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ESTIMATORS OF g-MONOTONE DEPENDENCE FUNCTIONS

Abstract. The notion of g-monotone dependence function introduced in [4] generalizes the notions of the monotone dependence function and the quantile monotone dependence function defined in [2], [3] and [6]. In this paper we study the asymptotic behaviour of sample g-monotone dependence functions and their strong properties.

1. Introduction. The g-monotone dependence function, introduced in [4], measures the monotone dependence, i.e. the tendency to associate large values of the first random variable with large values of the second, according to the nondecreasing but nonconstant real function g. The second parameter of this class, as in the case of the quantile monotone dependence function (cf. [5]), is $q \in (0,1)$, the "level of association". The main field of applications of the g-monotone dependence function is in investigation of dependence in statistics. This paper investigates statistical properties of the g-monotone dependence function. We introduce the estimators of this notion and study their strong asymptotic behaviour. First, however, we recall their definition.

Let $\mathcal{G}$ be a set of real-valued nondecreasing but nonconstant functions and let $\mathcal{C}_g$ ($g \in \mathcal{G}$) be the class of pairs of random variables (X, Y) satisfying $E|g(X)| < \infty$. For an arbitrary pair of random variables (X, Y) with marginal distribution functions F_X, F_Y, we define

$$F_p^{(1)}(x) := \frac{P[X < x, Y > y_p] + (1 - p - P[Y > y_p])P[X < x \mid Y = y_p]}{1 - p},$$

$$F_p^{(2)}(x) := \frac{P[X < x, Y < y_p] + (p - P[Y < y_p])P[X < x \mid Y = y_p]}{p},$$

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$$F_p^{(3)}(x) := \frac{P[X < x] - p}{1 - p} I_{(x_p,\infty)}(x),$$

$$F_p^{(4)}(x) := \frac{P[X < x]}{p} I_{(-\infty,x_p)}(x) + I_{[x_p,\infty)}(x),$$

where $I[\cdot]$ denotes the indicator function, $I_A(x) = I[x \in A]$, and $x_p = Q_p(F_X)$ and $y_p = Q_p(F_Y)$ stand for a pth quantile of F_X and F_Y, respectively. By convention, we assume that $E(X \mid A) = 0$ and $P[X < x \mid A] = 0$ if $P[A] = 0$.

For $g \in \mathcal{G}$, $(q,p) \in (0,1)^2$ and an arbitrary real function F such that

$$\int_{\mathbb{R}} |g(x)| \, dF(x) < \infty$$

we define

$$\phi_g(F,x) = \int_{\mathbb{R}} g(u - x) \, dF(u), \qquad x \in \mathbb{R}. \tag{1.1}$$

Definition 1. The *g-monotone dependence functions* $\mu_{X,Y}^{(j)}(g,q;\cdot)$, $j = 1,2$, for $g \in \mathcal{G}$, $(X,Y) \in \mathcal{C}_g$, $q \in (0,1)$, are defined as follows:

$$\mu_{X,Y}^{(1)}(g,q;p) = \begin{cases} \dfrac{x^{(1)}(g,q;p) - x_q}{x^{(3)}(g,q;p) - x_q} & \text{if } x^{(1)}(g,q;p) - x_q \ge 0, \\[2ex] \dfrac{x^{(1)}(g,q;p) - x_q}{x_q - x^{(4)}(g,q;1-p)} & \text{otherwise}, \end{cases} \tag{1.2}$$

$$\mu_{X,Y}^{(2)}(g,q;p) = \begin{cases} \dfrac{x^{(2)}(g,q;p) - x_q}{x^{(4)}(g,q;p) - x_q} & \text{if } x_q - x^{(2)}(g,q;p) \ge 0, \\[2ex] \dfrac{x^{(2)}(g,q;p) - x_q}{x_q - x^{(3)}(g,q;1-p)} & \text{otherwise}, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} x^{(i)}(g,q;p) = {}& \lambda \sup\{x : \phi_g(F_p^{(i)},x) \ge \phi_g(F_X,x_q)\} \\ &+ (1-\lambda) \inf\{x : \phi_g(F_p^{(i)},x) < \phi_g(F_X,x_q)\}, \\ & \qquad i = 1,2,3,4, \end{aligned} \tag{1.4}$$

$x_q = Q_q(F_X)$ and $\lambda \in [0,1]$ is a parameter constant throughout this paper.

The properties of this notion are listed in [4].

2. Sample g-monotone dependence functions. Let (X_1,Y_1), $(X_2,Y_2), \ldots, (X_n,Y_n)$ be a sequence of pairs of independent identically distributed random variables belonging to $\mathcal{C}_g$ for some $g \in \mathcal{G}$, and let (X,Y) denote a pair of random variables with the same distribution function. Fix $(q,p) \in (0,1)^2$.

Let $X_{k:n}, Y_{k:n}$, $k = 1, \dots, n$, be the kth order statistics of the sequences $\{X_i,\ 1 \le i \le n\}$ and $\{Y_i,\ 1 \le i \le n\}$, respectively. We additionally define $X_{n+1:n} = X_{n:n}$ and $Y_{n+1:n} = Y_{n:n}$, $n \ge 1$. Then the pth quantiles may be chosen as

$$\widehat{x}_p^{(n)} = X_{k:n}, \qquad \widehat{y}_p^{(n)} = Y_{k:n},$$

where $k = [np] + 1$ (for full discussion cf. [1]). The two-dimensional sample distribution function and the boundary sample distribution functions may be defined as

$$\widehat{F}_{X,Y}(x,y) = \widehat{P}[X < x, Y < y] = \sum_{j=1}^{n} I[X_j < x, Y_j < y]/n$$

and

$$\widehat{F}_X(x) = \widehat{P}[X < x] = \sum_{j=1}^{n} I[X_j < x]/n,$$
$$\widehat{F}_Y(y) = \widehat{P}[Y < y] = \sum_{j=1}^{n} I[Y_j < y]/n,$$

respectively. Let us define the sample distribution functions $\widehat{F}_{p,n}^{(k)}(\cdot)$, $k = 1, 2, 3, 4$, by

$$\widehat{F}_{p,n}^{(1)}(x) = \frac{\sum_{j=1}^{n} I[X_j < x, Y_j > \widehat{y}_p^{(n)}]}{n(1-p)} + \left(1 - p - \frac{\sum_{j=1}^{n} I[Y_j > \widehat{y}_p^{(n)}]}{n}\right) \frac{\sum_{j=1}^{n} I[X_j < x, Y_j = \widehat{y}_p^{(n)}]}{(1-p)\sum_{j=1}^{n} I[Y_j = \widehat{y}_p^{(n)}]},$$

$$\widehat{F}_{p,n}^{(2)}(x) = \frac{\sum_{j=1}^{n} I[X_j < x, Y_j < \widehat{y}_p^{(n)}]}{np} + \left(p - \frac{\sum_{j=1}^{n} I[Y_j < \widehat{y}_p^{(n)}]}{n}\right) \frac{\sum_{j=1}^{n} I[X_j < x, Y_j = \widehat{y}_p^{(n)}]}{p\sum_{j=1}^{n} I[Y_j = \widehat{y}_p^{(n)}]},$$

$$\widehat{F}_{p,n}^{(3)}(x) = \frac{\sum_{j=1}^{n} I[X_j < x] - np}{n(1-p)}\, I[\widehat{x}_p^{(n)} < x],$$

$$\widehat{F}_{p,n}^{(4)}(x) = \frac{\sum_{j=1}^{n} I[X_j < x]}{np}\, I[x < \widehat{x}_p^{(n)}] + I[x \ge \widehat{x}_p^{(n)}].$$

The appropriate sample function $\widehat{\phi}$ may be defined as follows:

$$\widehat{\phi}_g(\widehat{F}_{p,n}^{(1)}, x) = \frac{\sum_{j=1}^{n} g(X_j - x) I[Y_j > \widehat{y}_p^{(n)}]}{n(1-p)} + \left(1 - \frac{\sum_{j=1}^{n} I[Y_j > \widehat{y}_p^{(n)}]}{n(1-p)}\right) \frac{\sum_{j=1}^{n} g(X_j - x) I[Y_j = \widehat{y}_p^{(n)}]}{\sum_{j=1}^{n} I[Y_j = \widehat{y}_p^{(n)}]},$$

$$\begin{aligned}
\widehat{\phi}_g(\widehat{F}^{(2)}_{p,n}, x) &= \frac{\sum_{j=1}^n g(X_j - x) I[Y_j < \widehat{y}^{(n)}_p]}{np} \\
&\quad + \left(1 - \frac{\sum_{j=1}^n I[Y_j < \widehat{y}^{(n)}_p]}{np}\right) \frac{\sum_{j=1}^n g(X_j - x) I[Y_j = \widehat{y}^{(n)}_p]}{\sum_{j=1}^n I[Y_j = \widehat{y}^{(n)}_p]}, \\
\widehat{\phi}_g(\widehat{F}^{(3)}_{p,n}, x) &= \frac{\sum_{j=1}^n g(X_j - x) I[X_j > \widehat{x}^{(n)}_p]}{n(1-p)} \\
&\quad + \frac{g(\widehat{x}^{(n)}_p - x) \sum_{j=1}^n (I[X_j \le \widehat{x}^{(n)}_p] - p)}{n(1-p)}, \\
\widehat{\phi}_g(\widehat{F}^{(4)}_{p,n}, x) &= \frac{\sum_{j=1}^n g(X_j - x) I[X_j < \widehat{x}^{(n)}_p]}{np} \\
&\quad - \frac{g(\widehat{x}^{(n)}_p - x) \sum_{j=1}^n (1 - p - I[X_j \ge \widehat{x}^{(n)}_p])}{np}, \\
\widehat{\phi}_g(\widehat{F}_X, x) &= \frac{\sum_{j=1}^n g(X_j - x)}{n}.
\end{aligned}$$

Note that when observations are unique (i.e. $X_j \ne X_i$ and $Y_j \ne Y_i$, $j \ne i$, $1 \le i, j \le n$, a.s.), then

$$\begin{aligned}
\widehat{\phi}_g(\widehat{F}^{(1)}_{p,n}, x) &= \frac{\sum_{j=1}^n g(X_j - x)(I[Y_j \ge \widehat{y}^{(n)}_p] - \{np\} I[Y_j = \widehat{y}^{(n)}_p])}{n(1-p)}, \\
\widehat{\phi}_g(\widehat{F}^{(2)}_{p,n}, x) &= \frac{\sum_{j=1}^n g(X_j - x)(I[Y_j < \widehat{y}^{(n)}_p] + \{np\} I[Y_j = \widehat{y}^{(n)}_p])}{np},
\end{aligned}$$

where $\{x\} = x - [x]$ is the fractional part of x.

Define the integer-valued random variables $\Theta^{(j)}_n = \Theta^{(j)}_n(g, q; p)$, $1 \le j \le 4$, by the inequalities

$$\widehat{\phi}_g(\widehat{F}^{(j)}_{p,n}, X_{\Theta^{(j)}_n : n}) \ge \widehat{\phi}_g(\widehat{F}_X, \widehat{x}^{(n)}_q) > \widehat{\phi}_g(\widehat{F}^{(j)}_{p,n}, X_{\Theta^{(j)}_n + 1 : n}),$$

and the random variables $Z^{(j)}_n = Z^{(j)}_n(g, q; p)$, $V^{(j)}_n = V^{(j)}_n(g, q; p), 1 \le j \le 4$, as follows:

$$\begin{aligned}
Z^{(j)}_n(g, q; p) &= \lambda \sup\{x : \widehat{\phi}_g(\widehat{F}^{(j)}_{p,n}, x) \ge \widehat{\phi}_g(\widehat{F}_X, \widehat{x}^{(n)}_q)\} \\
&\qquad + (1-\lambda) \inf\{x : \widehat{\phi}_g(\widehat{F}^{(j)}_{p,n}, x) < \widehat{\phi}_g(\widehat{F}_X, \widehat{x}^{(n)}_q)\}, \\
V^{(j)}_n(g, q; p) &= \lambda X_{\Theta^{(j)}_n(g,q;p):n} + (1-\lambda) X_{\Theta^{(j)}_n(g,q;p)+1:n}, \qquad j = 1, 2, 3, 4.
\end{aligned}$$

Now we introduce two estimators of the g-monotone dependence functions.

Definition 2. For $(X, Y) \in \mathcal{C}_g$ $(g \in \mathcal{G})$ and $q \in (0, 1)$, we define the estimators $\widehat{\mu}^{(k)}_n(g, q; p)$ and $\widetilde{\mu}^{(k)}_n(g, q; p)$, $k = 1, 2$, as follows:

$$\widehat{\mu}_n^{(1)}(g,q;p) = \begin{cases} \dfrac{Z_n^{(1)}(g,q;p) - \widehat{x}_q^{(n)}}{Z_n^{(3)}(g,q;p) - \widehat{x}_q^{(n)}} & \text{if } Z_n^{(1)}(g,q;p) - \widehat{x}_q^{(n)} \geq 0, \\ \dfrac{Z_n^{(1)}(g,q;p) - \widehat{x}_q^{(n)}}{\widehat{x}_q^{(n)} - Z_n^{(4)}(g,q;1-p)} & \text{otherwise,} \end{cases} \tag{2.1}$$

$$\widehat{\mu}_n^{(2)}(g,q;p) = \begin{cases} \dfrac{Z_n^{(2)}(g,q;p) - \widehat{x}_q^{(n)}}{Z_n^{(4)}(g,q;p) - \widehat{x}_q^{(n)}} & \text{if } \widehat{x}_q^{(n)} - Z_n^{(2)}(g,q;p) \geq 0, \\ \dfrac{Z_n^{(2)}(g,q;p) - \widehat{x}_q^{(n)}}{\widehat{x}_q^{(n)} - Z_n^{(3)}(g,q;1-p)} & \text{otherwise,} \end{cases} \tag{2.2}$$

$$\widetilde{\mu}_n^{(1)}(g,q;p) = \begin{cases} \dfrac{V_n^{(1)}(g,q;p) - \widehat{x}_q^{(n)}}{V_n^{(3)}(g,q;p) - \widehat{x}_q^{(n)}} & \text{if } V_n^{(1)}(g,q;p) - \widehat{x}_q^{(n)} \geq 0, \\ \dfrac{V_n^{(1)}(g,q;p) - \widehat{x}_q^{(n)}}{\widehat{x}_q^{(n)} - V_n^{(4)}(g,q;1-p)} & \text{otherwise,} \end{cases} \tag{2.3}$$

$$\widetilde{\mu}_n^{(2)}(g,q;p) = \begin{cases} \dfrac{V_n^{(2)}(g,q;p) - \widehat{x}_q^{(n)}}{V_n^{(4)}(g,q;p) - \widehat{x}_q^{(n)}} & \text{if } \widehat{x}_q^{(n)} - V_n^{(2)}(g,q;p) \geq 0, \\ \dfrac{V_n^{(2)}(g,q;p) - \widehat{x}_q^{(n)}}{\widehat{x}_q^{(n)} - V_n^{(3)}(g,q;1-p)} & \text{otherwise.} \end{cases} \tag{2.4}$$

In general, $\widehat{\mu}_n^{(k)}(g,q;p)$ is a better estimator than $\widetilde{\mu}_n^{(k)}(g,q;p)$, although the second is easier in computations.

3. The main results. In this section we give conditions under which

$$\widehat{\mu}_n^{(k)}(g,q;p) \to \mu_{X,Y}^{(k)}(g,q;p) \quad \text{a.s.} \tag{3.1}$$

and

$$\widetilde{\mu}_n^{(k)}(g,q;p) \to \mu_{X,Y}^{(k)}(g,q;p) \quad \text{a.s.,} \tag{3.2}$$

as $n \to \infty$, $k = 1, 2$. We begin with some auxiliary definitions.

For every random variable X and distribution function F we denote by V_X and V_F the sets of points of continuity of F_X and F, respectively. Let $\mathcal{H}$ be the class of distribution functions F such that if $x_0 \notin V_F$ then there exist $\delta_1, \delta_2 > 0$ such that $x_0 - \delta_1 \leq x < x_0$ implies $F(x) = F(x_0)$, and $x_0 < x \leq x_0 + \delta_2$ implies $F(x) = F(x_0 + 0)$.

Let $\mathcal{K}$ be the class of nondecreasing functions such that:

(i) $\forall_{a \in \mathbb{R}}\ I_{(a,\infty)}(\cdot), I_{[a,\infty)}(\cdot) \in \mathcal{K}$;
(ii) If g is a continuous nondecreasing Lipschitz function then $g \in \mathcal{K}$;
(iii) $\forall_{a \in \mathbb{R}} \exp\{a\cdot\} \in \mathcal{K}$;
(iv) If $g_1, \ldots, g_k \in \mathcal{K}$ then $\forall_{\lambda_1,\ldots,\lambda_k \in \mathbb{R}}\ \lambda_1 g_1 + \lambda_2 g_2 + \ldots + \lambda_k g_k \in \mathcal{K}$;

(v) If $g \in \mathcal{K}$, $a, b \in \mathbb{R}$, $a < b$ then $g(a)I_{(-\infty,a]}(\cdot) + g(x)I_{(a,b)}(\cdot) + g(b)I_{[b,\infty)}(\cdot) \in \mathcal{K}$.

Let us now pass to the main results of this section.

Theorem 1. *Assume*

(i) $\phi_g(F_p^{(k)}, \cdot)$ *and* $\phi_g(F_{1-p}^{(k)}, \cdot)$ *are nonconstant and continuous in a sufficiently small neighbourhood of* $x^{(k)}(g, q; p)$, $k = 1, 2, 3, 4$, *and* $x^{(k)}(g, q; 1-p)$, $k = 3, 4$, *respectively*;

(ii) $\phi_g(F_X, \cdot)$ *is nonconstant in a sufficiently small neighbourhood of* x_q *and* F_X *is strictly increasing in* x_q;

(iii) $F_X, F_Y \in \mathcal{H}$;

(iv) $g \in \mathcal{K}$;

(v) $x_q \neq x^{(k)}(g, q; p)$ *and* $x_q \neq x^{(k)}(g, q; 1-p)$, $k = 3, 4$.

Then for $k = 1, 2$,

$$\widehat{\mu}_n^{(k)}(g, q; p) \to \mu_{X,Y}^{(k)}(g, q; p) \quad \text{a.s. as } n \to \infty. \tag{3.3}$$

For three special cases of the function g the convergence (3.3) will be considered in the following propositions:

Proposition 1. *Assume* $g(u) = I[u > 0]$ *and*

(i) $F_p^{(k)}(\cdot)$ *and* $F_{1-p}^{(k)}(\cdot)$ *are nonconstant and continuous in a sufficiently small neighbourhood of* $x^{(k)}(g, q; p)$, $k = 1, 2, 3, 4$, *and* $x^{(k)}(g, q; 1-p)$, $k = 3, 4$, *respectively*;

(ii) F_X *is strictly increasing in* x_q;

(iii) $F_X, F_Y \in \mathcal{H}$;

(iv) $x_q \neq x^{(k)}(g, q; p)$ *and* $x_q \neq x^{(k)}(g, q; 1-p)$, $k = 3, 4$.

Then for $k = 1, 2$,

$$\widehat{\mu}_n^{(k)}(g, q; p) \to \mu_{X,Y}^{(k)}(g, q; p) \quad \text{a.s. as } n \to \infty. \tag{3.4}$$

Proposition 2. *Assume* $g(u) = u$ *or* $g(u) = e^{\lambda u}$ *and* $F_X, F_Y \in \mathcal{H}$. *Then for* $k = 1, 2$,

$$\widehat{\mu}_n^{(k)}(g, q; p) \to \mu_{X,Y}^{(k)}(g, q; p) \quad \text{a.s. as } n \to \infty. \tag{3.5}$$

It is easy to see that this result improves Theorem 2 of [2].

Theorem 2. *In addition to the assumptions of Theorem* 1 (*or Propositions* 1, 2) *assume that*

(vi) $F_X(\cdot)$ *is continuous in a sufficiently small neighbourhood of* $x^{(k)}(g, q; p)$, $k = 1, 2, 3, 4$, *and* $x^{(k)}(g, q; 1-p)$, $k = 3, 4$.

Then

$$\widetilde{\mu}_n^{(k)}(g, q; p) \to \mu_{X,Y}^{(k)}(g, q; p) \quad \text{a.s. as } n \to \infty. \tag{3.6}$$

4. The proofs. We begin with the following auxiliary results.

PROPOSITION 3. *For every $g \in \mathcal{G}$ and sample $\{(X_j, Y_j) : j \geq 1\}$ drawn from the probability law from $\mathcal{C}_g$ the processes $\{\widehat{\phi}_g(\widehat{F}_{p,n}^{(k)}, x)\}_{x\in\mathbb{R}}$, $k = 1, 2, 3, 4$, and $\{\widehat{\phi}_g(\widehat{F}_X, x)\}_{x\in\mathbb{R}}$ have nonincreasing paths. Furthermore, for every pair $(X, Y) \in \mathcal{C}_g$ of random variables the function $\phi_g(F_X, \cdot)$ is nonincreasing.*

The proof is an easy consequence of $g(X - \cdot)$ being nonincreasing, for every random variable X.

PROPOSITION 4 [7, §2.3.2]. *Let $0 < p < 1$ be such that x_p and y_p are unique pth quantiles of F_X and F_Y, respectively. Then for every $\varepsilon > 0$ and $n \in \mathbb{N}$,*

$$P[|\widehat{x}_p^{(n)} - x_p| > \varepsilon] \leq 2e^{-2n\delta_\varepsilon^2(X)}$$

and

$$P[|\widehat{y}_p^{(n)} - y_p| > \varepsilon] \leq 2e^{-2n\delta_\varepsilon^2(Y)}$$

where

$$\delta_\varepsilon(X) = \min\{F_X(x_p + \varepsilon) - p, p - F_X(x_p - \varepsilon)\},$$
$$\delta_\varepsilon(Y) = \min\{F_Y(y_p + \varepsilon) - p, p - F_Y(y_p - \varepsilon)\}.$$

The following results are a generalization of the well-known Glivenko–Cantelli Theorem (cf. [7, §2.1.4]).

PROPOSITION 5. *Let $\{\widehat{f}_n(x) : x \in \mathbb{R}\}$ be a sequence of nonincreasing left-continuous random processes such that for some nonrandom function f and every $x \in \mathbb{R}$ we have*

$$\widehat{f}_n(x) \to f(x) \quad \text{a.s. as } n \to \infty.$$

If K is a compact subset of $\mathbb{R}$, then

$$\sup_{x\in f^{-1}(K)} |\widehat{f}_n(x) - f(x)| \to 0 \quad \text{a.s. as } n \to \infty.$$

COROLLARY 1. *We have*

$$\sup_{x\in\mathbb{R}} |\widehat{F}_X(x) - F_X(x)| \to 0 \quad \text{a.s. as } n \to \infty,$$
$$\sup_{x\in\mathbb{R}} |\widehat{F}_Y(x) - F_Y(x)| \to 0 \quad \text{a.s. as } n \to \infty.$$

If additionally $F_Y \in \mathcal{H}$, then for every $k = 1, 2, 3, 4$,

$$\sup_{x\in\mathbb{R}} |\widehat{F}_{p,n}^{(k)}(x) - F_p^{(k)}(x)| \to 0 \quad \text{a.s. as } n \to \infty.$$

The first part of Corollary 1 (the Glivenko–Cantelli Theorem) is an easy consequence of Proposition 5 and of the strong law of large numbers, but the proof of the second part is not so easy, mainly because in the definition of $\widehat{F}_{p,n}^{(k)}$ we use $\widehat{y}_p^{(n)}$ instead of y_p and in consequence $E\widehat{F}_{p,n}^{(k)}$ may be different from $F_p^{(k)}$. To prove the second part of Corollary 1 we must use the techniques developed later in the proof of Proposition 6.

Proof of Proposition 5. By a linear change of variables we may and do assume that $K = [0,1]$. Furthermore, for every $r, k \in \mathbb{N}$, $r < k$, we let $x_{r,k}$ be a solution of the inequalities

$$f(x-0) = f(x) \geq r/k \geq f(x+0),$$

and let $A_{r,k} = [f_n(x_{r,k}) \to f(x_{r,k})]$. From our assumptions we get

$$P\Big(\bigcap_{k\in\mathbb{N}} \bigcap_{1\leq r\leq k} A_{r,k}\Big) = 1$$

so that

$$\sup_{k\in\mathbb{N}} \max_{1\leq r\leq k} |f_n(x_{r,k}) - f(x_{r,k})| \to 0 \quad \text{a.s. as } n \to \infty.$$

On the other hand, for every $k \in \mathbb{N}$,

$$\sup_{x\in f^{-1}(K)} |f_n(x) - f(x)| \leq \max_{1\leq r\leq k} |f_n(x_{r,k}) - f(x_{r,k})| + 1/k,$$

and because K is arbitrary,

$$[\sup_{x\in f^{-1}(K)} |f_n(x) - f(x)| \to 0] \supset \bigcap_{k\in\mathbb{N}} \bigcap_{1\leq r\leq k} A_{r,k},$$

which completes the proof.

PROPOSITION 6. *Let $(X,Y) \in \mathcal{C}_g$ (for some $g \in \mathcal{G}$) and $F_X, F_Y \in \mathcal{H}$. Let $\{(X_i, Y_i) : i \geq 1\}$ be a sequence of independent random variables with the same distribution function as (X,Y). Then for every $q, p \in (0,1)$, $k = 1,2,3,4$,*

$$\widehat{\phi}_g(\widehat{F}_{p,n}^{(k)}, x) \to \phi_g(F_p^{(k)}, x) \quad \text{a.s. as } n \to \infty. \tag{4.1}$$

Moreover, if $g \in \mathcal{K}$ and F_X is strictly increasing in x_q, then

$$\widehat{\phi}_g(\widehat{F}_X, \widehat{x}_q^{(n)}) \to \phi_g(F_X, x_q) \quad \text{a.s. as } n \to \infty. \tag{4.2}$$

Proof. Assume $k = 1$ and remark that $\widehat{\phi}_g(\widehat{F}_{p,n}^{(1)}, x)$ and $\phi_g(F_p^{(1)}, x)$ may be rewritten in the following way:

$$\begin{aligned}
\widehat{\phi}_g(\widehat{F}_{p,n}^{(1)}, x) &= \alpha_{p,n}(\widehat{y}_p^{(n)}) \frac{\sum_{j=1}^n g(X_j - x) I[Y_j > \widehat{y}_p^{(n)}]}{n(1-p)} \\
&\quad + (1 - \alpha_{p,n}(\widehat{y}_p^{(n)})) \frac{\sum_{j=1}^n g(X_j - x) I[Y_j \geq \widehat{y}_p^{(n)}]}{n(1-p)}, \\
\phi_g(F_p^{(1)}, x) &= \alpha_p(y_p) \frac{Eg(X-x)I[Y > y_p]}{1-p} \\
&\quad + (1 - \alpha_p(y_p)) \frac{Eg(X-x)I[Y \geq y_p]}{1-p},
\end{aligned}$$

where

$$\alpha_{p,n}(y) = \begin{cases} \dfrac{\sum_{j=1}^{n}(1-p-I[Y_j>y])}{\sum_{j=1}^{n} I[Y_j=y]} & \text{if } \sum_{j=1}^{n} I[Y_j=y]>0,\\ 1 & \text{otherwise,}\end{cases}$$

$$\alpha_{p}(y) = \begin{cases} \dfrac{1-p-P[Y>y]}{P[Y=y]} & \text{if } P[Y=y]>0,\\ 1 & \text{otherwise.}\end{cases}$$

It is easily seen that $\alpha_{p,n}(y)$ and $\alpha_p(y)$ are continuous and $0 \le \alpha_{p,n}(y), \alpha_p(y) \le 1$.

In the proof we consider the following six cases:

(i) $\underline{y}_p = \overline{y}_p = y_p$, $y_p \in V_Y$;
(ii) $\underline{y}_p = \overline{y}_p = y_p$, $y_p \notin V_Y$;
(iii) $\underline{y}_p < \overline{y}_p$, $\underline{y}_p, \overline{y}_p \in V_Y$;
(iv) $\underline{y}_p < \overline{y}_p$, $\underline{y}_p \in V_Y$, $\overline{y}_p \notin V_Y$;
(v) $\underline{y}_p < \overline{y}_p$, $\underline{y}_p \notin V_Y$, $\overline{y}_p \in V_Y$;
(vi) $\underline{y}_p < \overline{y}_p$, $\underline{y}_p, \overline{y}_p \notin V_Y$;

here $\underline{y}_p = \sup\{y : F_Y(y) < p\}$ and $\overline{y}_p = \inf\{y : F_Y(y) > p\}$.

Case (i). From Proposition 4, for every $\varepsilon > 0$ there exists n_0 such that $\widehat{y}_p^{(n)} \in (y_p - \varepsilon, y_p + \varepsilon)$ for every $n > n_0$. Therefore for $n > n_0$ we get

$$\tag{4.3} \begin{aligned} &\left| \frac{\sum_{j=1}^{n} g(X_j - x) I[Y_j > \widehat{y}_p^{(n)}]}{n} - Eg(X-x)I[Y > y_p] \right| \\ &\quad\le \frac{\sum_{j=1}^{n} |g(X_j - x)| I[Y_j \in (y_p-\varepsilon, y_p+\varepsilon)]}{n} \\ &\qquad + \left| \frac{\sum_{j=1}^{n} g(X_j - x) I[Y_j > y_p]}{n} - Eg(X-x)I[Y > y_p] \right|. \end{aligned}$$

From the strong law of large numbers the first term on the right-hand side of (4.3) tends to $E|g(X-x)|I[y \in (y_p-\varepsilon, y_p+\varepsilon)]$, whereas the second term tends almost surely to 0. Because ε was chosen arbitrarily and $E|g(X-x)| < \infty$ we obtain

$$\tag{4.4} \frac{\sum_{j=1}^{n} g(X_j - x) I[Y_j > \widehat{y}_p^{(n)}]}{n} \to Eg(X-x)I[Y > y_p] \quad \text{a.s. as } n \to \infty,$$

and similarly

$$\tag{4.5} \frac{\sum_{j=1}^{n} g(X_j - x) I[Y_j \ge \widehat{y}_p^{(n)}]}{n} \to Eg(X-x)I[Y \ge y_p] \quad \text{a.s. as } n \to \infty.$$

Hence from the equation

$$Eg(X-x)I[Y>y_p] = Eg(X-x)I[Y \geq y_p]$$

we get the assertion, independently of the asymptotic behaviour of $\alpha_{p,n}(\widehat{y}_p^{(n)})$.

Case (ii). From Proposition 4 and the fact that $F_Y \in \mathcal{H}$, there exists n_0 such that for every $n > n_0$,

$$\begin{aligned} I[Y_n > \widehat{y}_p^{(n)}] &= I[Y_n > y_p], \\ I[Y_n = \widehat{y}_p^{(n)}] &= I[Y_n = y_p], \\ I[Y_n < \widehat{y}_p^{(n)}] &= I[Y_n < y_p]. \end{aligned} \tag{4.6}$$

Therefore, in this case (4.4) and (4.5) also hold, so that for (4.1) we must only prove that $\alpha_{p,n}(\widehat{y}_p^{(n)}) \to \alpha_p(y_p)$ a.s. as $n \to \infty$. But this is a consequence of (4.6) and of the strong law of large numbers applied to the numerator and denominator of $\alpha_{p,n}(\widehat{y}_p^{(n)})$.

Case (iii). For $\varepsilon > 0$, we define two sets A_1 and A_2 of positive integers such that $n \in A_1 \Rightarrow \widehat{y}_p^{(n)} \in (\underline{y}_p - \varepsilon, \underline{y}_p]$ and $n \in A_2 \Rightarrow \widehat{y}_p^{(n)} \in [\overline{y}_p, \overline{y}_p + \varepsilon)$. The proof runs similarly to case (i). We now have

$$\begin{aligned} \lim_{n\in A_1,\, n\to\infty} & \frac{\sum_{j=1}^n g(X_j - x)I[Y_j > \widehat{y}_p^{(n)}]}{n} \\ &= \lim_{n\in A_1,\, n\to\infty} \frac{\sum_{j=1}^n g(X_j - x)I[Y_j \geq \widehat{y}_p^{(n)}]}{n} \\ &\geq Eg(X-x)I[Y \leq \underline{y}_p] \quad \text{a.s.}, \\ \lim_{n\in A_2,\, n\to\infty} & \frac{\sum_{j=1}^n g(X_j - x)I[Y_j > \widehat{y}_p^{(n)}]}{n} \\ &= \lim_{n\in A_2,\, n\to\infty} \frac{\sum_{j=1}^n g(X_j - x)I[Y_j \geq \widehat{y}_p^{(n)}]}{n} \\ &\leq Eg(X-x)I[Y \geq \overline{y}_p] \quad \text{a.s.}, \end{aligned}$$

which concludes the proof of (iii), as $P[\underline{y}_p \leq Y \leq \overline{y}_p] = 0$.

Case (iv). It is possible to define, for every $\varepsilon > 0$, two infinite sets A_1 and A_2 of positive integers such that $\widehat{y}_p^{(n)} \in (\underline{y}_p - \varepsilon, \underline{y}_p]$ for every $n \in A_1$, and $\widehat{y}_p^{(n)} = \overline{y}_p$ for every $n \in A_2$. From cases (i) and (ii) we have, a.s.,

$$\begin{aligned} \lim_{n\in A_1,\, n\to\infty} \widehat{\phi}_g(\widehat{F}_{p,n}^{(1)}, x) &= \alpha_p(\underline{y}_p) \frac{Eg(X-x)I[Y > \underline{y}_p]}{1-p} \\ &\quad + (1 - \alpha_p(\underline{y}_p)) \frac{Eg(X-x)I[Y \geq \underline{y}_p]}{1-p}, \end{aligned} \tag{4.7}$$

$$\lim_{n\in A_2,\, n\to\infty} \widehat{\phi}_g(\widehat{F}^{(1)}_{p,n}, x) = \alpha_p(\overline{y}_p)\frac{Eg(X-x)I[Y>\overline{y}_p]}{1-p} + (1-\alpha_p(\overline{y}_p))\frac{Eg(X-x)I[Y\ge\overline{y}_p]}{1-p}. \tag{4.8}$$

Now, taking into account the equalities

$$\alpha_p(\underline{y}_p) = 1, \quad \alpha_p(\overline{y}_p) = 0,$$

$$Eg(X-x)I[Y>\underline{y}_p] = Eg(X-x)I[Y\ge\overline{y}_p] = (1-p)\phi_g(F^{(1)}_p, x),$$

we get (4.1).

Cases (v)–(vi) may be proved similarly to case (iv). We define $A_1 = \{n\in\mathbb{N} : \widehat{y}^{(n)}_p = \underline{y}_p\}$, $A_2 = \{n\in\mathbb{N} : \widehat{y}^{(n)}_p \in [\overline{y}_p, \overline{y}_p+\varepsilon)\}$ and $A_1 = \{n\in\mathbb{N} : \widehat{y}^{(n)}_p = \underline{y}_p\}$, $A_2 = \{n\in\mathbb{N} : \widehat{y}^{(n)}_p = \overline{y}_p\}$ in cases (v) and (vi), respectively, and prove (4.7) and (4.8).

The proof in case $k=2$ is similar. To prove (4.1) for $k=3,4$ it is enough to put $Y_j = X_j$, $j\ge 1$, in (4.1) for $k=1,2$.

Now we prove (4.2). From our assumptions and Proposition 4 we have

$$\widehat{x}^{(n)}_q \to x_q \quad \text{a.s. as } n\to\infty. \tag{4.9}$$

We prove (4.2) separately for every type (i)–(iii) of g in the definition of $\mathcal{K}$. For g as in (i) it is enough to prove that for every $a\in\mathbb{R}$ and $q\in(0,1)$,

$$\frac{\sum_{j=1}^n I[X_j > a+\widehat{x}^{(n)}_q]}{n} \to P[X > a+x_q],$$

$$\frac{\sum_{j=1}^n I[X_j \ge a+\widehat{x}^{(n)}_q]}{n} \to P[X \ge a+x_q],$$

a.s. as $n\to\infty$, which follows from Proposition 4 and Corollary 2.

In case g satisfies (ii), from Proposition 4, for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $\widehat{x}^{(n)}_q \in (x_q-\varepsilon, x_q+\varepsilon)$ for every $n>n_0$. We have

$$\left|\frac{\sum_{i=1}^n g(X_i-\widehat{x}^{(n)}_q)}{n} - Eg(X-x_q)\right| \le \left|\frac{\sum_{i=1}^n g(X_i-x_q)}{n} - Eg(X-x_q)\right| + \frac{\sum_{i=1}^n g(X_i-x_q+\varepsilon) - g(X_i-x_q-\varepsilon)}{n}.$$

From the strong law of large numbers the first term on the right-hand side tends to zero, and the second to $Eg(X-x_q+\varepsilon) - Eg(X-x_q-\varepsilon)$. This last expression tends to zero as $\varepsilon\to 0$, because g is a Lipschitz function.

If g satisfies (iii) then from Proposition 4,

$$e^{-\lambda\widehat{x}^{(n)}_q} \to e^{-\lambda x_q} \quad \text{a.s. as } n\to\infty \tag{4.10}$$

and

$$\frac{\sum_{j=1}^{n} e^{\lambda X_j}}{n} \to Ee^{\lambda X} \quad \text{a.s. as } n \to \infty \tag{4.11}$$

imply

$$\frac{\sum_{j=1}^{n} g(X_j - \widehat{x}_q^{(n)})}{n} \to Eg(X - x_q) \quad \text{a.s. as } n \to \infty. \tag{4.12}$$

The properties (iv)–(vi) of the class $\mathcal{K}$ follow from the construction of the function $\phi_g(F, x)$.

Proof of Theorem 1. From (v) it is enough to prove that for $j = 1, 2, 3, 4$,

$$Z_n^{(j)}(g, q; p) \to x^{(j)}(g, q; p) \quad \text{a.s. as } n \to \infty, \tag{4.13}$$

and for $j = 3, 4$,

$$Z_n^{(j)}(g, q; 1-p) \to x^{(j)}(g, q; 1-p) \quad \text{a.s. as } n \to \infty, \tag{4.14}$$

and

$$\widehat{x}_q^{(n)} \to x_q \quad \text{a.s. as } n \to \infty. \tag{4.15}$$

But (4.15) follows from Proposition 4 and (ii). To simplify notations, in this proof we omit the arguments $g, q; p$ and $g, q; 1-p$. To prove (4.13) choose $\varepsilon > 0$ and put

$$A_1 = [\widehat{\phi}_g(\widehat{F}_X, \widehat{x}_q^{(n)}) \to \phi_g(F_X, x_q)],$$
$$A_2 = [\sup_{x \in (x^{(j)} - \varepsilon, x^{(j)} + \varepsilon)} |\widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x) - \phi_g(F_p^{(j)}, x)| \to 0].$$

Then by Propositions 6 and 4 and assumptions (iii) and (iv), $P[A_1] = P[A_2] = 1$. Define

$$\delta(\varepsilon) = \min\{|\phi_g(F_p, x^{(j)} - \varepsilon) - \phi_g(F_p, x^{(j)})|, \\ |\phi_g(F_p, x^{(j)} + \varepsilon) - \phi_g(F_p, x^{(j)})|\}.$$

Then from Proposition 5 and (i),

$$\omega_n(\varepsilon) = \sup_{x \in (x^{(j)} - \varepsilon, x^{(j)} + \varepsilon)} |\widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x + 0) - \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x - 0)| \to 0$$

a.s. as $n \to \infty$.

On the other hand, putting

$$\underline{Z}_n^{(j)} = \sup\{x : \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x) \geq \widehat{\phi}_g(\widehat{F}_X, \widehat{x}_q^{(n)})\},$$
$$\overline{Z}_n^{(j)} = \inf\{x : \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x) < \widehat{\phi}_g(\widehat{F}_X, \widehat{x}_q^{(n)})\},$$

we prove that

$$(4.16)\qquad \begin{aligned} B_n &= [x^{(j)} - \varepsilon < \underline{Z}_n^{(j)} \le Z_n^{(j)} \le \overline{Z}_n(j) < x^{(j)} + \varepsilon] \\ &\supset B_n^1 \cap B_n^2 \cap B_n^3, \end{aligned}$$

where

$$\begin{aligned} B_n^1 &= [|\widehat{\phi}_g(\widehat{F}_X, \widehat{x}_q^{(n)}) - \phi_g(F_X, x_q)| < \delta(\varepsilon)/2], \\ B_n^2 &= [\sup_{x \in (x^{(j)}-\varepsilon, x^{(j)}+\varepsilon)} |\widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x) - \phi_g(F_p^j, x)| < \delta(\varepsilon)/6], \\ B_n^3 &= [\omega_n(\varepsilon) < \delta(\varepsilon)/3]. \end{aligned}$$

Indeed, from Proposition 3 we get

$$\begin{aligned} B_n &\supset [\widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x^{(j)} - \varepsilon) > \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, \underline{Z}_n^{(j)}) \ge \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, Z_n^{(j)}) \\ &\qquad \ge \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, \overline{Z}_n^{(j)}) > \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, x^{(j)} + \varepsilon)] \\ &\supset [\phi_g(F_p^{(j)}, x^{(j)} - \varepsilon) - \delta(\varepsilon)/6 > \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, \underline{Z}_n^{(j)}) \ge \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, Z_n^{(j)}) \\ &\qquad \ge \widehat{\phi}_g(\widehat{F}_{p,n}^{(j)}, \overline{Z}_n^{(j)}) > \phi_g(F_p^{(j)}, x^{(j)} + \varepsilon) + \delta(\varepsilon)/6] \cap B_n^2 \\ &\supset [\phi_g(F_p^{(j)}, x^{(j)}) + 5\delta(\varepsilon)/6 > \widehat{\phi}_g(\widehat{F}_X, \widehat{x}_q^{(n)}) + \delta(\varepsilon)/3 \\ &\qquad \ge \widehat{\phi}_g(\widehat{F}_X, \widehat{x}_q^{(n)}) - \delta(\varepsilon)/3 \ge \phi_g(F_p^{(j)}, x^{(j)}) - 5\delta(\varepsilon)/6] \cap B_n^2 \cap B_n^3 \\ &\supset B_n^1 \cap B_n^2 \cap B_n^3. \end{aligned}$$

Moreover, by the continuity of $\phi_g(F_p^{(j)}, \cdot)$ and the triangle inequality we have

$$B_n^3 \supset B_n^2.$$

Furthermore,

$$\lim_{n\to\infty} \lim_{m\to\infty} P\Big[\bigcup_{k=n}^{m} B_k^1\Big] = P[A_1] = 1$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} P\Big[\bigcup_{k=n}^{m} B_k^2\Big] = P[A_2] = 1$$

imply

$$\lim_{n\to\infty} \lim_{m\to\infty} P\Big[\bigcup_{k=n}^{m} B_k\Big] = 1,$$

which gives (4.13). The proof of (4.14) is similar.

Proof of Propositions 1 and 2. To prove Proposition 1 we remark that in this case $\phi_g(F, x) = F(x)$ for every distribution function F and $x \in \mathbb{R}$. For Proposition 2 it is easy to check (cf. [4]) that g-monotone dependence functions do not depend on q and x_q, so that we may omit the assumption (ii) of Theorem 1 which deals with x_q. Furthermore, because for every random variable X the equality $EXI[X > x] = EXP[X > x]$ is possible only when $P[X > x] = 0$ or $P[X > x] = 1$, the denominators of the g-monotone dependence functions in these cases are always nonzero and we may omit the assumption (v). The assumption (iv) is trivially satisfied and the fact that we may omit the assumption (i) follows immediately from Proposition 6.

Proof of Theorem 2. It is enough to prove that for $j = 1, 2, 3, 4$,

$$X_{\Theta_n^{(j)}(g,q;p)+1:n} \to x^{(j)}(g,q;p) \quad \text{a.s. as } n \to \infty,$$
$$X_{\Theta_n^{(j)}(g,q;p):n} \to x^{(j)}(g,q;p) \quad \text{a.s. as } n \to \infty,$$

and for $j = 3, 4$,

$$X_{\Theta_n^{(j)}(g,q;1-p)+1:n} \to x^{(j)}(g,q;1-p) \quad \text{a.s. as } n \to \infty,$$
$$X_{\Theta_n^{(j)}(g,q;1-p):n} \to x^{(j)}(g,q;1-p) \quad \text{a.s. as } n \to \infty;$$

but this follows from (vi) and Proposition 3.

5. An example of convergence. In practice, the computation of g-monotone dependence functions is difficult, but for some functions, e.g. $g(x) = \exp\{x\}$, it is easier than for the monotone dependence function. On the other hand, the numerical computations are almost always easy. The computation time depends only on the size of the sample. In this section we give an example of a pair (X, Y) of random variables for which we compute the g-monotone dependence functions. Moreover, for two samples drawn from the (X, Y)-distribution we compute the sample g-monotone dependence functions.

EXAMPLE. Let X be uniformly distributed on $[-1, 1]$ and let, for some parameter $k_m \in [-1, 1]$,

$$Y = \begin{cases} -\dfrac{X - k_m}{1 + k_m} & \text{if } -1 \le X \le k_m, \\ \dfrac{X - k_m}{1 - k_m} & \text{if } k_m < X \le 1. \end{cases} \tag{5.1}$$

The pair (X, Y) of random variables has the uniform distribution on the support given in Figure 1.

Now we draw two samples of size $N = 10$ and $N = 30$, respectively, from uniform distribution on $[-1, 1]$, and compute the sample g-monotone dependence functions.

Putting $q = 0.5$, $k_m = -0.5$, $g(x) = \operatorname{sign}(x)x^s$, $s = 2$, we obtain the results illustrated in Figure 2. The thick lines show the "exact" g-monotone dependence functions.

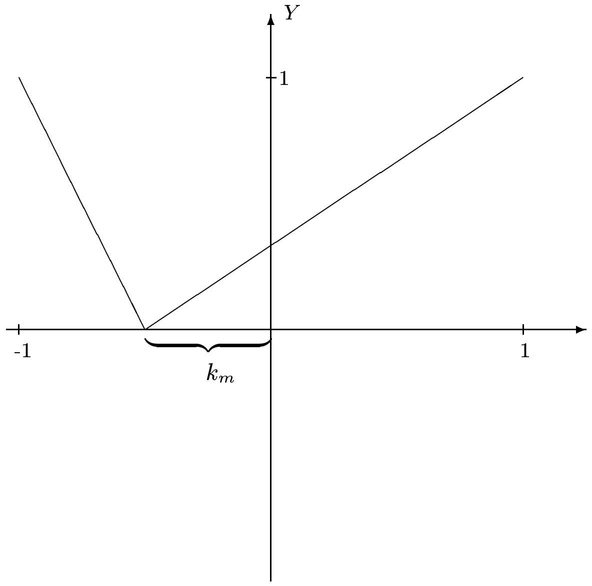

Fig. 1. The support of (X, Y) with parameter $k_m \in [-1, 1]$

If we consider the behaviour of Y with respect to X, we see that Y is strictly decreasing for $X < k_m$ and increasing for $X > k_m$. Therefore, the g-monotone dependence functions $\mu^{(i)}_{(Y,X)}(g, q, p)$, $i = 1, 2$, are first negative (decreasing tendency) but later positive (increasing tendency). The choice of k_m greater than 0.5 moves the zero of those functions to the right. The smaller values of q affect the beginning shape of the curves (they are more extended in this region), whereas the greater values of q affect the ends of the curves. If we consider the smaller s, the curves are vertically flattened, whereas for greater s they are vertically extended.

Let us consider the behaviour of X with respect to Y. We see the constant tendency to association of large values of X with large values of Y as the line $Y = (X - k_m)/(1 - k_m)$ takes 75% of the values of X whereas the line $Y = -(X - k_m)/(1+k_m)$ only 25%, irrespective of the interval of values of Y which we are considering. If $k_m = 0$ then $\mu^{(i)}_{(X,Y)}(g, 0.5, p) \equiv 0, i = 1, 2$. This constant tendency is illustrated in Figure 2 by the line which is almost straight. The choice of a different value of q does not affect the shape of the curves in this case, whereas the choice of a different s results in vertical translation.

The sample g-monotone dependence functions for $N = 10$ are drawn as thin lines and for $N = 30$ we only show dots on continuous curves. We see that the differences between the sample g-monotone dependence function for

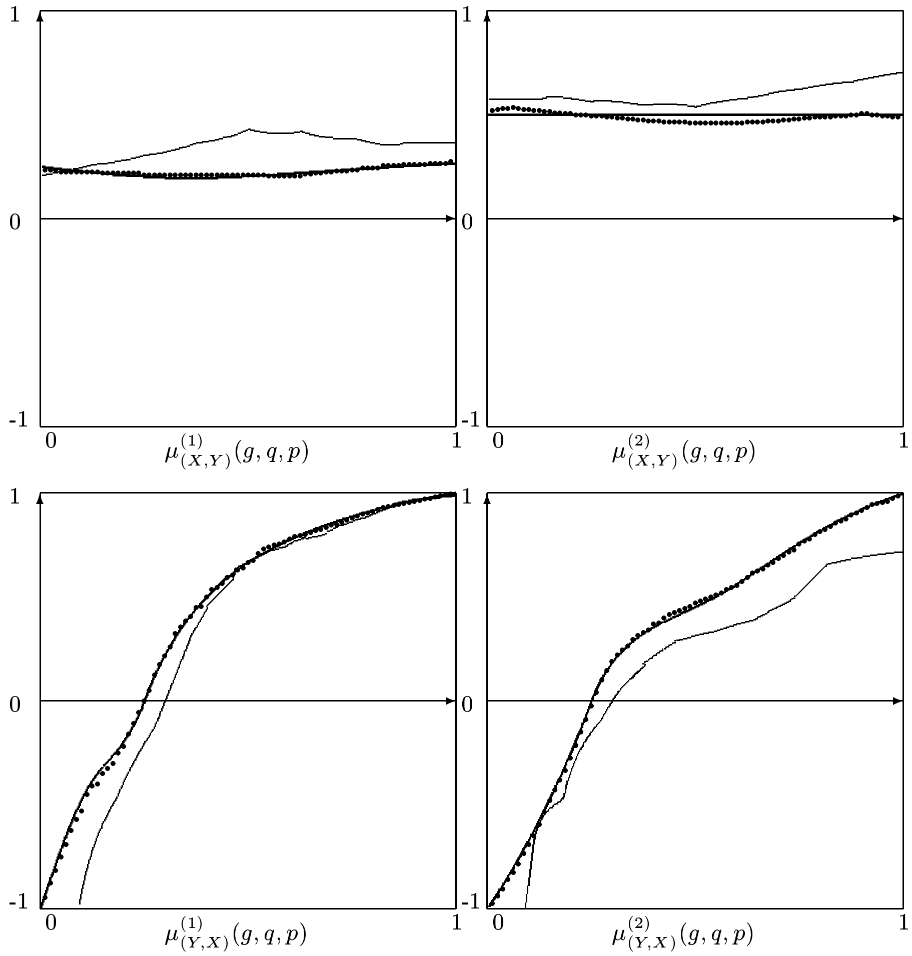

Fig. 2. The g-monotone dependence functions and sample g-monotone dependence functions for the pair (X, Y) defined in Figure 1

$N = 10$ and the exact g-monotone dependence function are visible, whereas for $N = 30$ they are negligibly small.

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