GLOBAL EXISTENCE AND BLOW UP OF SOLUTIONS FOR A COMPLETELY COUPLED FUJITA TYPE SYSTEM OF REACTION-DIFFUSION EQUATIONS

Abstract. We examine the parabolic system of three equations

\begin{align*}
  u_t - \Delta u &= v^p, \\
  v_t - \Delta v &= w^q, \\
  w_t - \Delta w &= u^r,
\end{align*}

with \( p, q, r \) positive numbers, \( N \geq 1 \), and nonnegative, bounded continuous initial values. We obtain global existence and blow up unconditionally (that is, for any initial data). We prove that if \( pqr \leq 1 \) then any solution is global; when \( pqr > 1 \) and \( \max(\alpha, \beta, \gamma) \geq N/2 \) (where \( \alpha, \beta, \gamma \) are defined in terms of \( p, q, r \)) then every nontrivial solution exhibits a finite blow up time.

1. Introduction and main results. In this paper we consider the system

\begin{align*}
  u_t - \Delta u &= v^p, \\
  v_t - \Delta v &= w^q, \\
  w_t - \Delta w &= u^r,
\end{align*}

for \( t > 0, \, x \in \mathbb{R}^N \) with \( N \geq 1, \, p, q, r > 0 \) and

\begin{align*}
  u(0, x) &= u_0(x), \\
  v(0, x) &= v_0(x), \\
  w(0, x) &= w_0(x),
\end{align*}

where \( u_0, v_0, w_0 \) are nonnegative, continuous, bounded functions.

Let us recall that the system (1.1)–(1.2) has a nonnegative classical solution in \( S_T = [0, T) \times \mathbb{R}^N \) for some \( T > 0 \) (cf. for instance a related

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argument in [EH]). Our primary concern is to describe two cases: $T = \infty$, when the system has bounded solutions for any $S_t, t > 0$ (global solutions), and $T < \infty$, when solutions are unbounded beyond $T$ (they blow up in a finite time $T$). In this paper, we discuss these cases in terms of $p, q, r$ and $N$ only. Some additional dependence on the initial data $u_0, v_0, w_0$ (which implies that both situations coexist) will be considered in another paper.

The Cauchy problem

$$u_t - \Delta u = u^p, \quad t > 0, \ x \in \mathbb{R}^N,$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N,$$

has been analyzed by several authors (see [F1], [F2]).

Also the system of two reaction-diffusion equations has been dealt with in case of coupled systems.

For instance, in [EH] and [AHV] global existence and blow up results were discussed for the problem

$$u_t - \Delta u = v^p, \quad (x,t) \in \mathbb{R}^N \times (0,T),$$
$$v_t - \Delta v = u^q, \quad (x,t) \in \mathbb{R}^N \times (0,T),$$
$$u(0,x) = u_0(x) \geq 0,$$
$$v(0,x) = v_0(x) \geq 0,$$

while in [EL] a more general system was studied:

$$u_t - \Delta u = u^{p_1}v^{q_1}, \quad (x,t) \in \mathbb{R}^N \times (0,T),$$
$$v_t - \Delta v = u^{p_2}v^{q_2}, \quad (x,t) \in \mathbb{R}^N \times (0,T),$$
$$u(0,x) = u_0(x) \geq 0,$$
$$v(0,x) = v_0(x) \geq 0.$$

Our goal is to extend Fujita type global existence-nonexistence theorems to systems of three equations.

Let

$$(1.3) \quad A = \begin{bmatrix} 0 & p & 0 \\ 0 & 0 & q \\ r & 0 & 0 \end{bmatrix}.$$

We denote by $(\alpha, \beta, \gamma)^t$ the solution of $(A - I)X = (1, 1, 1)^t$. We can easily find that

$$(1.4) \quad \alpha = \frac{1 + p + pq}{pqr - 1}, \quad \beta = \frac{1 + q + qr}{pqr - 1}, \quad \gamma = \frac{1 + r + rp}{pqr - 1},$$

where

$$(1.5) \quad \det(A - I) = pqr - 1.$$

We formulate
THEOREM 1. Suppose $\det(A-I) \leq 0$. Then every solution of (1.1)–(1.2) is global.

THEOREM 2. Suppose $\det(A-I) > 0$. If $\max(\alpha, \beta, \gamma) \geq N/2$ then (1.1)–(1.2) never has nontrivial global solutions.

We prove Theorem 1 in Section 3; Theorem 2 is proved by contradiction in Section 4. Section 2 contains some auxiliary tools.

2. Preliminaries. As we have mentioned, solutions of (1.1)–(1.2) are classical in some $S_T$ (that is, $(u, v, w)(x, t) \in C^{2,1}(\mathbb{R}^N \times (0, T))$). Such solutions satisfy the formulas

$$
\begin{align*}
    u(t) &= S(t)u_0 + \int_0^t S(t-s)v(s)^p \, ds, \\
    v(t) &= S(t)v_0 + \int_0^t S(t-s)w(s)^q \, ds, \\
    w(t) &= S(t)w_0 + \int_0^t S(t-s)u(s)^r \, ds,
\end{align*}
$$

where $S(t)$ is an operator semigroup and $S(t)\xi_0$ is the unique solution of $\xi_t - \Delta \xi = 0$, $\xi(0) = \xi_0(x)$, where

$$
S(t)\xi_0(x) = \int_{\mathbb{R}^N} (4\pi t)^{-N/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \xi_0(y) \, dy.
$$

**Remark 2.1.** If $(u, v, w)$ is a nontrivial solution of (1.1)–(1.2) on $[0, T]$, then there exists $t_0 \in (0, T)$ such that $u(x, \tau) > 0$, $v(x, \tau) > 0$ and $w(x, \tau) > 0$ for $(x, \tau) \in \mathbb{R}^N \times (t_0, T)$.

**Proof.** Let $(x_i, t_i)$, $i = 1, 2, 3$, be such that $u(x_1, t_1) > 0$, $v(x_2, t_2) > 0$ and $w(x_3, t_3) > 0$. Let $t_0 = \max(t_1, t_2, t_3)$. From formula (2.1) for $\tau \in (t_1, T-t_1)$,

$$
\begin{align*}
    u(\tau) &= S(\tau-t_1)u(t_1) + \int_0^{\tau-t_1} S(\tau-t_1-\eta)v(\eta)^q \, d\eta \\
    &\geq S(\tau-t_1)u(t_1),
\end{align*}
$$

and since $S(\tau) > 0$ we get $u(\tau) > 0$ on $\mathbb{R}^N$ for $\tau > t_1$. Similarly, $v(\tau) > 0$ on $\mathbb{R}^N$ for $\tau > t_2$ and $w(\tau) > 0$ on $\mathbb{R}^N$ for $\tau > t_3$.

We also need
Lemma 2.1. Let \((u_0, v_0, w_0) \neq (0, 0, 0)\) and \((u, v, w)\) be a solution of (1.1)–(1.2). Then we can choose \(\tau = \tau(u_0, v_0, w_0)\) and constants \(c, a > 0\) such that \(\min(u(\tau), v(\tau), w(\tau)) \geq ce^{-a|x|^2}\).

Proof. We use the same argument as in [EL] or [EH]. Let, for instance, \(u_0 \neq 0\). We can assume that for some \(R > 0\),
\[
\nu = \inf\{u_0(x) : |x| < R\} > 0.
\]
From (2.1) we have
\[
u(t) \geq S(t)u_0 \geq \nu(4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right) \int_{|y| \leq R} \exp\left(-\frac{|y|^2}{4\tau_0}\right) dy.
\]
Defining
\[
\overline{u}(t) = u(t + \tau_0) \quad \text{for some } \tau_0 > 0,
\]
\[
\alpha_1 = \frac{1}{4\tau_0}, \quad c_1 = \nu(4\pi \tau_0)^{N/2} \int_{|y| \leq R} \exp\left(-\frac{|y|^2}{4\tau_0}\right) dy
\]
we have
\[
\overline{u}(0) = u(\tau_0) > c_1 \exp(-\alpha_1|x|^2).
\]
In the same way we obtain
\[
\overline{v}(0) > c_2 \exp(-\alpha_2|x|^2), \quad \overline{w}(0) > c_3 \exp(-\alpha_3|x|^2).
\]
Finally, we have to choose \(\alpha\) and \(c\) suitable for \(u, v, w\) to ensure
\[
(u(x, \tau_0), v(x, \tau_0), w(x, \tau_0))^t \geq c \exp(-\alpha|x|^2)(1, 1, 1)^t
\]
and this concludes the proof.

3. Global existence. In this section we prove Theorem 1, considering separately the cases \(\det(A - I) = 0\) and \(\det(A - I) < 0\).

(a) \(\det(A - I) = 0\) (by (1.5), this is equivalent to \(pqr = 1\)). We want to find a global supersolution to system (1.1)–(1.2) of the form
\[
\begin{pmatrix}
\overline{u} \\
\overline{v} \\
\overline{w}
\end{pmatrix} =
\begin{pmatrix}
A e^{\alpha_1 t} \\
B e^{\beta_1 t} \\
C e^{\gamma_1 t}
\end{pmatrix},
\]
where, for given \(u_0, v_0, w_0\), we choose \(A, B, C\) so large that \(A \geq \|u_0\|_{L^\infty}, B \geq \|v_0\|_{L^\infty}\) and \(C \geq \|w_0\|_{L^\infty}\). Let \(\alpha_1, \beta_1, \gamma_1\) be positive constants such that
\[
\overline{u}_t - \Delta \overline{u} \geq \overline{u}^p, \quad \overline{v}_t - \Delta \overline{v} \geq \overline{v}^q, \quad \overline{w}_t - \Delta \overline{w} \geq \overline{w}^r.
\]
for all $x \in \mathbb{R}^N$ and $t > 0$. Then (3.1) satisfies (3.2) for all $t > 0$ if
\begin{align}
\alpha_1 &> A^{-1}B^p \exp((p\beta_1 - \alpha_1)t), \\
\beta_1 &> B^{-1}C^q \exp((q\gamma_1 - \beta_1)t), \\
\gamma_1 &> C^{-1}A^r \exp((r\alpha_1 - \gamma_1)t).
\end{align}
If we take $\beta_1 = \alpha_1/p$ and $\gamma_1 = \alpha_1/(pq) = r\alpha_1$ (the last equality follows from $pqr = 1$), then (3.2) holds for $\alpha_1$ large enough.

(b) $\det(A - I) < 0$ (by (1.5), that means $pqr < 1$). We are looking for a global supersolution of the form
\begin{equation}
\begin{pmatrix}
\tau \\
\theta \\
\varpi
\end{pmatrix} = \begin{pmatrix}
A(t + t_0)^{\alpha_1} \\
B(t + t_0)^{\beta_1} \\
C(t + t_0)^{\gamma_1}
\end{pmatrix},
\end{equation}
for some positive constants $A, B, C, \alpha_1, \beta_1, \gamma_1$ such that the inequalities (3.2) with $(\tau, \theta, \varpi)$ given by (3.4) hold for all $x \in \mathbb{R}^N$ and $t > 0$. We have to choose $t_0$ sufficiently large to satisfy
\begin{equation}
(\tau(x, 0), \theta(x, 0), \varpi(x, 0)) \geq (u_0, v_0, w_0)
\end{equation}
for $x \in \mathbb{R}^N$.

Substituting (3.4) into (3.2) we obtain the following conditions:
\begin{equation}
\alpha_1 - p\beta_1 \geq 1, \quad \beta_1 - q\gamma_1 \geq 1, \quad \gamma_1 - r\alpha_1 \geq 1.
\end{equation}
Let us remark that (3.6) has the form $(A - I)(-\alpha_1, -\beta_1, -\gamma_1)^t \geq (1, 1, 1)^t$ with $A$ given by (1.3). Set $(\alpha_1, \beta_1, \gamma_1) = (-\alpha, \beta, \gamma)$ for $\alpha, \beta, \gamma$ defined by (1.4). Since $pqr < 1$ we see that $(\alpha_1, \beta_1, \gamma_1)$ are positive. Thus, (3.4) satisfies (3.2) and (3.5) provided that
\begin{align}
A\alpha_1 &\geq B^p, \\
B\beta_1 &\geq C^q, \\
C\gamma_1 &\geq A^r,
\end{align}
and $t_0$ is large enough. Then every nonnegative solution of (1.1)–(1.2) with bounded initial values is global.

4. Blow up (proof of Theorem 2). Without loss of generality we assume henceforth $r \leq q \leq p$. Thus, by (1.4), $\max(\alpha, \beta, \gamma) = \alpha$ for $pqr > 1$.

LEMMA 4.1. Let $(u(t), v(t), w(t))$ be a bounded solution of (1.1)–(1.2) in some strip $S_T$ with $0 < T \leq \infty$. Let $pqr \geq 1$ and $r \geq 1$. Then there exists a positive constant $C$ such that
\begin{equation}
t^\alpha \|S(t)u_0\|_\infty \leq C, \quad t \in [0, T),
\end{equation}
where $C$ depends on $p, q, r$ only and $\alpha$ is given by (1.4).
Proof. Using (2.1) in (2.1) we get
\[ w(t) \geq \int_0^t S(t-s)(S(s)u_0)^r \, ds \]
and by the Jensen inequality for \( r \geq 1 \),
\[ w(t) \geq \int_0^t (S(t-s)S(s)u_0)^r \, ds = \int_0^t (S(t)u_0)^r \, ds = t(S(t)u_0)^r. \] \tag{4.2}

We substitute (4.2) in (2.1) (ignoring the first term on the right-hand side) and by the Jensen inequality we obtain
\[ v(t) \geq \int_0^t S(t-s)(S(S(s)u_0)^r)^q \, ds \geq \int_0^t s^q(S(t)u_0)^{rq} \, ds \]
\[ \geq \frac{1}{q+1} (S(t)u_0)^{rq} t^{q+1}. \] \tag{4.3}

Using (4.3) in (2.1) we can write
\[ u(t) \geq S(t-s) \left[ \frac{1}{q+1} (S(s)u_0)^{rq} s^{q+1} \right]^{\frac{1}{p}} \, ds \]
\[ \geq \left( \frac{1}{q+1} \right)^p (S(t)u_0)^{pqr} \frac{t^{p(q+1)+1}}{p(q+1)+1}. \] \tag{4.4}

Using again the lower bound (4.4) in (2.1) gives
\[ w(t) \geq \left( \frac{1}{q+1} \right)^{rp} (S(t)u_0)^{pqr^2} \frac{1}{(p(q+1)+1)^r} \cdot \frac{t^{rp(q+1)+r+1}}{rpq + rp + r + 1}. \] \tag{4.5}

Continuing this procedure gives
\[ v(t) \geq \frac{1}{(q+1)^{r^pq}} \cdot \frac{1}{(p(q+1)+1)^{rq}} \cdot \frac{1}{(rpq + rp + r + 1)^q} \cdot \frac{t^{rp(q+1)+r+1}}{(q+1)(rpq + 1) + rq}. \] \tag{4.6}

\[ u(t) \geq \frac{1}{(q+1)^{r^pq^2}} \cdot \frac{1}{(pq + p + 1)^{rpq}} \cdot \frac{1}{(rpq + rp + r + 1)^{pq}} \cdot \frac{1}{((q+1)(rpq + 1) + rq)^p} \cdot \frac{1}{(rpq + 1)(p(q+1)+1)} \cdot \frac{t^{rp(q+1)+r+1}}{(pq + 1)(p(q+1)+1)}. \]

Iterating this scheme, we obtain, using (1.4),
\[ u(t) \geq A_k B_k C_k (S(t)u_0)^{(pqr)^k} t^{nδ(1+rpq+...+(rpq)^{k-1})}, \] \tag{4.7}
where

\[ A_k = \frac{1}{(\alpha \delta)^{k-1}} \left[ \frac{1}{(rpq + 1)\alpha \delta} \right]^{(rpq)^{k-2}} \times \left[ \frac{1}{((rpq)^2 + rpq + 1)\alpha \delta} \right]^{(rpq)^{k-3}} \times \ldots \times \left[ \frac{1}{((rpq)^{k-1} + \ldots + rpq + 1)\alpha \delta} \right] \]

\[ B_k = \frac{1}{(q + 1)^{k-1}} \cdot \frac{1}{(r\alpha \delta + 1)} \cdot \frac{1}{(r\alpha \delta(rpq + 1) + 1)^{\frac{1}{2}(rpq)^{k-2}}} \times \frac{1}{(r\alpha \delta(1 + rpq + \ldots + (rpq)^{k-2}) + 1)^{pq}} \]

\[ C_k = \frac{1}{(rq\alpha \delta + q + 1)^{\frac{1}{2}(rpq)^{k-1}}} \times \frac{1}{(rq\alpha \delta(rpq + 1) + q + 1)^{\frac{1}{2}(rpq)^{k-2}}} \times \ldots \times \frac{1}{(rq\alpha \delta(1 + pqr + \ldots + (pqr)^{k-2}) + q + 1)^{pq}} \]

Setting \( pqr = z \) we can rewrite (4.8) as

\[ A_k = \frac{1}{(\alpha \delta)^{k-1}} \left( \frac{1}{1 + z} \right)^{z^{k-2}} \left( \frac{1}{1 + z + z^2} \right)^{z^{k-3}} \ldots \frac{1}{1 + z + \ldots + z^{k-1}} \]

so

\[ (4.9) \]

\[ A_k = \frac{1}{(\alpha \delta)^{k-1}} \prod_{j=1}^{k-1} \left( \frac{z - 1}{z^{j+1} - 1} \right)^{z^{k-j-1}}. \]

Using \( \alpha \delta(1 + z + \ldots + z^j) > 1 + q \) for \( j \geq 0 \) we can estimate:

\[ (4.10) \]

\[ B_k \geq \frac{1}{(q + 1)^{k-1}} \cdot \frac{1}{[(r + 1)\alpha \delta]^{\frac{1}{2}(rpq)^{k-2}}} \cdot \frac{1}{[(r + 1)\alpha \delta(1 + z)]^{\frac{1}{2}(rpq)^{k-2}}} \times \ldots \times \frac{1}{[(r + 1)\alpha \delta(1 + z + \ldots + (rpq)^{k-2})+ q + 1]^{pq}} \]

\[ = \frac{1}{(q + 1)^{k-1}} \cdot \frac{1}{[(r + 1)\alpha \delta]^{\frac{1}{2}(rpq)^{k-2}}} \left[ \prod_{j=1}^{k-2} \left( \frac{z - 1}{z^{j+1} - 1} \right)^{z^{k-j-1}} \right]^{\frac{1}{2}}. \]
where $c$.

Letting $k$.

Substituting (4.9)–(4.11) into (4.7) we get

$$u(t) \geq (S(t)u_0)^z \ell \alpha \delta \frac{1}{(r+1)^{n+1} \cdot \frac{(r+1)^{n+1}}{z^{k-1}}}$$

$$\times \prod_{j=1}^{k-2} \left( \frac{z-1}{z^j-1} \right)^{\frac{1}{z^{k-1}}} \cdot \frac{1}{\alpha \delta} \frac{1}{(r+1)^{n+1} \cdot \frac{(r+1)^{n+1}}{z^{k-1}}}$$

We infer that

$$u(t) \geq (S(t)u_0)^z \ell \alpha \delta \frac{1}{(r+1)^{n+1} \cdot \frac{(r+1)^{n+1}}{z^{k-1}}}$$

$$\times \prod_{j=1}^{k-2} \left( \frac{z-1}{z^j-1} \right)^{\frac{1}{z^{k-1}}} \cdot \frac{1}{\alpha \delta} \frac{1}{(r+1)^{n+1} \cdot \frac{(r+1)^{n+1}}{z^{k-1}}}$$

Letting $k \to \infty$ and using $\|u(t)\|_\infty < \infty$ we obtain in the limit

$$t^{q^a} \|S(t)u_0\|_\infty \leq c < \infty,$$

where $c = c(p, q, r)$ only.

**Lemma 4.2.** Assume that $pqr > 1$, $p > 1$ and $r \leq q < 1$. Let $(u(t), v(t), w(t))$ be as in Lemma 4.1. Then there exists a constant $C$ such that

$$t^{q^a} \|S(t)u_0\|_\infty \leq C \quad \text{and} \quad C = C(p, q, r).$$

**Proof.** By the Jensen inequality for $r < 1$ we have

$$w(t) \geq tS(t)u_0 \quad \text{and} \quad v(t) \geq \frac{1}{q+1} S(t)u_0^{q+1}.$$

Repeating the iteration as in Lemma 4.1 we obtain

$$u(t) \geq A_k B_k C_k (S(t)u_0)^{r^a} \ell \alpha \delta \frac{1}{(r+1)^{n+1} \cdot \frac{(r+1)^{n+1}}{z^{k-1}}}$$

and $A_k$, $B_k$, $C_k$ are given by (4.9)–(4.11). So we estimate as before and
letting $k \to \infty$ we conclude that
\[
 t^{\alpha r} \|S(t)u_0^r\|_\infty \leq C. \quad \blacksquare
\]

We complete the result by considering the case $r < 1 < q$.

**Lemma 4.3.** Let $pqr > 1$ with $p > 1$ and $r < 1 < q$, and let $u, v, w$ be as in Lemma 4.1. Then
\[
 t^{\alpha r} \|S(t)u_0^r\|_\infty \leq C,
\]
where the constant $C$ depends on $p, q, r$ only.

**Proof.** We argue as in the previous lemma, starting from $w(t) \geq tS(t)u_0^r$ and
\[
 v(t) \geq \frac{1}{q+1} (S(t)u_0^r)^q t^{q+1}
\]
so
\[
 u(t) \geq A_k B_k C_k (S(t)u_0^r)^{\frac{1}{k}(pqr)^k} t^{\alpha k(1+pqr+\ldots+(pqr)^{k-1})}
\]
and we get (4.14) as before. \hfill \blacksquare

**Lemma 4.4.** Let $pqr > 1$, and let $(u(t), v(t), w(t))$ be a bounded solution of (1.1)–(1.2) (as in Lemma 4.1). Then we can find a constant $C > 0$, such that for $t > 0$,
\[
 t^{\alpha r} \|S(t)u(t)\|_\infty \leq C < \infty \quad \text{if} \quad 1 < r \leq q \leq p,
\]
\[
 t^{\alpha q} \|S(t)u(t)^q\|_\infty \leq C < \infty \quad \text{if} \quad r \leq q < 1 < p,
\]
\[
 t^{\alpha r} \|S(t)u(t)^r\|_\infty \leq C < \infty \quad \text{if} \quad r < 1 < q \leq p.
\]

**Proof.** For $\tau, t \geq 0$ we can rewrite (2.1) in the form
\[
 w(t + \tau) = S(t + \tau)u_0 + \int_0^{t+\tau} S(t + \tau - s)u(s)^r \, ds
 = S(t)u(\tau) + \int_0^{\tau} S(t - s)u(\tau + s)^r \, ds
\]
and similarly for $v$ and $u$. Hence we can replace $u_0$ by $u(\tau)$ in (4.1), (4.12) and (4.14); setting $t = \tau$, we get the conclusion. \hfill \blacksquare

**Lemma 4.5.** Suppose that $\alpha \geq N/2$ and $pqr > 1$. Then every nontrivial solution of (1.1)–(1.2) blows up in finite time.

**Proof.** Assume that there exists a bounded solution of (1.1)–(1.2) with $(u_0, v_0, w_0) > (0, 0, 0)$ and $\alpha \geq N/2$, $pqr > 1$. By Lemma 2.1 we can find $c, a > 0$ such that
We will also use the following equality:

\[(4.17)\quad S(t)e^{-a|x|^2} = (1 + 4at)^{-N/2} \exp \left( \frac{-a|x|^2}{1 + 4at} \right).\]

First, we consider the case \(0 < r < 1 \leq q < p\). Using (4.16) in (4.17) we get

\[u(t) \geq S(t)u_0 \geq c(1 + 4at)^{-N/2} \exp \left( \frac{-a|x|^2}{1 + 4at} \right).\]

Now, from (2.1)_3, the last inequality and (4.17) we obtain

\[w(t) \geq \frac{c}{CK_0} \int_0^t S(t-s)u(s)^r \, ds \]

\[\geq c^r \int_0^t S(t-s)(1 + 4as)^{-N/r/2} \exp \left( \frac{-ar|x|^2}{1 + 4as} \right) \, ds \]

\[= c^r \int_0^t (1 + 4as)^{-N/r/2} \left( 1 + \frac{4ar}{1 + 4as} (t - s) \right)^{-N/2} \]

\[\times \exp \left( \frac{-ar|x|^2}{1 + 4as + 4ar(t - s)} \right) \, ds \]

\[= c^r \int_0^t (1 + 4as)^{-N(r-1)/2} (1 + 4as + 4ar(t - s))^{-N/2} \]

\[\times \exp \left( \frac{-ar|x|^2}{1 + 4as + 4ar(t - s)} \right) \, ds.\]

We note that \(f(s) = 1 + 4as + 4ar(t - s)\) is increasing (because \(f'(s) = 4a(1 - r) > 0\)) so that

\[w(t) \geq c^r (1 + 4at)^{-N/2} \exp \left( \frac{-ar|x|^2}{1 + 4ar(t)} \right) \int_0^t (1 + 4as)^{-N(r-1)/2} \, ds.\]

Integrating, we obtain

\[(4.18)\quad w(t) \geq \frac{c^r}{4a(1 - N(r-1)/2)} (1 + 4at)^{-N/2} \exp \left( \frac{-ar|x|^2}{1 + 4ar(t)} \right) (4at)^{1-N(r-1)/2}.\]

Now, we use (4.18) in (2.1)_2 to get (by (4.17))
\[ v(t) \geq \frac{c^q}{[4a(1 - N(r - 1)/2)]q} \int_0^t (1 + 4as)^{-Nq/2} (4as)^q (1 - N(r - 1)/2) \times S(t - s) \exp \left( \frac{-arq|x|^2}{1 + 4ars} \right) ds \]

\[ = \frac{c^q}{[4a(1 - N(r - 1)/2)]q} \int_0^t (1 + 4as)^{-Nq/2} (1 + 4ars)^{N/2} (4as)^q (1 - N(r - 1)/2) \times (1 + 4ars + 4arq(t - s))^{-N/2} \exp \left( \frac{-arq|x|^2}{1 + 4ar + 4arq(t - s)} \right) ds. \]

Consider \( g(s) = 1 + 4ars + 4arq(t - s) \); as \( g'(s) = 4ar(1 - q) < 0 \) we deduce that

\[ v(t) \geq \frac{c^q}{[4a(1 - N(r - 1)/2)]q} (1 + 4arqt)^{-N/2} \exp \left( \frac{-arq|x|^2}{1 + 4art} \right) \times \int_0^t (1 + 4as)^{-Nq/2} (1 + 4ars)^{N/2} (4as)^q (1 - N(r - 1)/2) ds. \]

To integrate, let us remark that \( (1 + 4ars)^{N/2} \geq r^{N/2}(1 + 4as)^{N/2} \) and \( 4as > \frac{1}{2}(1 + 4as) \) for \( s > 1/(4a) \). Denoting by \( c_1 \) the new constant such that

\[ c_1 := \frac{c^q}{[4a(1 - N(r - 1)/2)]q} r^{N/2} \left( \frac{1}{2} \right)^q (1 - N(r - 1)/2), \]

we have

\[ v(t) \geq c_1 (1 + 4arqt)^{-N/2} \exp \left( \frac{-arq|x|^2}{1 + 4art} \right) \int_0^t (1 + 4as)^q (1 - N(r - 1)/2) ds \]

and finally

\[ v(t) \geq c_1 (1 + 4arqt)^{-N/2} \exp \left( \frac{-arq|x|^2}{1 + 4art} \right), \]

where \( c = c(p, q, r, N/2, a) \) is a constant.

We need a lower bound for \( u(t) \), so we substitute (4.19) into (2.1)\textsubscript{1} to get

\[ u(t) \geq c^p \int_0^t (1 + 4arqs)^{-Np/2} (4as)^{p(1 + q - N(qr - 1)/2)} \times S(t - s) \exp \left( \frac{-apqr|x|^2}{1 + 4ars} \right) ds \]

\[ = c^p \int_0^t (1 + 4arqs)^{-Np/2} (4as)^{p(1 + q - N(qr - 1)/2)} \]
\[ \times \left(1 + \frac{4arpq(t-s)}{1+4ars}\right)^{-N/2} \]
\[ \times \exp\left(\frac{-apqr|x|^2}{1+4ars+4arpq(t-s)}\right) ds. \]

The last equality follows by (4.17). Now, consider \( h(s) = 1 + 4ars + 4arpq(t-s) \); note that \( h'(s) = 4ar(1-pq) < 0 \) so as before we get

\[
(4.19) \quad u(t) \geq c^p(1 + 4arpt)^{-N/2} \exp\left(\frac{-apqr|x|^2}{1+4art}\right) \times \int_0^t (1 + 4arqs)^{-Np/2}(4as)^{p(1+q-N(qr-1)/2)}(1 + 4ars)^{N/2} ds.
\]

Using again

\[ 4at > \frac{1}{2}(1 + 4at) \quad \text{for} \quad t > 1/(4a), \]
\[ (1 + 4ars)^{N/2} \geq r^{N/2}(1 + 4as)^{N/2} \quad \text{for} \quad r < 1, \]

and noting that

\[ (1 + 4arqt)^{-N/2} \geq (1 + 4apqrt)^{-N/2} \geq (pqr)^{-N/2}(1 + 4at)^{-N/2} \]

holds since \( p > 1 \) and \( pqr > 1 \), we obtain from (4.20), for \( t > 1/(4a) \),

\[ u(t) \geq c(1 + 4at)^{-N/2} \exp\left(\frac{-apqr|x|^2}{1+4art}\right) \int_{1/(4a)}^t (1 + 4as)^{\varrho} ds, \]

where

\[ \varrho = -Np/2 + p(1 + q - N(qr - 1)/2) + N/2 = p + pq - N(pqr - 1)/2 \geq -1 \]

by the assumption that \( \alpha \geq N/2 \).

So we infer that

\[ (4.20) \quad u(t) \geq c(1 + 4at)^{-N/2} \exp\left(\frac{-apqr|x|^2}{1+4art}\right) \log\left(\frac{4at + 1}{2}\right) \]

for \( t > 1/(4a) \).

It now follows by (4.17) that

\[ (4.21) \quad S(t)u(t)^r \geq c(1 + 4at)^{-Nr/2} \exp\left(\frac{-apqr^2|x|^2}{1+4art}\right) \times S(t) \left[ \log\left(\frac{1+4at}{2}\right) \right]^r \]
\[ = c(1 + 4at)^{-Nr/2}(1 + 4ar(1 + rpq)t)^{-N/2}(1 + 4art)^{N/2} \]
\[ \times \left[ \log\left(\frac{1+4at}{2}\right) \right]^r \exp\left(\frac{-ar^2pq|x|^2}{1+4ar(1 + rpq)t}\right) \]
\[
\begin{aligned}
\geq & \ c(1 + 4at)^{-Nr/2} \left( \frac{1 + 4art}{(1 + 4art)(1 + pqr)} \right)^{N/2} \\
& \times \left[ \log \left( \frac{1 + 4at}{2} \right) \right]^r \exp \left( \frac{-ar^2pq|x|^2}{1 + 4ar(1 + rpq)t} \right).
\end{aligned}
\]

Putting \( x = 0 \) in (4.22) we get

\[
(1 + 4at)^{Nr/2} S(t)u(t, 0)^r \geq \frac{c}{(1 + pqr)^{N/2}} \left[ \log \left( \frac{1 + 4at}{2} \right) \right]^r
\]

and therefore, for \( t > \max(1, 1/(4a)) \) and since \( \alpha \geq N/2 \),

\[
(4.22) \quad t^{\alpha} S(t)u(t, 0)^r \geq c \left[ \log \left( \frac{1 + 4at}{2} \right) \right]^r.
\]

It remains to notice that as \( t \to \infty \), the right-hand side of (4.23) diverges, and so does the left-hand side. But this contradicts (4.15)_3. Thus, \( u(t) \) must become unbounded, and by (2.1), \( v(t) \) and \( w(t) \) also blow up in finite time.

Now, we discuss the remaining cases.

In the case \( 0 < r \leq q < 1 < p \) we argue as before to get, instead of (4.21),

\[
u(t) \geq c(1 + 4at)^{-N/2} \exp \left( \frac{-arpq|x|^2}{1 + 4arqt} \right) \log \left( \frac{4at + 1}{2} \right)
\]

and

\[
S(t)u(t)^r \geq c(1 + 4at)^{-Nqr/2} \exp \left( \frac{-ar^2q^2p|x|^2}{1 + 4arqt} \right) \\
\times S(t) \left[ \log \left( \frac{1 + 4at}{2} \right) \right]^{qr} \\
= c(1 + 4at)^{-Nqr/2} \left( \frac{1 + 4art}{1 + 4arq(1 + pqr)t} \right)^{N/2} \\
\times \exp \left( \frac{-ar^2q^2p|x|^2}{1 + 4arq(1 + pqr)t} \right) \left[ \log \left( \frac{1 + 4at}{2} \right) \right]^{qr}.
\]

Thus, for \( x = 0, \)

\[
S(t)u(t, 0)^r(1 + 4at)^{qrN/2} \geq c \left[ \log \left( \frac{1 + 4at}{2} \right) \right]^{qr},
\]

which implies, for \( t > \max(1, 1/(4a)) \), as \( \alpha \geq N/2 \),

\[
(4.23) \quad t^{\alpha} S(t)u(t, 0)^{qr} \geq c \left[ \log \left( \frac{1 + 4at}{2} \right) \right]^{qr}.
\]

Now, we see that (4.24) is incompatible with \( (4.15)_2 \) for \( t \) large enough.
Finally, we consider the case $1 < r \leq q \leq p$. Then instead of (4.21) we infer that
\begin{equation*}
u(t) \geq c(1 + 4at)^{-N/2} \exp \left( \frac{-arpq|x|^2}{1 + 4at} \right) \log \left( \frac{4at + 1}{2} \right),
\end{equation*}
whence
\begin{equation*}
S(t)\nu(t)(1 + 4at)^{N/2} \geq c \exp \left( \frac{-arpq|x|^2}{1 + 4a(1 + pqr)t} \right) \log \left( \frac{4at + 1}{2} \right).
\end{equation*}
Setting $x = 0$ and using $\alpha \geq N/2$, we have
\begin{equation*}
t^\alpha S(t)\nu(t, 0) \geq c \log \left( \frac{4at + 1}{2} \right),
\end{equation*}
which contradicts (4.15) for $t$ large.

Thus, in each case, we have a contradiction and the proof is complete. \hfill \blacksquare

References


