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# EXTENSIONS OF CONVEX FUNCTIONALS ON CONVEX CONES

*Abstract.* We prove that under some topological assumptions (e.g. if $M$ has nonempty interior in $X$), a convex cone $M$ in a linear topological space $X$ is a linear subspace if and only if each convex functional on $M$ has a convex extension on the whole space $X$.

**1. Introduction.** Let $M$ be any convex cone in a linear topological space $X$ (i.e. $M + M \subset M$ and $\mathbb{R}^+ M \subset M$). Assume that every convex functional $\pi$ on $M$ ($\pi(\lambda m) = \lambda\pi(m)$, $\pi(m + m_{1)} \leq \pi(m) + \pi(m_1)$ for $\lambda \in \mathbb{R}^+$, $m, m_1 \in M$) can be extended to a convex functional $\pi^*$ on $X$. It is not difficult to notice that this condition on $M$ is rather restrictive. In fact, we shall prove that it can be valid only for $M$ being a linear subspace of $X$. But some auxiliary constructions are necessary. We also need some topological assumptions, e.g. that $M$ has nonempty interior.

The possibility of extending any convex functional is a natural question important in a number of applications of functional analysis. We only point out that convex functionals appear in a natural way in the pricing of contingent claims on market with transaction costs [3], [4]. We explain this in the simple case of a finite set of trading times $(0, 1, \ldots, T)$ (cf. [1]). Let $X_0, X_1, \ldots, X_T$ be random prices of a unit of one stock. Thus $(X_t)$ is adapted to increasing $\sigma$-fields $F_0 = \{\emptyset, \Omega\} \subset F_1 \subset \ldots \subset F_T$. For any trading strategy, which is a predictable sequence $\theta_1, \ldots, \theta_T$ (that is, $\theta_t$ is $F_{t-1}$-measurable), and for the number $V_0$ denoting initial investments, one defines the payoff of the strategy by the formula $V_T = V_0 + \theta_1(X_1 - X_0) + \ldots + \theta_T(X_T - X_{T-1})$.

It was Harrison and Kreps [1] who showed the role of models of this type.

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They proved that, for $X_0, X_1, \ldots, X_T$ satisfying the so-called no arbitrage condition, all the payoffs $V_T$ form a linear subspace. The initial investment $V_0$ is uniquely determined by $V_T$ and $V_0$ is a linear functional of $V_T$.

To describe a more realistic case, one assumes that bid and ask prices of a stock are given by different $(F_0, \ldots, F_T)$-adapted processes $(X_0, \ldots, X_T)$, $(X'_0, \ldots, X'_T)$ with $X_t \le X'_t$. In such a case, one describes a trading strategy by two nonnegative, nondecreasing processes $(\theta_1, \ldots, \theta_T)$ and $(\theta'_1, \ldots, \theta'_T)$, predictable with respect to the filtration $(F_1, \ldots, F_T)$. At each trading time $t$, the investor sells $\theta_{t+1} - \theta_t$ units of the stock and buys $\theta'_{t+1} - \theta'_t$ units. Therefore $\theta_t$ and $\theta'_t$ are respectively the cumulative long position and cumulative short position in the stock just before time $t = 0, \ldots, T$.

Denote by $V_t$ the amount of the investor's money (in a bank or riskless bonds) just before trading time $t = 0, \ldots, T$. We assume that $\theta_0 = \theta'_0 = 0$ and, moreover, that after time $T$, our net position in the stock should be equal to zero. More precisely, $\theta_{T+1} - \theta_T = \theta'_T$, $\theta'_{T+1} - \theta'_T = \theta_T$. The final payoff is given by the formula

$$
\begin{aligned}
(1)\quad V = V_0 &+ [\theta_1 X_0 - \theta'_1 X'_0] + [(\theta_2 - \theta_1)X_1 - (\theta'_2 - \theta'_1)X'_1] + \ldots \\
&+ [(\theta_T - \theta_{T-1})X_{T-1} - (\theta'_T - \theta'_{T-1})X'_{T-1}] + \theta'_T X_T - \theta_T X'_T.
\end{aligned}
$$

This example is, in fact, a special case of the model described in [3].

We assume that prices are discounted in such a way that the interest rate of a bank deposit (bond) equals zero (and that the short position in a bond is possible with no restrictions). The investor cumulates all his income in the bank and the self-financing condition is satisfied automatically.

Obviously, $V_0$ still describes an initial investment. Assume that $V$ is a possible final payoff. Then $\pi(V) = \inf V_0$, where the infimum is taken over all $\theta', \theta, V_0$ satisfying (1), is a necessary initial investment, uniquely determined by $V$. In this situation the $V$'s form a convex cone $M$ with $\pi$ being a convex functional on $M$. Extension of this functional has a very important interpretation as the pricing of all possible "contingent claims" (see [4] for general explanations).

From this point of view, our result is negative. Extension of pricing functionals cannot be obtained by general (geometrical) methods dealing with all cones and functionals.

**2. Extensions in linear topological spaces.** We use the following terminology. A *convex cone* means any set $M$ contained in a real linear space $X$, satisfying $M + M \subset M$ and $\mathbb{R}^+ M \subset M$. A *convex functional* means a function $\pi : M \to \mathbb{R}$ satisfying $\pi(\lambda x) = \lambda \pi(x)$ and $\pi(x + y) \le \pi(x) + \pi(y)$ for $\lambda \ge 0$ and $x, y \in M$ (see [5] for more information).

THEOREM 2.1. *Let $X$ be a linear space, $M \subset X$ a convex cone, $\psi$ a linear functional on $X$, and let $a$, $b$ be two elements of $X$ such that*:

(i) $\psi(x) \le 0$ *for* $x \in M$;
(ii) $\psi(a) = 0$, $a \neq 0$;
(iii) $\psi(b) < 0$;
(iv) $a + t(b - a) \in M$ *for* $t \in (0, 1]$.

*Then there exists a convex functional* $\pi : M \to \mathbb{R}$ *without a convex extension to* $X$. *If* $X$ *is a linear topological space and* $\psi$ *is continuous*, *then* $\pi$ *can be chosen continuous.*

Proof. Define $L = \mathrm{lin}(a, b)$. Let $\overline{\varphi} : L \to \mathbb{R}$ be a linear functional with $\overline{\varphi}(a) = \overline{\varphi}(b) = 1$ and let $\varphi : X \to \mathbb{R}$ be a linear extension of $\overline{\varphi}$ on $X$, i.e. $\varphi_{|L} = \overline{\varphi}$. Define $D = M \cap \{x \in X : \varphi(x) = 1\}$. It is clear that $a, b \in D$. Let $f : (-\infty, 0] \to \mathbb{R}$ be given by

$$f(t) = \begin{cases} 1 - \sqrt{1 - (t+1)^2}, & -1 \le t \le 0, \\ 0, & t \le -1. \end{cases}$$

Notice that $f$ is a convex non-decreasing function with non-negative values. Let $\pi$ be the function on $M$ given by

$$\pi(x) = \begin{cases} \lambda f(\psi(d)) & \text{for } x = \lambda d \text{ with some } \lambda > 0 \text{ and } d \in D, \\ 0 & \text{for the remaining } x \in M. \end{cases}$$

Observe that $\pi$ is well defined and convex.

We prove that $\pi$ cannot be extended to $X$. It is enough to show that there exists no convex functional $\pi^*$ defined on $L = \mathrm{lin}(a, b)$ such that

$$\pi^*_{|L \cap M} = \pi_{|L \cap M}. \tag{2}$$

So, suppose that a convex functional $\pi^*$ satisfies (2). Consider the function $g(t) = \pi^*(a + t(b - a))$, $t \in \mathbb{R}$. Note that $g$ is convex. Moreover, for $t \in (0, 1]$, we have $g(t) = \pi(a + t(b - a))$ because $\varphi(a + t(b - a)) = \varphi(a) + t(\varphi(b) - \varphi(a)) = 1$, i.e. $a + t(b - a) \in L \cap M$. Hence $g(t) = f(\psi(a + t(b - a))) = f(t\psi(b))$, which means that

$$g(t) = 1 - \sqrt{1 - (t\psi(b) + 1)^2}, \qquad t \in (0, -1/\psi(b)]. \tag{3}$$

This is impossible because the function (3) cannot be extended to a convex function on the whole real line. ∎

THEOREM 2.2. *Let* $X$ *be a real linear topological space and* $M$ *a cone in* $X$ *with nonempty interior. The following conditions are equivalent*:

(i) $M = X$;
(ii) *every convex functional* $\pi : M \to \mathbb{R}$ *can be extended to a convex functional* $\pi^*$ *on* $X$;
(iii) *every continuous convex functional* $\pi : M \to \mathbb{R}$ *can be extended to a convex functional* $\pi^*$ *on* $X$.

Proof. It is enough to prove the implication (iii)⇒(i).

Let $M \subset X$ be a convex cone with nonempty interior such that $M \neq X$.

Suppose that $b \in \operatorname{int} M$, and $c \notin M$ (i.e. $c \neq 0$). We may assume that $b, c$ are linearly independent. Let

$$a = b + \sup\{t \in [0,1] : b + t(c-b) \in M\}(c-b).$$

Then $a \neq 0$ and $a \in \overline{M} \setminus \operatorname{int} M$. If $-a \in \operatorname{int} M$, then $a - a = 0 \in \operatorname{int} M$, i.e. $M = X$. Thus $\operatorname{lin} a \cap \operatorname{int} M = \emptyset$. By the Mazur theorem [5], there exists a linear functional such that $\psi(a) = 0$ and $\psi(x) < 0$ for each $x \in \operatorname{int} M$. To complete the proof, it is now enough to apply the previous theorem. ■

Corollary 2.3. *If $M$ is a convex cone in $\mathbb{R}^n$ containing at least two linearly independent vectors, then the following conditions are equivalent*:

(i) *$M$ is a linear subspace*;

(ii) *every continuous convex functional $\pi$ defined on $M$ can be extended to a convex functional on $\mathbb{R}^n$.*

Proof. It suffices to prove (ii)⇒(i). Let $x_1, \ldots, x_k$ be a maximal system of linearly independent vectors in $M$. Observe that (ii) is equivalent to

(ii′) every continuous convex functional on $M$ can be extended to a convex functional on $\operatorname{lin}(x_1, \ldots, x_k)$.

First we prove that (ii′) implies (ii). Let $\pi^*$ be a convex functional on $\operatorname{lin}(x_1, \ldots, x_k)$ and let $x \in \mathbb{R}^n$. Then $x = \alpha_1 x_1 + \ldots + \alpha_k x_k + \alpha_{k+1} x_{k+1} + \ldots + \alpha_n x_n$ for some linear basis $x_1, \ldots, x_k, \ldots, x_n$. Define a functional $\pi^{**}(x) = \pi^*(\alpha_1 x_1 + \ldots + \alpha_k x_k)$. Since $\pi^*$ is convex, so is $\pi^{**}$. In view of (ii′), without loss of generality we may assume that $k = n$. So, it remains to show that $M = \mathbb{R}^n$. This, however, follows from Theorem 2.2 since $\operatorname{int} M \neq \emptyset$. ■

## 3. Extensions in a Hilbert space

Theorem 3.1. *Let $H$ be a Hilbert space and let $M \subset H$ be a closed convex cone containing at least two linearly independent vectors. Then the following conditions are equivalent*:

(i) *$M$ is a closed linear subspace*;

(ii) *every continuous convex functional $\pi$ defined on $M$ can be extended to a convex functional $\pi^*$ on $H$.*

Proof. (i)⇒(ii) is obvious; it is sufficient to take $\pi^*(x) = \pi(P_M x)$, $x \in H$, where $P_M$ is the orthogonal projection onto $M$.

(ii)⇒(i). Suppose that (i) does not hold. Let $H_1 = \overline{\operatorname{lin} M}$. There exist vectors $c \in H_1 \setminus M$ and $b \in M$, $b \neq 0$. We may assume $b, c$ to be linearly independent. Let $a = b + t_0(c-b)$ where $t_0 = \sup\{t \in (0,1] : b + t(c-b) \in M\}$.

There exists $t_1 \in (t_0, 1]$ such that, for $y = b + t_1(c - b) \in H_1$, we have

$$0 < \|y - a\| < \min_{t\in[0,1]} \|b + t(c - b)\| \le \|y\|. \tag{4}$$

Denote by $P_M y$ the orthogonal projection of $y$ onto the closed convex set $M$ (see [2]). From (4) we have $P_M y \neq 0$. Let $\psi = y - P_M y$.

Then $\psi x := (x, \psi) \le 0$ for $x \in M$. Hence $\psi P_M y = (P_M y, y - P_M y) = 0$.

Since $H_1 = \overline{\operatorname{lin} M}$, there exists $\widetilde{b} \in M$ such that $\psi\widetilde{b} < 0$. It is now enough to take $\widetilde{a} = P_M y$ and apply Theorem 2.1. ∎

Definition 3.1. A real function $\pi$ on a topological space $X$ satisfying the condition

$$\limsup_{x\to x_0} \pi(x) \le \pi(x) \quad \text{for any } x_0 \in X$$

is said to be *upper semicontinuous* (u.s.c.).

Theorem 3.2. *Let $H$ be a Hilbert space and let $M \subset H$ be a convex cone containing at least two linearly independent vectors. Then the following statements are equivalent*:

(i) $\overline{M}$ *is a linear subspace*;

(ii) *every continuous convex functional $\pi : M \to \mathbb{R}$ can be extended to a convex functional $\pi^*$ which is u.s.c.*

Proof. We prove that (ii)⇒(i) ((i)⇒(ii) is obvious). The idea of the proof is similar to that of Theorem 2.1. Assume that $\overline{M}$ is not a linear subspace and let $H_1 = \overline{\operatorname{lin} M}$. Then there exists some $y \in H_1 \setminus \overline{M}$. Let $P_{\overline{M}} y$ be the orthogonal projection of $y$ onto $\overline{M}$ and let $\psi := y - P_{\overline{M}} y$, $\psi x := (x, \psi)$, $x \in M$. Note that

$$\forall x \in M : \psi(x) \le 0; \qquad \exists b \in M : \psi(b) < 0.$$

Let $a = P_{\overline{M}} y$. There exists a function $\varphi'$ such that $\varphi' = kb + (1 - k)a$ for some $k$ and $\varphi' \perp (b - a)$. Notice that $(\varphi', a) = (\varphi, b)$. As $a \in \overline{M}$, there is a sequence $(a'_n) \subset M$ such that $a'_n \to a$. Hence $\varphi(a'_n) \to \varphi(a) = 1$. Now, consider the sequence $a_n = a'_n/\varphi(a'_n)$. It is obvious that $(a_n) \subset M$ and $\varphi(a_n) = 1$. As in the proof of Theorem 2.1, consider the set $L = M \cap \{x \in H_1 : \varphi(x) = 1\}$ (obviously, $a_n, b \in L$) and let $\pi : M \to \mathbb{R}$ be given by

$$\pi(x) = \begin{cases} \lambda f(\psi(l)) & \text{if } x = \lambda l \text{ for some } \lambda > 0, \\ 0 & \text{for other } x \in M, \end{cases}$$

where

$$f(t) = \begin{cases} 1 - \sqrt{1 - (t + 1)^2}, & -1 \le t \le 0, \\ 0, & t < -1. \end{cases}$$

Let $\pi^* : H \to \mathbb{R}$ be a convex u.s.c. functional with $\pi^*_{|M} = \pi$. Define $g_n(\tau) = \pi^*(a_n + \tau(a_n - b))$, $\tau \in \mathbb{R}$.

Notice that for all $\tau \in [-1,0]$, $a_n + \tau(a_n - b) \in M \cap L$ and $g_n(\tau) = \pi(a_n + \tau(a_n - b)) = f(\psi(a_n) + \tau\psi(a_n - b))$, $\psi(a_n) \to 0$. From the existence of $g_n'(0^-)$ and from the convexity of $g_n$ we obtain $g_n(\tau) \ge g_n(0) + g_n'(0^-)\tau$ for all $\tau \in \mathbb{R}$. In particular, for $\tau = 1$, we have $g_n(1) \ge g_n(0) + g_n'(0) \ge g_n' = f'(\psi(a_n))\psi(a_n - b) \xrightarrow{n} \infty$, contrary to the assumption that $\pi^*$ is u.s.c. ∎

EXAMPLE 3.1 The functional $\pi(m) = \sup m$ defined on some convex cone of functions bounded from above is an elementary example of a convex functional (for example in $L_2$) for which the lack of a convex extension is possible.

Let $(\Omega, F, P) = ((0,1), B_{(0,1)}, \lambda)$ and

$$M = \left\{\alpha(1 - \sqrt{1-\varpi^2}) - \beta\frac{1}{1-\varpi^2} : \varpi \in (0,1),\ \alpha,\beta \ge 0\right\}.$$

For $\alpha = 1$,

$$\begin{aligned} f(\beta) &= \sup_{\varpi\in(0,1)} \left(\alpha(1-\sqrt{1-\varpi^2}) - \beta\frac{1}{1-\varpi^2}\right) \\ &= 1 - [2^{1/3} + 2^{-2/3}]\beta^{1/3}, \qquad \text{when } 0 \le \beta \le 1/2. \end{aligned}$$

Thus $f(\beta)$ cannot be extended to a convex function for $\beta \in \mathbb{R}$, and $\pi(x) = \sup_{\varpi\in(0,1)} x(\varpi)$ cannot be extended from $M$ to $\mathrm{lin}(1-\sqrt{1-\varpi^2}, 1/(1-\varpi^2))$.

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