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M. WIŚNIEWSKI (Kielce)

EXTREMES IN MULTIVARIATE STATIONARY NORMAL SEQUENCES

Abstract. This paper deals with a weak convergence of maximum vectors built on the base of stationary and normal sequences of relatively strongly dependent random vectors. The discussion concentrates on the normality of limits and extends some results of McCormick and Mittal [4] to the multivariate case.

1. Introduction and notation. The classical monographs on weak convergence of maximum variables M_n in stationary normal sequences are Galambos [2] and Leadbetter *et al.* [3]. The existence and type of the limit distribution depend only on the asymptotic behaviour of the sequence $\{r(n)\ln n : n \in \mathbb{N}\}$, where $r(n)$ is the covariance between the first and nth variable of the normal stationary sequence considered. We will focus our attention on the normality of limits. McCormick and Mittal [4] have proved the following result:

THEOREM 1. *Suppose that the stationary normal sequence has covariances $\{r(n)\}$ such that $r(n) \to 0$ monotonically and $r(n)\ln n \to \infty$ monotonically for large n. Then*

$$P[r(n)^{-1/2}(M_n - (1-r(n))^{1/2}b_n) \le x] \to \Phi(x) \text{ as } n \to \infty \text{ for all } x \in \mathbb{R},$$

where Φ denotes a standard normal distribution function, and

$$b_n = (2\ln n)^{1/2} - \tfrac{1}{2}(2\ln n)^{-1/2}\ln(4\pi\ln n).$$

In the paper [5] by Mittal and Ylvisaker, where the above result was first proved under the extra assumption that $\{r(n)\}$ is convex, it is also shown that the normal limit distribution is by no means the only possible one; they exhibit a further class of limit distributions which occur when the covariance decreases irregularly. However, it is obvious from the Normal Comparison

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Lemma (see Theorem 4.2.1 of Leadbetter *et al.* [3]) that the assumptions of Theorem 1 can be replaced by a variety of somewhat weaker conditions without affecting the conclusion.

The present work is devoted to the study of the multidimensional aspect of Theorem 1. The result obtained below (Theorem 2) and its proof are based on the concepts of Theorem 3.8.4 of Galambos [2] and the results of Wiśniewski [6].

Let $\mathbb{N}(p)$ denote the set $\{1,\ldots,p\}$ for $p\in\mathbb{N}$. Fix $d\in\mathbb{N}$. All the vectors considered will be d-dimensional, for example, $\mathbf{X}=(X_i : i\in\mathbb{N}(d))$. We emphasize that arithmetical operations and other relations will always be meant componentwise. We denote the covariance coefficient of a stationary sequence $\{\mathbf{X}_n : n\in\mathbb{N}\}$ by $r_{ij}(n)=\operatorname{cov}(X_{1i},X_{nj})$ for $i,j\in\mathbb{N}(d), n\in\mathbb{N}$. Set $\mathbf{r}(n)=(r_{ii}(n) : i\in\mathbb{N}(d))$ and $m(n)=\max\{r_{ij}(n) : i,j\in\mathbb{N}(d)\}$. Let $\mathbf{M}_n=\max\{\mathbf{X}_1,\ldots,\mathbf{X}_n\}$ denote the vector maximum and write $\mathbf{b}_n=(b_n,\ldots,b_n)$, $\mathbf{1}=(1,\ldots,1)$. All the limits in the sequel will be considered as $n\to\infty$.

2. Main result

THEOREM 2. *Suppose that the stationary standard normal sequence $\{\mathbf{X}_n\}$ has covariances $\{r_{ij}(n)\}$ such that for each $i,j\in\mathbb{N}(d)$,*

$$r_{ii}(n)\ln n\to\infty, \tag{1}$$

$$r_{ij}(n)\ln n \ \textit{increases}, \tag{2}$$

$$r_{ij}(n) \ \textit{decreases}, \tag{3}$$

$$r_{ij}(n)[r_{ii}(n)r_{jj}(n)]^{-1/2}\to\varrho_{ij}, \tag{4}$$

$$m(2)<1, \tag{5}$$

$$m(n)(\ln n)^{1/3}\to 0. \tag{6}$$

Then

$$P[\mathbf{r}(n)^{-1/2}(\mathbf{M}_n-(\mathbf{1}-\mathbf{r}(n))^{1/2}\mathbf{b}_n)\le\mathbf{x}]\to\Phi(\mathbf{x})\quad \textit{for all } \mathbf{x}\in\mathbb{R}^d,$$

where Φ denotes the normal distribution function with zero mean vector and covariance matrix (ϱ_{ij}).

REMARK. It is easy to check that

$$r_{ij}(n)=[\ln(n+1)]^{p(i,j)},\quad \text{where } -1<p(i,j)<-1/3,$$

is a covariance sequence which satisfies the assumptions of Theorem 2.

Proof (of Theorem 2). Write $\mathbf{x}_n=\mathbf{r}(n)\mathbf{x}+(\mathbf{1}-\mathbf{r}(n))^{1/2}\mathbf{b}_n$. Denote by $\{\mathbf{Y}_k^n\},\{\mathbf{Z}_k^n\},\{\mathbf{W}_k^n\}$, for $k\in\mathbb{N}(n)$, $n\in\mathbb{N}$, three auxiliary arrays of random vectors. It is required that the rows of the arrays are standard stationary normal sequences and for $n \in \mathbb{N}$:

$$\begin{aligned}
\operatorname{cov}(\mathbf{Y}_k^n) = \operatorname{cov}(\mathbf{Z}_k^n) = \operatorname{cov}(\mathbf{W}_k^n) &= \operatorname{cov}(\mathbf{X}_1) \quad \text{for } k \in \mathbb{N}(n),\\
\operatorname{cov}(\mathbf{Y}_1^n, \mathbf{Y}_k^n) &= \operatorname{cov}(\mathbf{X}_1, \mathbf{X}_n) \quad \text{for } k \in \mathbb{N}(n),\\
\operatorname{cov}(\mathbf{Z}_1^n, \mathbf{Z}_k^n) &= \operatorname{cov}(\mathbf{X}_1, \mathbf{X}_k) \quad \text{for } k \in \mathbb{N}(s),\\
\operatorname{cov}(\mathbf{Z}_1^n, \mathbf{Z}_k^n) &= \operatorname{cov}(\mathbf{X}_1, \mathbf{X}_s) \quad \text{for } k \in \mathbb{N}(n) \setminus \mathbb{N}(s),\\
\operatorname{cov}(\mathbf{W}_1^n, \mathbf{W}_k^n) &= \operatorname{cov}(\mathbf{X}_1, \mathbf{X}_s) \quad \text{for } k \in \mathbb{N}(n),
\end{aligned}$$

where

$$s = s(n) = \left\langle \exp\left\{\left[1 - 2\left(1 + \frac{1}{t}\right)m(n)\right]\ln n - (\ln n)^{1/2}\right\}\right\rangle, \qquad t \in \left(0, \frac{1 - m(2)}{1 + m(2)}\right),$$

and $\langle a\rangle$ denotes the integer part of a.

Set $\mathbf{M}_n^Y = \max\{\mathbf{Y}_1^n, \ldots, \mathbf{Y}_n^n\}$, $\mathbf{M}_n^Z = \max\{\mathbf{Z}_1^n, \ldots, \mathbf{Z}_n^n\}$ and $\mathbf{M}_n^W = \max\{\mathbf{W}_1^n, \ldots, \mathbf{W}_n^n\}$. Since a normal distribution function is a monotonic function of its covariances (see Berman [1]) we conclude that

$$P[\mathbf{M}_n^Y \le \mathbf{x}_n] \le P[\mathbf{M}_n \le \mathbf{x}_n] \le P[\mathbf{M}_n^Z \le \mathbf{x}_n].$$

Hence, to complete the proof it is sufficient to show that

$$P[\mathbf{M}_n^Y \le \mathbf{x}_n] \to \Phi(\mathbf{x}), \tag{7}$$

$$P[\mathbf{M}_n^Z \le \mathbf{x}_n] \to \Phi(\mathbf{x}). \tag{8}$$

By the assumptions of Theorem 2 (without (2) and (6)), Theorem 3 of Wiśniewski [6] yields (7). The proof of (8) falls naturally into the following two parts:

$$|P[\mathbf{M}_n^Z \le \mathbf{x}_n] - P[\mathbf{M}_n^W \le \mathbf{x}_n]| \to 0, \tag{9}$$

$$P[\mathbf{M}_n^W \le \mathbf{x}_n] \to \Phi(\mathbf{x}). \tag{10}$$

To deal with (9) we note that

$$x_{ni}^2 = 2(1 - r_{ii}(n))\ln n + o((\ln n)^{1/2}), \qquad \max\{r_{ij}(s), r_{ij}(k)\} = r_{ij}(k),$$
$$|r_{ij}(s) - r_{ij}(k)| \le r_{ij}(k), \qquad 0 < r_{ij}(k) < 1 \text{ for } k \in \mathbb{N}(s) \setminus \{1\}$$

and apply the Normal Comparison Lemma:

$$\begin{aligned}
&|P[\mathbf{M}_n^Z \le \mathbf{x}_n] - P[\mathbf{M}_n^W \le \mathbf{x}_n]|\\
&\quad \le C \sum_{i,j} \sum_{k=2}^{s} n \exp\{-[2 - r_{ii}(n) - r_{jj}(n)][1 + r_{ij}(k)]^{-1} \ln n + o((\ln n)^{1/2})\}.
\end{aligned}$$

Divide the above sum into $\Sigma_1(n)$ and $\Sigma_2(n)$, where $k \in \mathbb{N}(T) \setminus \{1\}$ for $\Sigma_1(n)$

and $k \in \mathbb{N}(s) \setminus \mathbb{N}(T)$ for $\Sigma_2(n)$, and $T = \langle n^t \rangle$. Then

$$\Sigma_1(n) \le K \sum_{i,j} \exp\{(1 + t - [2 - r_{ii}(n) - r_{jj}(n)][1 + m(2)]^{-1}) \ln n + o((\ln n)^{1/2})\}.$$

Since $r_{ii}(n) \to 0$ for $i \in \mathbb{N}(d)$ the definition of t ensures the existence of $\varepsilon > 0$ such that for all sufficiently large n we have

$$\Sigma_1(n) \le K d^2 \exp\{-\varepsilon \ln n + o((\ln n)^{1/2})\} = o(1).$$

We now turn to the proof of $\Sigma_2(n) = o(1)$. Let $w_{ij}(n)$ denote the exponents of the components occurring in $\Sigma_2(n)$. Then

$$w_{ij}(n) \le [r_{ij}(T) + r_{ii}(n) + r_{jj}(n) - 1] \ln n + o((\ln n)^{1/2}).$$

From (2) we conclude that

$$r_{ij}(T) \le \frac{r_{ij}(n) \ln n}{\ln T} \le \frac{2}{t} r_{ij}(n).$$

This gives

$$w_{ij}(n) \le w(n) = \left[2\left(1 + \frac{1}{t}\right) m(n) - 1\right] \ln n + o((\ln n)^{1/2}).$$

By the definition of s we have

$$\Sigma_2(n) \le K d^2 s(n) e^{w(n)} \le K d^2 \exp[-(\ln n)^{1/2} + o((\ln n)^{1/2})] = o(1)$$

and (9) is proved.

We next show (10). Since Theorem 3 of Wiśniewski [6] implies

$$P[\mathbf{M}_n^W \le \mathbf{x}_s] \to \Phi(\mathbf{x}),$$

according to a multidimensional version of Khinchin's theorem it is sufficient to show that for $i \in \mathbb{N}(d)$,

$$\frac{r_{ii}(s)}{r_{ii}(n)} \to 1, \tag{11}$$

$$b_n r_{ii}(n)^{-1/2} [(1 - r_{ii}(n))^{1/2} - (1 - r_{ii}(s))^{1/2}] \to 0. \tag{12}$$

We deduce from (2) that

$$\frac{\ln s}{\ln n} \le \frac{r_{ii}(n)}{r_{ii}(s)} \le 1.$$

On the other hand, the definitions of s and (3) give

$$\frac{\ln s}{\ln n} = 1 - 2\left(1 + \frac{1}{t}\right) m(n) - (\ln n)^{-1/2} \to 1,$$

which establishes (11).

Since $(1-r)^{1/2} = 1-\frac{1}{2}r+O(r^2)$ as $r \to 0$ it follows that for all sufficiently large n,

$$\begin{aligned}
0 &\le b_n r_{ii}(n)^{-1/2}[(1-r_{ii}(n))^{1/2} - (1-r_{ii}(s))^{1/2}] \\
&\le b_n r_{ii}(n)^{-1/2}[r_{ii}(s)) - r_{ii}(n)] + o(1) \\
&\le (2r_{ii}(n)\ln n)^{1/2}(r_{ii}(n)\ln n)^{-1}[r_{ii}(s)\ln n - r_{ii}(n)\ln n] + o(1) \\
&\le (2r_{ii}(n)\ln n)^{1/2}\left(\frac{\ln n}{\ln s} - 1\right) + o(1) \\
&\le (2r_{ii}(n)\ln n)^{1/2}\left[2\left(1+\frac{1}{t}\right)m(n) + o((\ln n)^{-1/2})\right] + o(1) \\
&\le 2^{3/2}\left(1+\frac{1}{t}\right)[m(n)(\ln n)^{1/3}]^{3/2} + o(1).
\end{aligned}$$

From the above inequalities and (6) we obtain (12). This completes the proof.

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Mateusz Wiśniewski
Technical University of Kielce
Al. 1000-Lecia Państwa Polskiego 7
25-314 Kielce, Poland
E-mail: ztpmw@eden.tu.kielce.pl

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