{3.13}$$

Next, using the same argument as in the proof of Lemma 5.2 of [23] we obtain the estimate

$$\|p_\sigma\|_{0,\Omega_t}^2 \le c_{16}\alpha_1\delta' + c(\delta')\delta_3, \tag{3.14}$$

where $c_{16} > 0$ is a constant, α_1 is the constant from Lemma 3.1, $\delta' \in (0,1)$ is a sufficiently small constant, and $c(\delta')$ is a decreasing function of δ'.

Finally, the estimate

$$\|R(\omega,t)-R_t\|_{1,S^1}^2 \le c_{17}\delta_3 \quad \text{for } t\in[0,T] \tag{3.15}$$

follows from Lemma 2.4 of [23]. Hence by (3.15) and (2.18) we have

$$\begin{aligned}\|R(\omega,t)-R(0,t)\|_{1,S^1}^2 &\le \|R(\omega,t)-R_t\|_{1,S^1}^2 \\ &\quad + c_{18}|R_t-R_0|^2+\|R(0,t)-R_0\|_{1,S^1}^2 \\ &\le c_{19}\delta_3+c_{20}\delta,\end{aligned} \tag{3.16}$$

where $R_0=\big(\frac{3}{4\pi}|\Omega|\big)^{1/3}$.

By (3.10), (3.13), (3.14) and (3.16) we get (3.9).

This completes the proof. ■

REMARK 3.3. In the case $\varepsilon=c_v\theta$ ($c_v>0$ is a constant), $p=R\varrho\theta$ ($R>0$ is a constant) assumption (3.7) is satisfied if

$$M|\Omega_e|^\alpha\bigg|(\beta-1)c_v\theta_1\bigg(\frac{\varrho_e}{\varrho_2}\bigg)^\alpha-R\theta_e\bigg|<c_{21}\delta_5, \tag{3.17}$$

where $c_{21}>0$ is a constant and $\delta_5>0$ is sufficiently small.

Proof. Consider Q_i (where $i=1,2,3$) and set $x_0=Q_i$. Then the equation determining x_0 has the form (see equation (40) of [15])

$$p_0x_0^\beta+\widetilde{c}\sigma(\beta-1/3)\beta^{-1}x_0^{\beta-1/3}-(\beta-1)\beta^{-1}\,dx_0^{\beta-1}=0. \tag{3.18}$$

Moreover, from assumption $(2.17)_i$ it follows (see [15]) that

$$0\le -y(x_0)=-\bigg(p_0x_0^\beta+\widetilde{c}\sigma x_0^{\beta-1/3}-dx_0^{\beta-1}+\frac{\varepsilon_1}{\varrho_2^\alpha}M^\beta\bigg)\le\delta_0. \tag{3.19}$$

Applying (3.18) in (3.19) we get

$$0\le p_0x_0^\beta+\frac{2}{3}\widetilde{c}\sigma x_0^{\beta-1/3}-(\beta-1)\frac{\varepsilon_1}{\varrho_2^\alpha}M^\beta\le(\beta-1)\delta_0. \tag{3.20}$$

In this case equation (1.3) takes the form

$$R\varrho_e\theta_e=\frac{2\sigma}{R_e}+p_0.$$

Hence

$$p_0|\Omega_e|^\beta+\frac{2}{3}\widetilde{c}\sigma|\Omega_e|^{\beta-1/3}-R\varrho_e\theta_e|\Omega_e|^\beta=0, \tag{3.21}$$

where we have used the fact that $2\big(\frac{4}{3}\pi\big)^{1/3}=\frac{2}{3}\widetilde{c}$.

In view of (3.20)–(3.21) we see that assumption (3.7) is satisfied if (3.17) holds and δ_0 is sufficiently small. ■

Now, Lemmas 3.1, 3.2 and inequality (3.5) imply

LEMMA 3.4 (see Lemma 5.4 of [19]). *Let the assumptions of Lemmas* 3.1–3.2 *be satisfied. Moreover, assume*

$$\varphi(0) \le \alpha_1, \qquad \|H(\cdot,0) + 2/R_e\|^2_{2,S^1} \le \overline{\alpha}.$$

Then for sufficiently small α_1, $\overline{\alpha}$, $\overline{\delta}$, δ_3 (*where* $\overline{\delta}$ *is the constant from assumption* (3.6) *and* δ_3 *is the constant from Lemma* 3.2) *we have*

$$\varphi(t) \le \alpha_1 \quad \text{for } t \le T.$$

Now, we formulate the main result of the paper.

THEOREM 3.5. *Let* $\nu > \frac{1}{3}\mu > 0$, $\varkappa > 0$, $c_v = \varepsilon_\theta > 0$, $c_v \in C^2(\mathbb{R}^2_+)$, $\varepsilon \in C^1(\mathbb{R}^2_+)$, $p \in C^3(\mathbb{R}^2_+)$, $p_\varrho > 0$, $p_\theta > 0$, $f = 0$, $\overline{\theta} \ge 0$. *Suppose the assumptions of the local existence theorem* (*Theorem* 4.2 *of* [12]) *with* $r, \overline{\theta} \in C^{2,1}_B(\mathbb{R}^3_+ \times [0,\infty))$ *are satisfied and the following compatibility conditions hold*:

$$D^\alpha_\xi \partial^i_t(\mathbb{T}\overline{n} - \sigma H\overline{n} + p_0\overline{n})|_{t=0,S} = 0, \qquad |\alpha| + i \le 2,$$
$$D^\alpha_\xi \partial^i_t(\overline{n}\cdot\nabla\theta - \overline{\theta})|_{t=0,S} = 0, \qquad |\alpha| + i \le 2.$$

Let $(v, \vartheta_0, \varrho_\sigma) \in \mathfrak{N}(0)$ *and*

$$\varphi(0) \le \alpha_1, \tag{3.22}$$

$$\|v_0\|^2_{4,\Omega} \le \alpha_1. \tag{3.23}$$

Assume that

(3.24) $\quad l > 0$ *is a constant such that* $\varrho_e - l > 0$, $\theta_e - l > 0$ *and*

$$\varepsilon_1 < \varepsilon(\varrho,\theta) < \varepsilon_2 \quad \text{for } \varrho \in (\varrho_1,\varrho_2),\ \theta \in (\theta_1,\theta_2),$$

where $\varrho_1 = \varrho_e - l$, $\theta_1 = \theta_e - l$, $\varrho_2 = \varrho_e + l$, $\theta_2 = \theta_e + l$,

$$F(t) \le \overline{\delta} \quad \text{for } t \ge 0, \tag{3.25}$$

F *occurs in inequality* (3.5), *and*

$$\int_\Omega \varrho_0\, d\xi = M, \qquad \int_\Omega \varrho_0 \xi\, d\xi = 0, \qquad \int_\Omega \varrho_0 v_0 (a + b\times\xi)\, d\xi = 0, \tag{3.26}$$

for all constant vectors a, b.

Moreover, let the parameters ν_0, μ_0, β, ε_1, ϱ_2, M *satisfy one of the relations*

$$\nu_0 \in I_i \quad \text{and} \quad 0 < \Phi_i(\mu_0, \varphi_i, p_0, \beta, \varepsilon_1, \varrho_2, M) \le \delta_0, \tag{3.27$_i$}$$

$i = 1,2,3$, (*where* I_i *are defined by* (2.8)–(2.10) *and* Φ_i *are defined by* (2.14)–(2.16)) *and assume the following conditions*:

$$|Q_i - |\Omega_e|| \le \delta_1 \tag{3.28}$$

and

$$\int_\Omega \varrho_0 \frac{v_0^2}{2}\,d\xi + \int_\Omega \varrho_0(\varepsilon(\varrho_0,\theta_0)-\varepsilon_1)\,d\xi + p_0(|\Omega_t| - Q_i + \delta_2) + \sigma[|S| - \widetilde{c}(Q_i-\delta_2)^{2/3}] + \varkappa \sup_t \int_0^t dt' \int_{S_{t'}} \overline{\theta}(s,t')\,ds \le \delta_3 \quad \textit{for } \nu_0 \in I_i, \tag{3.29}$$

where $\delta_2 \in (0,1/2]$ *is a constant so small that* $Q_i - \delta_2 > 0$.

Moreover, assume that Ω *is diffeomorphic to a ball and let* S *be described by* $|\xi| = \widetilde{R}(\omega)$, $\omega \in S^1$ (S^1 *is the unit sphere*), *where* $\widetilde{R}$ *satisfies*

$$\sup_{S^1} |\nabla \widetilde{R}|^2 + \|\widetilde{R} - R_e\|^2_{0,S^1} \le \alpha_2, \tag{3.30}$$

and R_e *is the solution of equation* (1.3).

Finally, assume that $S \in W_2^{4+1/2}$ *and it is very close to a sphere, so*

$$\|H(\cdot,0) + 2/R_e\|^2_{2,S^1} \le \alpha_3. \tag{3.31}$$

Then for sufficiently small constants α_i $(i=1,2,3)$, δ_0, δ_i $(i=1,2,3)$ *and* $\overline{\delta}$ *there exists a global solution of problem* (1.1) *such that* $(v,\vartheta_0,\varrho_\sigma) \in \mathfrak{M}(t)$ *for* $t \in \mathbb{R}_+$, $S_t \in W_2^{4+1/2}$ *for* $t \in \mathbb{R}_+$ *and*

$$\varphi(t) \le \alpha_1, \quad \|H(\cdot,t) + 2/R_e\|^2_{2,S^1} \le \alpha_3 \quad \textit{for } t \in \mathbb{R}_+. \tag{3.32}$$

P r o o f. Similarly to the case when $\sigma = 0$ (see [21]) the theorem is proved step by step using the local existence in a fixed time interval.

By Remark 3.2 of [19] the local solution of problem (1.1) satisfies the estimate

$$\begin{aligned}\|u\|^2_{4,\Omega^T} + \|\eta_\sigma\|^2_{3,\Omega^T} + |\eta_\sigma|^2_{3,0,\infty,\Omega^T} + \|\gamma_0\|^2_{4,\Omega^T} &\le \varphi_1(T,K_0)(\|v_0\|^2_{3,\Omega} + \|\varrho_{\sigma 0}\|^2_{3,\Omega} + \|\vartheta_{00}\|^2_{3,\Omega} \\ &\quad + \|k\|^2_{2,\Omega^T} + \|\overline{\Gamma}\|^2_{3-1/2,S^T} + |D^2_{\xi,t}\overline{\Gamma}|^2_{1/2,S^T} + \|k(0)\|^2_{1,\Omega} \\ &\quad + \|H(\cdot,0) + 2/R_e\|^2_{2+1/2,S^1}) \\ &\le \varphi_2(T,K_0)(\alpha_1 + \overline{\delta}),\end{aligned} \tag{3.33}$$

where $\varrho_{\sigma 0} = \varrho_0 - \varrho_e$, $\vartheta_{00} = \theta_0 - \theta_e$, $u(\xi,t) = v(X_u(\xi,t),t)$, $\eta_\sigma(\xi,t) = \varrho_\sigma(X_u(\xi,t),t)$, $\gamma_0(\xi,t) = \vartheta(X_u(\xi,t),t)$, $k(\xi,t) = r(X_u(\xi,t),t)$, $\overline{\Gamma}(\xi,t) = \overline{\theta}(X_u(\xi,t),t)$; φ_1 and φ_2 are continuous increasing functions of their arguments, K_0 is a constant such that $K_0 > c(\|\varrho_0\|_{3,\Omega} + |\varrho_0|_{\infty,\Omega} + |1/\varrho_0|_{\infty,\Omega} + \|v_0\|_{3,\Omega} + \|\theta_0\|_{3,\Omega} + \|u_t(0)\|_{1,\Omega} + \|\gamma_{0t}(0)\|_{1,\Omega})$, $c > 0$ is a constant.

Next, using (3.33) and assumption (3.22) we get

$$\begin{aligned}(3.34)\quad & \|u(t)\|_{3,\Omega}^2+\|\eta_\sigma(t)\|_{3,\Omega}^2+\|\gamma_0(t)\|_{3,\Omega}^2\\ &\le c_{22}(\|u\|_{4,\Omega^T}^2+\|\eta_\sigma\|_{3,\Omega^T}^2+|\eta_\sigma|_{3,0,\infty,\Omega^T}^2+\|\gamma_0\|_{4,\Omega^T}^2+\|v_0\|_{3,\Omega}^2\\ &\quad+\|\vartheta_{00}\|_{3,\Omega}^2+\|\varrho_{\sigma 0}\|_{3,\Omega}^2)\\ &\le c_{22}(\alpha_1+\overline{\delta})\varphi_2+c_{22}\alpha_1\le\varphi_3(T,K_0)(\alpha_1+\overline{\delta}),\end{aligned}$$

where φ_3 is a continuous increasing functions of its arguments.

Hence

$$(3.35)\qquad |u|_{\infty,\Omega^T}^2+|\eta_\sigma|_{\infty,\Omega^T}^2+|\gamma_0|_{\infty,\Omega^T}^2\le(\alpha_1+\overline{\delta})c(\Omega)\varphi_3,$$

where $c(\Omega)>0$ is a constant from the imbedding lemma.

Assume now that α_1 and $\overline{\delta}$ are so small that

$$(3.36)\qquad [(\alpha_1+\overline{\delta})c(\Omega)\varphi_3]^{1/2}<l$$

(where l is the constant from assumption (3.24)). Then by (3.35) we have

$$(3.37)\qquad \varrho_1<\varrho(x,t)<\varrho_2\quad\text{for } x\in\overline{\Omega}_t,\ t\in[0,T],$$

$$(3.38)\qquad \theta_1<\theta(x,t)<\theta_2\quad\text{for } x\in\overline{\Omega}_t,\ t\in[0,T],$$

where ϱ_1, ϱ_2, θ_1 and θ_2 are defined in assumption (3.24).

Now notice that assumptions (3.22)–(3.23) and the boundary condition

$$\mathbb{T}(v_0,p_\sigma(\varrho_{\sigma 0},\vartheta_{00}))\overline{n}_0=H(\cdot,0)+2/R_e$$

yield

$$(3.39)\qquad \|H(\cdot,0)+2/R_e\|_{2+1/2,S^1}^2<c_{23}\alpha_1,$$

where $c_{23}>0$ is a constant depending on μ, ν and the constants from the imbedding theorem (which depend on $|\Omega|$ and the shape of Ω).

Next, in the same way as in the proof of Theorem 5.5 of [19], using assumptions (3.22)–(3.24), $(3.27)_i$, (3.30), (3.31), inequalities (3.37), (3.38), (3.33) and Lemma 2.4 of [23] we deduce

$$(3.40)\qquad \|R(\cdot,t)-R_e\|_{0,S^1}^2\le\alpha_4\quad\text{for } t\le T,$$

where $\alpha_4\to 0$ as $\delta_0\to 0$ and $\delta_3\to 0$. Moreover, from assumptions $(3.27)_i$, (3.24), inequalities (3.37)–(3.38) and Theorem 2.3 it follows that

$$(3.41)\qquad \operatorname*{var}_{0\le t\le T}|\Omega_t|\le c_1\delta,$$

where $\delta\to 0$ if $\delta_0\to 0$.

Thus, estimates (3.40)–(3.41) yield that the volume and the shape of Ω_t do not change much in $[0,T]$.

Now, the assumptions of the theorem, estimates (3.39)–(3.40), Lemmas 3.1–3.2 and 3.4 and boundary condition $(1.1)_4$ yield

$$(3.42)\qquad \varphi(t)\le\alpha_1,\quad \|H(\cdot,t)+2/R_e\|_{2,S^1}^2\le\alpha_3\quad\text{for } t\le T.$$

Hence, by the local existence theorem (Th. 4.2 of [12]) and Remark 3.2 of [19] we obtain the existence of a local solution of problem (1.1) in $[T,2T]$ satisfying the estimate

$$\|u\|^2_{4,\Omega_T\times(T,2T)}+\|\eta_\sigma\|^2_{3,\Omega_T\times(T,2T)}+\boldsymbol{|}\eta_\sigma\boldsymbol{|}^2_{3,0,\infty,\Omega_T\times(T,2T)} + \|\gamma_0\|^2_{4,\Omega_T\times(T,2T)}\le\varphi_2(T,K_0)(\alpha_1+\overline{\delta}), \tag{3.43}$$

where $\eta_\sigma=\eta-\varrho_e$, $\gamma_0=\Gamma-\theta_e$, and u,η and Γ denote v,ϱ and θ written in the Lagrangian coordinates $\xi\in\Omega_T$ connected with the Eulerian coordinates x by the relation

$$x=\xi+\int_T^t v(x,t')\,dt'=\xi+\int_T^t u(x,t')\,dt'.$$

In view of (3.42), (3.43) and the inequality

$$\begin{aligned}&\|u(t)\|^2_{3,\Omega_T}+\|\eta_\sigma(t)\|^2_{3,\Omega_T}+\|\gamma_0(t)\|^2_{3,\Omega_T}\\ &\quad\le c_{22}(\|u\|^2_{4,\Omega_T\times(T,2T)}+\|\eta_\sigma\|^2_{3,\Omega_T\times(T,2T)}+\boldsymbol{|}\eta_\sigma\boldsymbol{|}^2_{3,0,\infty,\Omega_T\times(T,2T)}\\ &\qquad+\|\gamma_0\|^2_{4,\Omega_T\times(T,2T)}+\|u(T)\|^2_{3,\Omega_T}+\|\eta_\sigma(T)\|^2_{3,\Omega_T}+\|\gamma_0(T)\|^2_{3,\Omega_T})\end{aligned}$$

(where by estimate (3.40) the constant c_{22} is the same as in (3.34)), we have

$$|u|^2_{\infty,\Omega_T\times(T,2T)}+|\eta_\sigma|^2_{\infty,\Omega_T\times(T,2T)}+|\gamma_0|^2_{\infty,\Omega_T\times(T,2T)} \le(\alpha_1+\overline{\delta})c(\Omega_T)\varphi_3, \tag{3.44}$$

where $c(\Omega_T)>0$ is a constant from the imbedding lemma and by (3.40) the estimate

$$[(\alpha_1+\overline{\delta})c(\Omega_T)\varphi_3]^{1/2}<l$$

holds (here l is the constant from assumption (3.24)).

Then by (3.44) we have

$$\varrho_1<\varrho(x,t)<\varrho_2\quad\text{for } x\in\overline{\Omega}_t,\ t\in[0,2T] \tag{3.45}$$

$$\theta_1<\theta(x,t)<\theta_2\quad\text{for } x\in\overline{\Omega}_t,\ t\in[0,2T], \tag{3.46}$$

where ϱ_1, ϱ_2, θ_1 and θ_2 are defined in assumption (3.24).

Finally, in the same way as in [19] (see the proof of Th. 5.5) by using Lemma 5.3 of [19] we obtain the estimate

$$\|H(\cdot,T)+2/R_e\|^2_{2+1/2,S^1}<c_{23}\alpha_1, \tag{3.47}$$

where $c_{23}>0$ is the same constant as in (3.39).

Next, in view of (3.45), (3.46), (3.42) and assumptions (3.24), $(3.27)_i$, (3.30), (3.31) and Lemma 2.4 of [23] we get

$$\|R(\cdot,t)-R_e\|^2_{0,S^1}\le\alpha_4\quad\text{for } t\le 2T. \tag{3.48}$$

Hence estimates (3.47), (3.48), the assumptions of the theorem and Lemmas 3.1, 3.2 and 3.4 yield

$$\varphi(t) \le \alpha_1, \quad \|H(\cdot,t) + 2/R_e\|^2_{2,S^1} \le \alpha_3 \quad \text{for } t \le 2T.$$

Continuing in the same way we prove the global existence and estimates (3.32).

This completes the proof. ∎

Remark 3.6. In the case $\varepsilon = c_v\theta$ ($c_v > 0$ is a constant) and $p = R\varrho\theta$ ($R > 0$ is a constant) instead of (3.28) we assume (3.17).

In the case $p_0 = 0$ using Theorem 2.5 we obtain

Theorem 3.7. *Let* $\nu > \frac{1}{3}\mu > 0$, $\varkappa > 0$, $c_v = \varepsilon_\theta > 0$, $c_v \in C^2(\mathbb{R}^2_+)$, $\varepsilon \in C^1(\mathbb{R}^2_+)$, $p \in C^3(\mathbb{R}^2_+)$, $p_\varrho > 0$, $p_\theta > 0$, $f = 0$, $\overline{\theta} \ge 0$. *Suppose the assumptions of the local existence theorem* (*Theorem* 4.2 *of* [12]) *with* $r, \overline{\theta} \in C^{2,1}_B(\mathbb{R}^3 \times [0,\infty))$ *are satisfied and the following compatibility conditions hold*:

$$D^\alpha_\xi \partial^i_t (\mathbb{T}\overline{n} - \sigma H \overline{n})|_{t=0,S} = 0, \quad |\alpha| + i \le 2,$$
$$D^\alpha_\xi \partial^i_t (\overline{n} \cdot \nabla\theta - \overline{\theta})|_{t=0,S} = 0, \quad |\alpha| + i \le 2.$$

Let $(v, \vartheta_0, \varrho_\sigma) \in \mathfrak{N}(0)$ *and assumptions* (3.22), (3.23), (3.24)–(3.26), (3.28), (3.30), (3.31) *hold. Moreover*, *assume that*

$$\Big| \int_\Omega \varrho_0 \frac{v_0^2}{2}\, d\xi + \int_\Omega \varrho_0(\varepsilon(\varrho_0,\theta_0) - \varepsilon(\varrho_e,\theta_e))\, d\xi + \varkappa \sup_t \int_0^t dt' \int_{S_{t'}} \overline{\theta}(s,t')\, ds \Big| \le \delta_0,$$

$$\int_\Omega |\varrho_0 - \varrho_e|\, d\xi \le \delta_0,$$

$$|\,|S| - |S_e|\,| \le \delta_0,$$

$$0 < \bigg[\frac{2}{3}(\beta-1)^{3(\beta-1)/2}(\beta - 1/3)^{-(3\beta-1)/2}(\widetilde{c}\sigma)^{-(3\beta-1)/2}$$
$$\cdot (\widetilde{c}\sigma|\Omega_e|^{1/3} + \varrho_e\varepsilon(\varrho_e,\theta_e))^{(3\beta-1)/2}|\Omega_e|^{(\beta-1)/2} - \frac{\varepsilon_1}{\varrho_2^\alpha}\varrho_e^\beta\bigg]|\Omega_e|^\beta \le \delta_0.$$

Then for sufficiently small constants α_i $(i = 1,2,3)$, δ_0, δ_i $(i = 1,2,3)$ *and* $\overline{\delta}$ *there exists a global solution of problem* (1.1) *such that* $(v, \vartheta_0, \varrho_\sigma) \in \mathfrak{M}(t)$ *for* $t \in \mathbb{R}_+$, $S_t \in W_2^{4+1/2}$ *for* $t \in \mathbb{R}_+$ *and*

$$\varphi(t) \le \alpha_1, \quad \|H(\cdot,t) + 2/R_e\|^2_{2,S^1} \le \alpha_3 \quad \textit{for } t \in \mathbb{R}_+.$$

4. Global existence in the case of barotropic fluid. In this section we consider the motion of a drop of a viscous compressible barotropic fluid whose free boundary is governed by surface tension. The motion of such a drop is described by the following system of equations:

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = 0 &&\text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 &&\text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} &&\text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\varphi_t/|\nabla\varphi| &&\text{on } \widetilde{S}^T,\\
&\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0, &&\text{in } \Omega,
\end{aligned}
\tag{4.1}
$$

where $p = p(\varrho)$.

Assume that $p'(\varrho) > 0$ for $\varrho > 0$ and consider the equation

$$
p\biggl(\frac{M}{\frac{4}{3}\pi R_e^3}\biggr) = p_0 + \frac{2\sigma}{R_e}. \tag{4.2}
$$

We assume that (4.2) is solvable with respect to $R_e > 0$ and introduce the following definition.

Definition 4.1. By a *constant* (*equilibrium*) *state* we mean a solution (v,ϱ,Ω_t) of problem (4.1) such that $v = 0$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \ge 0$, where $\varrho_e = M/\bigl(\frac{4}{3}\pi R_e^3\bigr)$, Ω_e is a ball of radius R_e and R_e is a solution of equation (4.2).

Similarly to the case of a heat conducting fluid, in this case inequality (3.5) also holds (see Theorem 4.13 of [23]) with

$$
c'\overline{\varphi}_0(t) \le \overline{\varphi}(t) \le c''\overline{\varphi}_0(t)
$$

(where $c', c'' > 0$ are constants depending on $\varrho_1, \varrho_2, p, \Omega_t$ and S_t for $t \le T$) and

$$
\begin{aligned}
\overline{\varphi}_0(t) &= |v|^2_{3,0,\Omega_t} + |\varrho_\sigma|^3_{3,0,\Omega_t} + \Bigl\|\int_0^t v\,dt'\Bigr\|^2_{4,S_t} - \Bigl\|\int_0^t v\,dt'\Bigr\|^2_{0,S_t}\\
&\quad + |v|^2_{3,1,S_t} + \|H(\cdot,0) + 2/R_e\|^2_{2,S^1},\\
\overline{\Phi}(t) &= |v|^2_{4,1,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t},\\
\overline{X}(t) &= |v|^2_{3,0,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t} + + \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt',\\
\overline{\psi}(t) &= \|v\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t} + \|R(\cdot,t) - R(\cdot,0)\|^2_{0,S^1}.
\end{aligned}
$$

In this case we use the spaces

$$
\begin{aligned}
\overline{\mathfrak{N}}(t) &= \{(v,\varrho_\sigma) : \overline{\varphi}_0(t) < \infty\},\\
\overline{\mathfrak{M}}(t) &= \Bigl\{(v,\varrho_\sigma) : \overline{\varphi}_0(t) + \int_0^t \overline{\Phi}(t')\,dt' < \infty\Bigr\}.
\end{aligned}
$$

Now assume

$$\varrho_1 < \varrho(x,t) < \varrho_2 \quad \text{for all } x \in \overline{\Omega}_t \text{ and } t \in [0,T], \tag{4.3}$$

where T is the time of existence of a solution of problem (4.1); $0 < \varrho_1 < \varrho_2$. Moreover, set

$$|\Omega_1| = \frac{M}{\varrho_2}, \quad |\Omega_2| = \frac{M}{\varrho_1} \quad \text{and} \quad |S_1| = 4\pi R_1^2,$$

where R_1 is the radius of a ball of volume $|\Omega_1|$, i.e. $|\Omega_1| = \frac{4}{3}\pi R_1^3$. Then by (4.3) we have

$$|\Omega_1| < |\Omega_t| < |\Omega_2| \quad \text{for } t \in [0,T]. \tag{4.4}$$

Next, by (4.4),

$$|S_t| - |S_1| = |S_t| - 4\pi R_t^2 + \widetilde{c}(|\Omega_t|^{2/3} - |\Omega_1|^{2/3}) > 0 \quad \text{for } t \in [0,T],$$

where $\widetilde{c} = (36\pi)^{1/3}$.

In [23] (see Lemma 2.1) it is proved that sufficiently smooth solutions of problem (4.1) satisfy

$$\frac{d}{dt}\bigg[\int\limits_{\Omega_t} \varrho\bigg(\frac{v^2}{2} + h(\varrho)\bigg)\,dx + p_0|\Omega_t| + \sigma|S_t|\bigg] + \frac{\mu}{2}E(v) + (\nu-\mu)\|\operatorname{div} v\|_{0,\Omega_t}^2 = 0, \tag{4.5}$$

where $h(\varrho) = \int (p(\varrho)/\varrho^2)\,d\varrho$ and $E(v) = \int_{\Omega_t} (v_{ix_j} + v_{jx_i})^2\,dx$. Identity (4.5) is analogous to the energy conservation law (2.1).

In order to prove the global existence theorem we use (4.5) and Lemmas 5.1 and 5.4 of [23] which are similar to Lemmas 3.1 and 3.4. We also use Lemma 5.3 of [23] and the following lemma analogous to Lemma 3.2.

LEMMA 4.2. *Assume that there exists a local solution of problem* (4.1) *which belongs to* $\overline{\mathfrak{M}}(t)$ *for* $t \le T$. *Let the assumptions of Theorem* 6.2 *of* [24] *be satisfied. Let the initial data* v_0, ϱ_0, S *of problem* (4.1) *be such that* $(v, \varrho_\sigma) \in \overline{\mathfrak{N}}(0)$ *and* $S \in W_2^{4+1/2}$. *Let assumptions* (A), (3.22) *with* φ *replaced by* $\overline{\varphi}$ *and* (4.3) *be satisfied. Moreover, let*

$$h \in C^1(\mathbb{R}_+), \tag{4.6}$$

$$h_1 < h(\varrho) < h_2 \quad \textit{for all } \varrho \in (\varrho_1, \varrho_2), \tag{4.7}$$

$$\int\limits_{\Omega} \varrho_0 \frac{v_0^2}{2}\,d\xi + \int\limits_{\Omega} \varrho_0(h(\varrho_0) - h_1)\,d\xi + p_0(|\Omega| - |\Omega_1|) + \sigma(|S| - |S_1|) \le \delta_0. \tag{4.8}$$

Then

$$\overline{\psi}(t) \le \delta \quad \textit{for } t \le T,$$

where $\delta = c(\delta_0 + \delta'\alpha_1)$.

P r o o f. Integrating (4.5) with respect to t in an interval $(0,t)$ $(t\le T)$ we get

$$\int\limits_{\Omega_t}\varrho\frac{v^2}{2}\,dx+\int\limits_{\Omega_t}\varrho h(\varrho)\,dx+p_0|\Omega_t|+\sigma|S_t| = \int\limits_{\Omega}\varrho_0\frac{v_0^2}{2}\,d\xi+\int\limits_{\Omega}\varrho_0 h(\varrho_0)\,d\xi+p_0|\Omega|+\sigma|S|. \tag{4.9}$$

Hence, in view of (4.3), (4.4), (4.7), (4.8) and (1.2), identity (4.9) yields

$$\int\limits_{\Omega_t}\varrho\frac{v^2}{2}\,dx+\int\limits_{\Omega_t}\varrho(h(\varrho)-h_1)\,dx+p_0(|\Omega_t|-|\Omega_1|)+\sigma(|S_t|-|S_1|)\le\delta_0.$$

Therefore

$$\|v\|^2_{0,\Omega_t}\le\frac{2}{\varrho_1}\delta_0.$$

The estimates for $\|p_\sigma\|^2_{0,\Omega_t}$ and $\|R(\omega,t)-R(0,t)\|^2_{1,S^1}$ are obtained in the same way as in Lemma 3.2.

This completes the proof. ■

Now, using Lemmas 5.1, 5.3, 5.4 of [23] and Lemma 4.2 we get

THEOREM 4.3. *Let* $\nu>\frac{1}{3}\mu>0$, $p\in C^3(\mathbb{R}_+)$ *and* $p'>0$ *for* $\varrho>0$. *Let the assumptions of the local existence theorem* (*Theorem* 6.2 *of* [24]) *be satisfied and assume the following compatibility condition holds*:

$$D^\alpha_\xi\partial^i_t(\mathbb{T}\overline{n}-\sigma H\overline{n}+p_0\overline{n})|_{t=0,S}=0,\qquad |\alpha|+i\le 2.$$

Let $(v,\varrho_\sigma)\in\overline{\mathfrak{N}}(0)$ *and*

$$\overline{\varphi}(0)\le\alpha_1,\qquad \|v_0\|^2_{4,\Omega}\le\alpha_1.$$

Assume that

$$\varrho_1<h(\varrho)<\varrho_2\quad \textit{for any } \varrho\in(\varrho_1,\varrho_2),$$

where $\varrho_1=\varrho_e-l$, $\varrho_2=\varrho_e+l$, ϱ_e *is introduced in Definition* 1.1 *and* $l>0$ *is a constant such that* $\varrho_e-l>0$, *and*

$$\int\limits_{\Omega_t}\varrho_0\frac{v_0^2}{2}\,d\xi+\int\limits_{\Omega_t}\varrho_0(h(\varrho)-h_1)\,d\xi+p_0(|\Omega_t|-|\Omega_1|)+\sigma(|S_t|-|S_1|)\le\delta_0,$$

$$\int\limits_{\Omega}\varrho_0\,d\xi=M,\qquad\int\limits_{\Omega}\varrho_0\xi\,d\xi=0,\qquad\int\limits_{\Omega}\varrho_0v_0(a+b\times\xi)\,d\xi=0,$$

for all constant vectors a, b. *Moreover*, *assume that* Ω *is diffeomorphic to a ball and let* S *be described by* $|\xi|=\widetilde{R}(\omega)$, $\omega\in S^1$ (S^1 *is the unit sphere*), *where* $\widetilde{R}$ *satisfies* (3.30) *with* R_e *which is the solution of* (4.2).

Finally, *assume that* $S\in W_2^{4+1/2}$ *and that condition* (3.31) *is satisfied.*

Then for sufficiently small constants α_i $(i = 1, 2, 3)$ *and* δ_0 *there exists a global solution of problem* (4.1) *such that* $(v, \varrho_\sigma) \in \overline{\mathfrak{M}}(t)$ *for* $t \in \mathbb{R}_+$, $S \in W_2^{4+1/2}$ *for* $t \in \mathbb{R}_+$ *and*

$$\overline{\varphi}(t) \le \alpha_1, \quad \|H(\cdot, t) + 2/R_e\|^2_{2,S^1} \le \alpha_3 \quad \textit{for } t \in \mathbb{R}_+.$$

The proof is analogous to the proof of Theorem 3.5.

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