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A. S. NOWAK (Wrocław)

A GENERALIZATION OF UENO'S INEQUALITY FOR n-STEP TRANSITION PROBABILITIES

Abstract. We provide a generalization of Ueno's inequality for n-step transition probabilities of Markov chains in a general state space. Our result is relevant to the study of adaptive control problems and approximation problems in the theory of discrete-time Markov decision processes and stochastic games.

Let $(S, \mathcal{F})$ be a measurable space and let P and Q be transition probabilities from S into S. The composition of P and Q, denoted by PQ, is the transition probability defined by

$$PQ(s, B) = \int_S Q(z, B)\, P(s, dz),$$

where $s \in S$, $B \in \mathcal{F}$. For any integer $n \geq 2$, we write Q^n to denote the n-step transition probability QQ^{n-1} from S into S, induced by $Q^1 = Q$. By $\|\cdot\|$, we denote the total variation norm in the vector space of all finite signed measures on $(S, \mathcal{F})$. Recall that if μ_1 and μ_2 are probability measures on $(S, \mathcal{F})$, then

$$\|\mu_1 - \mu_2\| = 2 \sup_{B \in \mathcal{F}} |\mu_1(B) - \mu_2(B)|.$$

In the sequel, we prove the following result.

THEOREM. *Let P and Q be transition probabilities from S into S and let*

$$\varepsilon = \sup_{s \in S} \|P(s, \cdot) - Q(s, \cdot)\|.$$

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Then for $s, z \in S$ *and* $n \ge 1$ *we have*

$$\|P^n(s,\cdot) - Q^n(z,\cdot)\| \le \varepsilon\,(1+\beta+\ldots+\beta^{n-1}) + 2\beta^n, \tag{1}$$

where

$$\beta = \tfrac{1}{2}\sup\nolimits_{x,y\in S}\ \|P(x,\cdot) - P(y,\cdot)\|. \tag{2}$$

Remark 1. If $\varepsilon = 0$, then (1) is exactly Ueno's inequality [9].

Corollary 1. *If* $\beta < 1$, *then* (1) *implies that for* n *sufficiently large we have*

$$\|P^n(s,\cdot) - Q^n(z,\cdot)\| \le \frac{2\varepsilon}{1-\beta}$$

for each $s, z \in S$.

Suppose that S is the state space for Markov chains having transition probabilities P and Q respectively. If there exists a probability measure π_P on $(S, \mathcal{F})$ such that

$$\sup_{s\in S}\ \|P^n(s,\cdot) - \pi_P(\cdot)\| \to 0 \quad \text{as } n \to \infty,$$

at a geometric rate, then the Markov chain with transition probability P is called *uniformly ergodic* and π_P is the unique *invariant probability measure* for P.

Corollary 2. *Let* π_P *and* π_Q *be the invariant probability measures for* P *and* Q *respectively. Assume that the Markov chains with transition probabilities* P *and* Q *are uniformly ergodic. If* $\beta < 1$, *then*

$$\|\pi_P - \pi_Q\| \le \frac{\varepsilon}{1-\beta}.$$

It is well known that the Markov chain with transition probability T is uniformly ergodic if and only if there exist a constant $c \in (0,1)$ and a positive integer m such that

$$\|T^m(s,\cdot) - T^m(z,\cdot)\| \le 2c \tag{3}$$

for every $s, z \in S$. For a proof see, e.g., [2].

Put $P = T^m$ and fix a transition probability Q. Define

$$\varepsilon = \sup_{s\in S}\ \|T^m(s,\cdot) - Q(s,\cdot)\|.$$

Assume that (3) holds and consider β defined by (2). Then $\beta < 1$, and using Corollary 1, we infer that for n sufficiently large, we have

$$\|Q^n(s,\cdot) - Q^n(z,\cdot)\| \le \|Q^n(s,\cdot) - T^{mn}(s,\cdot)\| + \|T^{mn}(s,\cdot) - Q^n(z,\cdot)\| \le \frac{4\varepsilon}{1-\beta}.$$

This enables us to state the following result.

Corollary 3. *If* (3) *holds and* $2\varepsilon/(1-\beta) < 1$, *then the Markov chain with transition probability* Q *is also uniformly ergodic. Moreover*,

$$\|\pi_T - \pi_Q\| \le \frac{\varepsilon}{1-\beta} \tag{4}$$

where π_T (π_Q) *is the unique invariant probability measure for the transition probability* T (Q).

Remark 2. Our main result and Corollaries 1–3 may have applications to approximation problems or adaptive control problems as studied in [3], [5], [6], [7] and [8]. A result closely related to Corollary 2 was proved by Stettner in [8], but our inequality (5) has a more elementary form. Also, our proof is quite elementary while the method of proof in [8] is based on the theory of bounded transition operators considered in [4]. However, Stettner's proof [8] can be used for studying some uniform convergence problems of n-step transition probabilities in different norms on the state space [6].

Proof of Theorem. We proceed by induction on n. It is easy to see that (1) holds for $n = 1$. Suppose it holds for a positive integer n. Note that

$$\begin{aligned} &\|P^{n+1}(s,\cdot) - Q^{n+1}(z,\cdot)\| \\ &\quad = \|P^nP(s,\cdot) - Q^nQ(z,\cdot)\| \\ &\quad \le \|P^nP(s,\cdot) - Q^nP(z,\cdot)\| + \|Q^nP(z,\cdot) - Q^nQ(z,\cdot)\| \\ &\quad \le \|P^nP(s,\cdot) - Q^nP(z,\cdot)\| + \varepsilon. \end{aligned} \tag{5}$$

Moreover, we have

$$\|P^nP(s,\cdot) - Q^nP(z,\cdot)\| = 2\sup_{B\in\mathcal{F}} |L(B)|, \tag{6}$$

where

$$L(B) = \int_S P(x,B)\,\mu(s,z)(dx)$$

for any $B \in \mathcal{F}$ and $\mu(s,z)(\cdot) = P^n(s,\cdot) - Q^n(z,\cdot)$.

Define

$$\varphi(x) = P(x,B) - \inf_{y\in S} P(y,B).$$

Note that $\varphi \ge 0$ on S and

$$L(B) = \int_S \varphi(x)\,\mu(s,z)(dx).$$

Fix $B \in \mathcal{F}$. Without loss of generality, we can assume that $|L(B)| = L(B)$ (otherwise, use $-\mu(s,z)(dx)$ instead of $\mu(s,z)(dx)$). By the Hahn decomposition theorem [1], there exists a set $D \in \mathcal{F}$ such that

$$\begin{aligned} &\mu(s,z)(E) \ge 0 \quad \text{for all } E \in \mathcal{F},\ E \subset D, \\ &\mu(s,z)(E) \le 0 \quad \text{for all } E \in \mathcal{F},\ E \subset S \setminus D. \end{aligned}$$

Note that

$$
\begin{aligned}
|L(B)| = L(B) &= \int_D \varphi(x)\,\mu(s,z)(dx) + \int_{S\setminus D} \varphi(x)\,\mu(s,z)(dx)\\
&\le \int_D \varphi(x)\,\mu(s,z)(dx) \le \mu(s,z)(D)\sup_{x\in S}\varphi(x)\\
&\le \tfrac{1}{2}\mu(s,z)(D)\sup_{x,y\in S} 2|P(x,B)-P(y,B)|.
\end{aligned}
$$

Hence,

$$
L(B) \le \mu(s,z)(D)\cdot \tfrac{1}{2}\sup_{x,y\in S}\|P(x,\cdot)-P(y,\cdot)\| = \mu(s,z)(D)\cdot\beta. \tag{7}
$$

But

$$
\begin{aligned}
\mu(s,z)(D) = P^n(s,D) - Q^n(z,D) &\le \tfrac{1}{2}2\sup_{F\in\mathcal{F}}|P^n(s,F)-Q^n(z,F)|\\
&= \tfrac{1}{2}\|P^n(s,\cdot)-Q^n(z,\cdot)\|.
\end{aligned}
$$

This and (7) imply that

$$
|L(B)| = L(B) \le \tfrac{1}{2}\|P^n(s,\cdot)-Q^n(z,\cdot)\|\cdot\beta. \tag{8}
$$

By (6) and (8) we obtain

$$
\|P^nP(s,\cdot)-Q^nP(z,\cdot)\| \le \|P^n(s,\cdot)-Q^n(z,\cdot)\|\cdot\beta.
$$

Applying this inequality, (5) and our induction hypothesis we finally get

$$
\begin{aligned}
\|P^{n+1}(s,\cdot)-Q^{n+1}(z,\cdot)\| &\le \varepsilon + \|P^n(s,\cdot)-Q^n(z,\cdot)\|\cdot\beta\\
&\le \varepsilon + \beta(\varepsilon+\varepsilon\beta+\ldots+\varepsilon\beta^{n-1}+2\beta^n)\\
&= \varepsilon(1+\beta+\ldots+\beta^n)+2\beta^{n+1},
\end{aligned}
$$

which we wanted to prove.

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Andrzej S. Nowak
Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: nowak@im.pwr.wroc.pl

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