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GLOBAL EXISTENCE AND BLOW UP OF SOLUTIONS FOR A COMPLETELY COUPLED FUJITA TYPE SYSTEM OF REACTION-DIFFUSION EQUATIONS

Abstract. We examine the parabolic system of three equations

$$\begin{aligned} u_t - \Delta u &= v^p, \\ v_t - \Delta v &= w^q, \qquad x \in \mathbb{R}^N,\ t > 0, \\ w_t - \Delta w &= u^r, \end{aligned}$$

with p, q, r positive numbers, $N \ge 1$, and nonnegative, bounded continuous initial values. We obtain global existence and blow up unconditionally (that is, for any initial data). We prove that if $pqr \le 1$ then any solution is global; when $pqr > 1$ and $\max(\alpha, \beta, \gamma) \ge N/2$ (where α, β, γ are defined in terms of p, q, r) then every nontrivial solution exhibits a finite blow up time.

1. Introduction and main results. In this paper we consider the system

$$\begin{aligned} u_t - \Delta u &= v^p, \\ v_t - \Delta v &= w^q, \\ w_t - \Delta w &= u^r, \end{aligned} \tag{1.1}$$

for $t > 0$, $x \in \mathbb{R}^N$ with $N \ge 1$, $p, q, r > 0$ and

$$\begin{aligned} u(0, x) &= u_0(x), \\ v(0, x) &= v_0(x), \qquad x \in \mathbb{R}^N, \\ w(0, x) &= w_0(x), \end{aligned} \tag{1.2}$$

where u_0, v_0, w_0 are nonnegative, continuous, bounded functions.

Let us recall that the system (1.1)–(1.2) has a nonnegative classical solution in $S_T = [0, T) \times \mathbb{R}^N$ for some $T > 0$ (cf. for instance a related

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argument in [EH]). Our primary concern is to describe two cases: $T=\infty$, when the system has bounded solutions for any S_t, $t>0$ (global solutions), and $T<\infty$, when solutions are unbounded beyond T (they blow up in a finite time T). In this paper, we discuss these cases in terms of p, q, r and N only. Some additional dependence on the initial data u_0, v_0, w_0 (which implies that both situations coexist) will be considered in another paper.

The Cauchy problem

$$\begin{aligned} &u_t-\Delta u=u^p, && t>0,\ x\in\mathbb{R}^N,\\ &u(0,x)=u_0(x), && x\in\mathbb{R}^N, \end{aligned}$$

has been analyzed by several authors (see [F1], [F2]).

Also the system of two reaction-diffusion equations has been dealt with in case of coupled systems.

For instance, in [EH] and [AHV] global existence and blow up results were discussed for the problem

$$\begin{aligned} &u_t-\Delta u=v^p, && (x,t)\in\mathbb{R}^N\times(0,T),\\ &v_t-\Delta v=u^q, && (x,t)\in\mathbb{R}^N\times(0,T),\\ &u(0,x)=u_0(x)\ge 0,\\ &v(0,x)=v_0(x)\ge 0, \end{aligned}$$

while in [EL] a more general system was studied:

$$\begin{aligned} &u_t-\Delta u=u^{p_1}v^{q_1}, && (x,t)\in\mathbb{R}^N\times(0,T),\\ &v_t-\Delta v=u^{p_2}v^{q_2}, && (x,t)\in\mathbb{R}^N\times(0,T),\\ &u(0,x)=u_0(x)\ge 0,\\ &v(0,x)=v_0(x)\ge 0. \end{aligned}$$

Our goal is to extend Fujita type global existence-nonexistence theorems to systems of three equations.

Let

$$A=\begin{bmatrix}0&p&0\\0&0&q\\r&0&0\end{bmatrix}. \tag{1.3}$$

We denote by $(\alpha,\beta,\gamma)^t$ the solution of $(A-I)X=(1,1,1)^t$. We can easily find that

$$\alpha=\frac{1+p+pq}{pqr-1},\qquad \beta=\frac{1+q+qr}{pqr-1},\qquad \gamma=\frac{1+r+rp}{pqr-1}, \tag{1.4}$$

where

$$\det(A-I)=pqr-1. \tag{1.5}$$

We formulate

THEOREM 1. *Suppose* $\det(A-I) \le 0$. *Then every solution of* (1.1)–(1.2) *is global.*

THEOREM 2. *Suppose* $\det(A - I) > 0$. *If* $\max(\alpha, \beta, \gamma) \ge N/2$ *then* (1.1)–(1.2) *never has nontrivial global solutions.*

We prove Theorem 1 in Section 3; Theorem 2 is proved by contradiction in Section 4. Section 2 contains some auxiliary tools.

2. Preliminaries. As we have mentioned, solutions of (1.1)–(1.2) are classical in some S_T (that is, $(u, v, w)(x, t) \in C^{2,1}(\mathbb{R}^N \times (0, T))$). Such solutions satisfy the formulas

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(t-s)v(s)^p\,ds,\\ v(t) &= S(t)v_0 + \int_0^t S(t-s)w(s)^q\,ds, \\ w(t) &= S(t)w_0 + \int_0^t S(t-s)u(s)^r\,ds, \end{aligned} \tag{2.1}$$

where $S(t)$ is an operator semigroup and $S(t)\xi_0$ is the unique solution of $\xi_t - \Delta\xi = 0$, $\xi(0) = \xi_0(x)$, where

$$S(t)\xi_0(x) = \int_{\mathbb{R}^N} (4\pi t)^{-N/2} \exp\left(-\frac{|x-y|^2}{4t}\right)\xi_0(y)\,dy.$$

REMARK 2.1. *If* (u, v, w) *is a nontrivial solution of* (1.1)–(1.2) *on* $[0, T]$, *then there exists* $t_0 \in (0, T)$ *such that* $u(x, \tau) > 0$, $v(x, \tau) > 0$ *and* $w(x, \tau) > 0$ *for* $(x, \tau) \in \mathbb{R}^N \times (t_0, T)$.

Proof. Let (x_i, t_i), $i = 1, 2, 3$, be such that $u(x_1, t_1) > 0$, $v(x_2, t_2) > 0$ and $w(x_3, t_3) > 0$. Let $t_0 = \max(t_1, t_2, t_3)$. From formula $(2.1)_1$ for $\tau \in (t_1, T - t_1)$,

$$\begin{aligned} u(\tau) &= S(\tau - t_1)u(t_1) + \int_0^{\tau - t_1} S(\tau - t_1 - \eta)v(\eta)^q\,d\eta \\ &\ge S(\tau - t_1)u(t_1), \end{aligned}$$

and since $S(\tau) > 0$ we get $u(\tau) > 0$ on $\mathbb{R}^N$ for $\tau > t_1$. Similarly, $v(\tau) > 0$ on $\mathbb{R}^N$ for $\tau > t_2$ and $w(\tau) > 0$ on $\mathbb{R}^N$ for $\tau > t_3$. ∎

We also need

LEMMA 2.1. *Let* $(u_0, v_0, w_0) \neq (0,0,0)$ *and* (u,v,w) *be a solution of* (1.1)–(1.2). *Then we can choose* $\tau = \tau(u_0, v_0, w_0)$ *and constants* $c, a > 0$ *such that* $\min(u(\tau), v(\tau), w(\tau)) \geq ce^{-a|x|^2}$.

Proof. We use the same argument as in [EL] or [EH]. Let, for instance, $u_0 \neq 0$. We can assume that for some $R > 0$,

$$\nu = \inf\{u_0(x) : |x| < R\} > 0.$$

From $(2.1)_1$ we have

$$u(t) \geq S(t)u_0 \geq \nu(4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right) \int\limits_{|y|\leq R} \exp\left(-\frac{|y|^2}{4t}\right) dy.$$

Defining

$$\overline{u}(t) = u(t+\tau_0) \quad \text{for some } \tau_0 > 0,$$
$$\alpha_1 = \frac{1}{4\tau_0}, \quad c_1 = \nu(4\pi\tau_0)^{N/2} \int\limits_{|y|\leq R} \exp\left(-\frac{|y|^2}{4\tau_0}\right) dy$$

we have

$$\overline{u}(0) = u(\tau_0) > c_1 \exp(-\alpha_1 |x|^2).$$

In the same way we obtain

$$\overline{v}(0) > c_2 \exp(-\alpha_2|x|^2), \quad \overline{w}(0) > c_3 \exp(-\alpha_3|x|^2).$$

Finally, we have to choose α and c suitable for u, v, w to ensure

$$(u(x,\tau_0), v(x,\tau_0), w(x,\tau_0))^t > c\exp(-\alpha|x|^2)(1,1,1)^t$$

and this concludes the proof. ■

3. Global existence. In this section we prove Theorem 1, considering separately the cases $\det(A-I) = 0$ and $\det(A-I) < 0$.

(a) $\det(A-I) = 0$ (by (1.5), this is equivalent to $pqr = 1$). We want to find a global supersolution to system (1.1)–(1.2) of the form

$$\begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} Ae^{\alpha_1 t} \\ Be^{\beta_1 t} \\ Ce^{\gamma_1 t} \end{pmatrix}, \tag{3.1}$$

where, for given u_0, v_0, w_0, we choose A, B, C so large that $A \geq \|u_0\|_{L^\infty}$, $B \geq \|v_0\|_{L^\infty}$ and $C \geq \|w_0\|_{L^\infty}$. Let α_1, β_1, γ_1 be positive constants such that

$$\overline{u}_t - \Delta\overline{u} \geq \overline{v}^p, \quad \overline{v}_t - \Delta\overline{v} \geq \overline{w}^q, \quad \overline{w}_t - \Delta\overline{w} \geq \overline{u}^r, \tag{3.2}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Then (3.1) satisfies (3.2) for all $t > 0$ if

$$\begin{aligned} \alpha_1 &> A^{-1}B^p \exp((p\beta_1 - \alpha_1)t), \\ \beta_1 &> B^{-1}C^q \exp((q\gamma_1 - \beta_1)t), \\ \gamma_1 &> C^{-1}A^r \exp((r\alpha_1 - \gamma_1)t). \end{aligned} \tag{3.3}$$

If we take $\beta_1 = \alpha_1/p$ and $\gamma_1 = \alpha_1/(pq) = r\alpha_1$ (the last equality follows from $pqr = 1$), then (3.2) holds for α_1 large enough. ■

(b) $\det(A - I) < 0$ (by (1.5), that means $pqr < 1$). We are looking for a global supersolution of the form

$$\begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} A(t+t_0)^{\alpha_1} \\ B(t+t_0)^{\beta_1} \\ C(t+t_0)^{\gamma_1} \end{pmatrix}, \tag{3.4}$$

for some positive constants $A, B, C, \alpha_1, \beta_1, \gamma_1$ such that the inequalities (3.2) with $(\overline{u}, \overline{v}, \overline{w})$ given by (3.4) hold for all $x \in \mathbb{R}^N$ and $t > 0$. We have to choose t_0 sufficiently large to satisfy

$$\overline{u}(x,0), \overline{v}(x,0), \overline{w}(x,0)) \geq (u_0, v_0, w_0) \tag{3.5}$$

for $x \in \mathbb{R}^N$.

Substituting (3.4) into (3.2) we obtain the following conditions:

$$\alpha_1 - p\beta_1 \geq 1, \quad \beta_1 - q\gamma_1 \geq 1, \quad \gamma_1 - r\alpha_1 \geq 1. \tag{3.6}$$

Let us remark that (3.6) has the form $(A-I)(-\alpha_1, -\beta_1, -\gamma_1)^t \geq (1,1,1)^t$ with A given by (1.3). Set $(\alpha_1, \beta_1, \gamma_1) = -(\alpha, \beta, \gamma)$ for α, β, γ defined by (1.4). Since $pqr < 1$ we see that $(\alpha_1, \beta_1, \gamma_1)$ are positive. Thus, (3.4) satisfies (3.2) and (3.5) provided that

$$A\alpha_1 \geq B^p, \quad B\beta_1 \geq C^q, \quad C\gamma_1 \geq A^r,$$

and t_0 is large enough. Then every nonnegative solution of (1.1)–(1.2) with bounded initial values is global. ■

4. Blow up (proof of Theorem 2). Without loss of generality we assume henceforth $r \leq q \leq p$. Thus, by (1.4), $\max(\alpha, \beta, \gamma) = \alpha$ for $pqr > 1$.

LEMMA 4.1. *Let $(u(t), v(t), w(t))$ be a bounded solution of (1.1)–(1.2) in some strip S_T with $0 < T \leq \infty$. Let $pqr \geq 1$ and $r \geq 1$. Then there exists a positive constant C such that*

$$t^\alpha \|S(t)u_0\|_\infty \leq C, \quad t \in [0, T), \tag{4.1}$$

where C depends on p, q, r only and α is given by $(1.4)_1$.

P r o o f. Using $(2.1)_1$ in $(2.1)_3$ we get

$$w(t) \geq \int_0^t S(t-s)(S(s)u_0)^r\,ds$$

and by the Jensen inequality for $r \geq 1$,

$$w(t) \geq \int_0^t (S(t-s)S(s)u_0)^r\,ds = \int_0^t (S(t)u_0)^r\,ds = t(S(t)u_0)^r. \tag{4.2}$$

We substitute (4.2) in $(2.1)_2$ (ignoring the first term on the right-hand side) and by the Jensen inequality we obtain

$$\begin{aligned} v(t) &\geq \int_0^t S(t-s)(s(S(s)u_0)^r)^q\,ds \geq \int_0^t s^q(S(t)u_0)^{rq}\,ds \\ &\geq \frac{1}{q+1}(S(t)u_0)^{rq}t^{q+1}. \end{aligned} \tag{4.3}$$

Using (4.3) in $(2.1)_1$ we can write

$$\begin{aligned} u(t) &\geq S(t-s)\left[\frac{1}{q+1}(S(s)u_0)^{rq}s^{q+1}\right]^p ds \\ &\geq \left(\frac{1}{q+1}\right)^p (S(t)u_0)^{pqr}\frac{t^{p(q+1)+1}}{p(q+1)+1}. \end{aligned} \tag{4.4}$$

Using again the lower bound (4.4) in $(2.1)_3$ gives

$$w(t) \geq \left(\frac{1}{q+1}\right)^{rp}(S(t)u_0)^{pqr^2}\frac{1}{(p(q+1)+1)^r}\cdot\frac{t^{rp(q+1)+r+1}}{rpq+rp+r+1}. \tag{4.5}$$

Continuing this procedure gives

$$\begin{aligned} v(t) &\geq \frac{1}{(q+1)^{rpq}}\cdot\frac{1}{(p(q+1)+1)^{rq}}\cdot\frac{1}{(rp(q+1)+r+1)^q} \\ &\quad\times\frac{t^{(q+1)(rpq+1)+rq}(S(t)u_0)^{pq^2r^2}}{(q+1)(rpq+1)+rq}, \\ u(t) &\geq \frac{1}{(q+1)^{rp^2q^2}}\cdot\frac{1}{(pq+p+1)^{rpq}}\cdot\frac{1}{(rpq+rp+r+1)^{pq}} \\ &\quad\times\frac{1}{((q+1)(rpq+1)+rq)^p}\cdot\frac{1}{(rpq+1)(p(q+1)+1)} \\ &\quad\times(S(t)u_0)^{p^2q^2r^2}t^{(rpq+1)(pq+p+1)}. \end{aligned} \tag{4.6}$$

Iterating this scheme, we obtain, using (1.4),

$$u(t) \geq A_kB_kC_k(S(t)u_0)^{(pqr)^k}t^{\alpha\delta(1+rpq+\ldots+(rpq)^{k-1})}, \tag{4.7}$$

where

$$(4.8)\qquad \begin{aligned} A_k &= \frac{1}{(\alpha\delta)^{(rpq)^{k-1}}}\left[\frac{1}{(rpq+1)\alpha\delta}\right]^{(rpq)^{k-2}} \\ &\quad\times\left[\frac{1}{((rpq)^2+rpq+1)\alpha\delta}\right]^{(rpq)^{k-3}} \\ &\quad\times\ldots\times\frac{1}{((rpq)^{k-1}+\ldots+rpq+1)\alpha\delta}, \\ B_k &= \frac{1}{(q+1)^{p(rpq)^{k-1}}}\cdot\frac{1}{(r\alpha\delta+1)^{\frac{1}{r}(pqr)^{k-1}}} \\ &\quad\times\frac{1}{(r\alpha\delta(rpq+1)+1)^{\frac{1}{r}(rpq)^{k-2}}} \\ &\quad\times\frac{1}{(r\alpha\delta(1+rpq+\ldots+(rpq)^{k-2})+1)^{pq}}, \\ C_k &= \frac{1}{(rq\alpha\delta+q+1)^{\frac{1}{rq}(rpq)^{k-1}}} \\ &\quad\times\frac{1}{(rq\alpha\delta(rpq+1)+q+1)^{\frac{1}{rq}(rpq)^{k-2}}} \\ &\quad\times\ldots\times\frac{1}{(rq\alpha\delta(1+pqr+\ldots+(pqr)^{k-2})+q+1)^{p}}. \end{aligned}$$

Setting $pqr=z$ we can rewrite $(4.8)_1$ as

$$A_k=\frac{1}{(\alpha\delta)^{\frac{z^k-1}{z-1}}}\left(\frac{1}{1+z}\right)^{z^{k-2}}\left(\frac{1}{1+z+z^2}\right)^{z^{k-3}}\cdots\frac{1}{1+z+\ldots+z^{k-1}}$$

so

$$(4.9)\qquad A_k=\frac{1}{(\alpha\delta)^{\frac{z^k-1}{z-1}}}\prod_{j=1}^{k-1}\left(\frac{z-1}{z^{j+1}-1}\right)^{z^{k-j-1}}.$$

Using $\alpha\delta(1+z+\ldots+z^j)>1+q$ for $j\geq 0$ we can estimate:

$$(4.10)\qquad \begin{aligned} B_k &\geq \frac{1}{(q+1)^{pz^{k-1}}}\cdot\frac{1}{[(r+1)\alpha\delta]^{\frac{1}{r}z^{k-1}}}\cdot\frac{1}{[(r+1)\alpha\delta(1+z)]^{\frac{1}{r}z^{k-2}}} \\ &\quad\times\ldots\times\frac{1}{[(r+1)\alpha\delta(1+z+\ldots+z^{k-2})]^{pq}} \\ &= \frac{1}{(q+1)^{pz^{k-1}}}\cdot\frac{1}{[(r+1)\alpha\delta]^{\frac{1}{r}\cdot\frac{z^k-z}{z-1}}}\left[\prod_{j=1}^{k-2}\left(\frac{z-1}{z^{j+1}-1}\right)^{z^{k-j-1}}\right]^{\frac{1}{r}}, \end{aligned}$$

$$C_k \ge \frac{1}{[(rq+1)\alpha\delta]^{\frac{1}{rq}z^{k-1}}} \cdot \frac{1}{[(rq+1)\alpha\delta(z+1)]^{\frac{1}{rq}z^{k-2}}} \times \ldots \times \frac{1}{[(rq+1)\alpha\delta(1+z+\ldots+z^{k-2})]^{p}} = \frac{1}{[(rq+1)\alpha\delta]^{\frac{1}{rq}\cdot\frac{z^k-z}{z-1}}}\Bigg[\prod_{j=1}^{k-2}\bigg(\frac{z-1}{z^{j+1}-1}\bigg)^{z^{k-j-1}}\Bigg]^{\frac{1}{rq}}. \tag{4.11}$$

Substituting (4.9)–(4.11) into (4.7) we get

$$u(t) \ge (S(t)u_0)^{z^k} t^{\alpha\delta\frac{z^k-1}{z-1}} \frac{1}{(q+1)^{pz^{k-1}}} \cdot \frac{1}{(r+1)^{\frac{1}{r}\cdot\frac{z^k-z}{z-1}}} \times \frac{1}{(rq+1)^{\frac{1}{rq}\cdot\frac{z^k-z}{z-1}}} \cdot \frac{1}{(\alpha\delta)^{\frac{z^k-1}{z-1}(1+\frac{1}{r}+\frac{1}{rq})-\frac{1}{r}(1+\frac{1}{q})}} \times \Bigg[\prod_{j=1}^{k-2}\bigg(\frac{z-1}{z^{j+1}-1}\bigg)^{z^{k-j-1}}\Bigg]^{1+\frac{1}{r}+\frac{1}{rq}} \frac{z-1}{z^k-1}.$$

We infer that

$$t^{\alpha\delta\frac{z^k-1}{z^k(z-1)}} S(t)u_0 \le (q+1)^{\frac{p}{z}}(r+1)^{\frac{1}{rz^k}\cdot\frac{z^k-z}{z-1}} \times (rq+1)^{\frac{1}{rq}\cdot\frac{z^k-z}{z^k(z-1)}}(\alpha\delta)^{\frac{1}{z^k}[\frac{z^k-1}{z-1}(1+\frac{1}{r}+\frac{1}{rq})-\frac{1}{r}(1+\frac{1}{q})]} \times \Bigg[\prod_{j=1}^{k-2}\bigg(\frac{z-1}{z^{j+1}-1}\bigg)^{z^{j+1}}\Bigg]^{1+\frac{1}{r}+\frac{1}{rq}} \bigg(\frac{z^k-1}{z-1}\bigg)^{1/z^k} \|u(t)\|_\infty^{1/z^k}.$$

Letting $k\to\infty$ and using $\|u(t)\|_\infty < \infty$ we obtain in the limit

$$t^\alpha \|S(t)u_0\|_\infty \le c < \infty,$$

where $c = c(p,q,r)$ only. ■

LEMMA 4.2. *Assume that* $pqr > 1$, $p > 1$ *and* $r \le q < 1$. *Let* $(u(t), v(t), w(t))$ *be as in Lemma* 4.1. *Then there exists a constant* C *such that*

$$t^{rq\alpha}\|S(t)u_0^{rq}\|_\infty \le C \quad \textit{and} \quad C = C(p,q,r). \tag{4.12}$$

Proof. By the Jensen inequality for $r < 1$ we have

$$w(t) \ge tS(t)u_0^r \quad \text{and} \quad v(t) \ge \frac{1}{q+1} S(t)u_0^{rq} t^{q+1}.$$

Repeating the iteration as in Lemma 4.1 we obtain

$$u(t) \ge A_k B_k C_k (S(t)u_0^{rq})^{\frac{1}{rq}(pqr)^k} t^{\alpha\delta(1+\ldots+(pqr)^{k-1})} \tag{4.13}$$

and A_k, B_k, C_k are given by (4.9)–(4.11). So we estimate as before and letting $k \to \infty$ we conclude that

$$t^{rq\alpha}\|S(t)u_0^{rq}\|_\infty \le C. \ \blacksquare$$

We complete the result by considering the case $r < 1 < q$.

LEMMA 4.3. *Let $pqr > 1$ with $p > 1$ and $r < 1 < q$, and let u, v, w be as in Lemma* 4.1. *Then*

$$t^{r\alpha}\|S(t)u_0^r\|_\infty \le C, \tag{4.14}$$

where the constant C depends on p, q, r only.

Proof. We argue as in the previous lemma, starting from $w(t) \ge tS(t)u_0^r$ and

$$v(t) \ge \frac{1}{q+1}(S(t)u_0^r)^q t^{q+1}$$

so

$$u(t) \ge A_k B_k C_k (S(t)u_0^r)^{\frac{1}{r}(pqr)^k} t^{\alpha\delta(1+pqr+\ldots+(pqr)^{k-1})}$$

and we get (4.14) as before. $\blacksquare$

LEMMA 4.4. *Let $pqr > 1$, and let $(u(t), v(t), w(t))$ be a bounded solution of* (1.1)–(1.2) (*as in Lemma* 4.1). *Then we can find a constant $C > 0$, $C = C(p, q, r)$, such that for $t > 0$,*

$$\begin{aligned} &t^\alpha\|S(t)u(t)\|_\infty \le C < \infty && \text{if } 1 < r \le q \le p,\\ &t^{rq\alpha}\|S(t)u(t)^{rq}\|_\infty \le C < \infty && \text{if } r \le q < 1 < p,\\ &t^{r\alpha}\|S(t)u(t)^r\|_\infty \le C < \infty && \text{if } r < 1 < q \le p. \end{aligned} \tag{4.15}$$

Proof. For $\tau, t \ge 0$ we can rewrite $(2.1)_3$ in the form

$$\begin{aligned} w(t+\tau) &= S(t+\tau)u_0 + \int_0^{t+\tau} S(t+\tau-s)u(s)^r\,ds \\ &= S(t)u(\tau) + \int_0^t S(t-s)u(\tau+s)^r\,ds \end{aligned}$$

and similarly for v and u. Hence we can replace u_0 by $u(\tau)$ in (4.1), (4.12) and (4.14); setting $t = \tau$, we get the conclusion. $\blacksquare$

LEMMA 4.5. *Suppose that $\alpha \ge N/2$ and $pqr > 1$. Then every nontrivial solution of* (1.1)–(1.2) *blows up in finite time.*

Proof. Assume that there exists a bounded solution of (1.1)–(1.2) with $(u_0, v_0, w_0) > (0, 0, 0)$ and $\alpha \ge N/2$, $pqr > 1$. By Lemma 2.1 we can find $c, a > 0$ such that

$$u_0(x) \ge ce^{-a|x|^2}. \tag{4.16}$$

We will also use the following equality:

$$S(t)e^{-a|x|^2} = (1+4at)^{-N/2} \exp\left(\frac{-a|x|^2}{1+4at}\right). \tag{4.17}$$

First, we consider the case $0 < r < 1 \le q < p$. Using (4.16) in (4.17) we get

$$u(t) \ge S(t)u_0 \ge c(1+4at)^{-N/2} \exp\left(\frac{-a|x|^2}{1+4at}\right).$$

Now, from $(2.1)_3$, the last inequality and (4.17) we obtain

$$\begin{aligned} w(t) &\ge \int_0^t S(t-s)u(s)^r\, ds \\ &\ge c^r \int_0^t S(t-s)(1+4as)^{-Nr/2} \exp\left(\frac{-ar|x|^2}{1+4as}\right) ds \\ &= c^r \int_0^t (1+4as)^{-Nr/2}\left(1+\frac{4ar}{1+4as}(t-s)\right)^{-N/2} \\ &\quad\times \exp\left(\frac{-ar|x|^2}{1+4as+4ar(t-s)}\right) ds \\ &= c^r \int_0^t (1+4as)^{-N(r-1)/2}(1+4as+4ar(t-s))^{-N/2} \\ &\quad\times \exp\left(\frac{-ar|x|^2}{1+4as+4ar(t-s)}\right) ds. \end{aligned}$$

We note that $f(s) = 1+4as+4ar(t-s)$ is increasing (because $f'(s) = 4a(1-r) > 0$) so that

$$w(t) \ge c^r(1+4at)^{-N/2} \exp\left(\frac{-ar|x|^2}{1+4art}\right) \int_0^t (1+4as)^{-N(r-1)/2}\, ds.$$

Integrating, we obtain

$$\begin{aligned} &w(t) \\ &\quad \ge \frac{c^r}{4a(1-N(r-1)/2)}(1+4at)^{-N/2} \exp\left(\frac{-ar|x|^2}{1+4art}\right)(4at)^{1-N(r-1)/2}. \end{aligned} \tag{4.18}$$

Now, we use (4.18) in $(2.1)_2$ to get (by (4.17))

$$v(t) \geq \frac{c^{rq}}{[4a(1-N(r-1)/2)]^q} \int_0^t (1+4as)^{-Nq/2}(4as)^{q(1-N(r-1)/2)} \times S(t-s)\exp\left(\frac{-arq|x|^2}{1+4ars}\right) ds$$

$$= \frac{c^{rq}}{[4a(1-N(r-1)/2)]^q} \times \int_0^t (1+4as)^{-Nq/2}(1+4ars)^{N/2}(4ars)^{q(1-N(r-1)/2)} \times (1+4ars+4arq(t-s))^{-N/2}\exp\left(\frac{-arq|x|^2}{1+4ar+4arq(t-s)}\right) ds.$$

Consider $g(s) = 1+4ars+4arq(t-s)$; as $g'(s) = 4ar(1-q) < 0$ we deduce that

$$v(t) \geq \frac{c^{rq}}{[4a(1-N(r-1)/2)]^q}(1+4arqt)^{-N/2}\exp\left(\frac{-arq|x|^2}{1+4art}\right) \times \int_0^t (1+4as)^{-Nq/2}(1+4ars)^{N/2}(4as)^{q(1-N(r-1)/2)}\, ds.$$

To integrate, let us remark that $(1+4ars)^{N/2} \geq r^{N/2}(1+4as)^{N/2}$ and $4as > \frac{1}{2}(1+4as)$ for $s > 1/(4a)$. Denoting by c_1 the new constant such that

$$c_1 := \frac{c^{rq}}{[4a(1-N(r-1)/2)]^q} r^{N/2}\left(\frac{1}{2}\right)^{q(1-N(r-1)/2)},$$

we have

$$v(t) \geq c_1(1+4arqt)^{-N/2}\exp\left(\frac{-arq|x|^2}{1+4art}\right)\int_0^t (1+4as)^{q-N(qr-1)/2}\, ds$$

and finally

$$v(t) \geq c(1+4arqt)^{-N/2}(4at)^{1+q-N(qr-1)/2}\exp\left(\frac{-arq|x|^2}{1+4art}\right), \tag{4.18}$$

where $c = c(p,q,r,N/2,a)$ is a constant.

We need a lower bound for $u(t)$, so we substitute (4.19) into $(2.1)_1$ to get

$$u(t) \geq c^p \int_0^t (1+4arqs)^{-Np/2}(4as)^{p(1+q-N(qr-1)/2)} \times S(t-s)\exp\left(\frac{-apqr|x|^2}{1+4ars}\right) ds$$

$$= c^p \int_0^t (1+4arqs)^{-Np/2}(4as)^{p(1+q-N(qr-1)/2)}$$

$$\times \left(1 + \frac{4arpq(t-s)}{1+4ars}\right)^{-N/2}$$
$$\times \exp\left(\frac{-apqr|x|^2}{1+4ars+4arpq(t-s)}\right) ds.$$

The last equality follows by (4.17). Now, consider $h(s) = 1 + 4ars + 4arpq(t-s)$; note that $h'(s) = 4ar(1-pq) < 0$ so as before we get

$$u(t) \ge c^p (1+4arpqt)^{-N/2} \exp\left(\frac{-apqr|x|^2}{1+4art}\right) \times \int_0^t (1+4arqs)^{-Np/2} (4as)^{p(1+q-N(qr-1)/2)} (1+4ars)^{N/2}\, ds. \tag{4.19}$$

Using again

$$4at > \tfrac{1}{2}(1+4at) \quad \text{for } t > 1/(4a),$$
$$(1+4ars)^{N/2} \ge r^{N/2}(1+4as)^{N/2} \quad \text{for } r < 1,$$

and noting that

$$(1+4arqt)^{-N/2} \ge (1+4apqrt)^{-N/2} \ge (pqr)^{-N/2}(1+4at)^{-N/2}$$

holds since $p > 1$ and $pqr > 1$, we obtain from (4.20), for $t > 1/(4a)$,

$$u(t) \ge c(1+4at)^{-N/2} \exp\left(\frac{-apqr|x|^2}{1+4art}\right) \int_{1/(4a)}^{t} (1+4as)^{\varrho}\, ds,$$

where

$$\begin{aligned}\varrho &= -Np/2 + p(1+q-N(qr-1)/2) + N/2 \\ &= p + pq - N(pqr-1)/2 \ge -1\end{aligned}$$

by the assumption that $\alpha \ge N/2$.

So we infer that

$$u(t) \ge c(1+4at)^{-N/2} \exp\left(\frac{-apqr|x|^2}{1+4art}\right) \log\left(\frac{4at+1}{2}\right) \tag{4.20}$$

for $t > 1/(4a)$.

It now follows by (4.17) that

$$\begin{aligned} S(t)u(t)^r &\ge c(1+4at)^{-Nr/2} \exp\left(\frac{-apqr^2|x|^2}{1+4art}\right) \\ &\quad \times S(t)\left[\log\left(\frac{1+4at}{2}\right)\right]^r \\ &= c(1+4at)^{-Nr/2}(1+4ar(1+rpq)t)^{-N/2}(1+4art)^{N/2} \\ &\quad \times \left[\log\left(\frac{1+4at}{2}\right)\right]^r \exp\left(\frac{-ar^2pq|x|^2}{1+4ar(1+rpq)t}\right)\end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&\geq c(1+4at)^{-Nr/2}\left(\frac{1+4art}{(1+4art)(1+pqr)}\right)^{N/2}\\
&\quad\times\left[\log\left(\frac{1+4at}{2}\right)\right]^{r}\exp\left(\frac{-ar^2pq|x|^2}{1+4ar(1+rpq)t}\right).
\end{aligned}
$$

Putting $x=0$ in (4.22) we get

$$
(1+4at)^{Nr/2}S(t)u(t,0)^r \geq \frac{c}{(1+pqr)^{N/2}}\left[\log\left(\frac{1+4at}{2}\right)\right]^{r}
$$

and therefore, for $t>\max(1,1/(4a))$ and since $\alpha\geq N/2$,

$$
t^{r\alpha}S(t)u(t,0)^r \geq c\left[\log\left(\frac{1+4at}{2}\right)\right]^{r}. \tag{4.22}
$$

It remains to notice that as $t\to\infty$, the right-hand side of (4.23) diverges, and so does the left-hand side. But this contradicts $(4.15)_3$. Thus, $u(t)$ must become unbounded, and by (2.1), $v(t)$ and $w(t)$ also blow up in finite time.

Now, we discuss the remaining cases.

In the case $0<r\leq q<1<p$ we argue as before to get, instead of (4.21),

$$
u(t)\geq c(1+4at)^{-N/2}\exp\left(\frac{-arpq|x|^2}{1+4arqt}\right)\log\left(\frac{4at+1}{2}\right)
$$

and

$$
\begin{aligned}
S(t)u(t)^{rq} &\geq c(1+4at)^{-Nqr/2}\exp\left(\frac{-ar^2q^2p|x|^2}{1+4arqt}\right)\\
&\quad\times S(t)\left[\log\left(\frac{1+4at}{2}\right)\right]^{qr}\\
&= c(1+4at)^{-Nqr/2}\left(\frac{1+4arqt}{1+4arq(1+pqr)t}\right)^{N/2}\\
&\quad\times\exp\left(\frac{-ar^2q^2p|x|^2}{1+4arq(1+pqr)t}\right)\left[\log\left(\frac{1+4at}{2}\right)\right]^{qr}.
\end{aligned}
$$

Thus, for $x=0$,

$$
S(t)u(t,0)^{qr}(1+4at)^{qrN/2}\geq c\left[\log\left(\frac{1+4at}{2}\right)\right]^{qr},
$$

which implies, for $t>\max(1,1/(4a))$, as $\alpha\geq N/2$,

$$
t^{qr\alpha}S(t)u(t,0)^{qr}\geq c\left[\log\left(\frac{1+4at}{2}\right)\right]^{qr}. \tag{4.23}
$$

Now, we see that (4.24) is incompatible with $(4.15)_2$ for t large enough.

Finally, we consider the case $1 < r \le q \le p$. Then instead of (4.21) we infer that

$$u(t) \ge c(1+4at)^{-N/2} \exp\left(\frac{-arpq|x|^2}{1+4at}\right) \log\left(\frac{4at+1}{2}\right),$$

whence

$$S(t)u(t)(1+4at)^{N/2} \ge c \exp\left(\frac{-arpq|x|^2}{1+4a(1+pqr)t}\right) \log\left(\frac{4at+1}{2}\right).$$

Setting $x = 0$ and using $\alpha \ge N/2$, we have

$$t^{\alpha} S(t)u(t,0) \ge c \log\left(\frac{4at+1}{2}\right), \tag{4.24}$$

which contradicts $(4.15)_1$ for t large.

Thus, in each case, we have a contradiction and the proof is complete. ∎

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