APPLICATIONES MATHEMATICAE
25,2 (1998), pp. 221–252

J. RENCŁAWOWICZ and W. M. ZAJĄCZKOWSKI (Warszawa)

LOCAL EXISTENCE OF SOLUTIONS OF THE MIXED PROBLEM FOR THE SYSTEM OF EQUATIONS OF IDEAL RELATIVISTIC HYDRODYNAMICS

Abstract. Existence and uniqueness of local solutions for the initial-boundary value problem for the equations of an ideal relativistic fluid are proved. Both barotropic and nonbarotropic motions are considered. Existence for the linearized problem is shown by transforming the equations to a symmetric system and showing the existence of weak solutions; next, the appropriate regularity is obtained by applying Friedrich's mollifiers technique. Finally, existence for the nonlinear problem is proved by the method of successive approximations.

1. Introduction. In this paper we prove the local existence of solutions to the equations of ideal relativistic hydrodynamics which are the following system of conservation laws:

$$T^{ij}_{,x^j} = 0, \quad i,j = 0,1,2,3, \tag{1.1}$$

and

$$(\delta u^i)_{,x^i} = 0, \tag{1.2}$$

where the summation convention over repeated indices is assumed and

$$T^{ij} = wu^i u^j + pg^{ij} \tag{1.3}$$

is the energy-momentum tensor, and g^{ij} is the space-time metric tensor of

1991 *Mathematics Subject Classification*: 35L60, 35Q75, 35A07, 35L65.
Key words and phrases: relativistic hydrodynamics, existence, initial-boundary value problem, system of hyperbolic equations of the first order, symmetrization.

the form

$$\{g^{ij}\} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.4}$$

Moreover, $w = e + p$, where w is the density of enthalpy, e the density of the internal energy and δ the density of the fluid particles in a suitable system of coordinates in which the volume element does not move. We denote by p the pressure and by $u = \{u^i\}_{i=0,1,2,3}$ the four-velocity: $u_\alpha = v_\alpha/(c\beta)$, $\alpha = 1, 2, 3$, $u_0 = -1/\beta$, where $\beta = \sqrt{1 - v^2/c^2}$, c is the speed of light, $v^2 = v_1^2 + v_2^2 + v_3^2$, where $v = (v_1, v_2, v_3)$ is the velocity vector, and $u^i = g^{ij}u_j$, $g^{ij} = g_{ij}$, $g^{ij}g_{jk} = \delta^i_k$.

In the above notation the energy-momentum tensor takes the form

$$\begin{aligned} &T_{\alpha\gamma} = w\frac{v_\alpha v_\gamma}{c^2\beta^2} + p\delta_{\alpha\gamma}, \quad \alpha, \gamma = 1, 2, 3, \\ &T_{\alpha 0} = -w\frac{v_\alpha}{c\beta^2}, \quad T_{00} = \frac{w}{\beta^2} - p. \end{aligned} \tag{1.5}$$

We consider problem (1.1)–(1.2) for $t \in [0, T]$ and $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$, with initial and boundary conditions

$$(p, u, \delta)|_{t=0} = (p_0, u_0, \delta_0), \tag{1.6}$$

$$Mz|_{\partial\Omega} = g(x', t), \tag{1.7}$$

where $z = (p, u, \delta)$, and the matrix M is defined in Section 4.

To prove the existence of solutions to (1.1)–(1.2), we have to transform our problem to a symmetric hyperbolic system (2.2). We present this symmetrization in Section 2. In Section 3 we introduce the necessary spaces and norms; moreover, we rewrite the symmetric system (2.2) in the form (3.1) (with the initial-boundary conditions (1.6)–(1.7) suitably transformed).

In Section 4 we consider the linearized problem (3.1); first in 4(a) we prove the existence of solutions in a half-space, in 4(b) we obtain the regularity of solutions and in the last part of the section, using a partition of unity and a localized problem, we transform the results of 4(a) and 4(b) to the case of a bounded domain. Using the properties of the solutions obtained, we prove the existence and uniqueness of local solutions to the nonlinear problem (3.1) by the method of successive approximations in Section 5.

Finally, in Section 6 we specify the results of Sections 4 and 5 for problem (2.2). In Section 7 the barotropic case is considered.

To prove existence of solutions to problem (1.1), (1.2), (1.6), (1.7) we need to know that the form (6.1) is uniformly positive definite. To show it we choose a state equation (here $p = R\delta T$). This implies strong restrictions on the initial velocity (see Remark 6.1). In the barotropic case we do not have such restrictions so we can also consider near light motions.

2. Symmetrization. To symmetrize equations (1.1)–(1.2) we use considerations from [1], [2]. We have a system of conservation laws; now we write a new conservation law, which is a consequence of the old ones. (1.1) implies

$$u_i \frac{\partial(wu^k)}{\partial x^k} + wu^k \frac{\partial u_i}{\partial x^k} + \frac{\partial p}{\partial x_i} = 0.$$

Multiplying by u^i, summing over i and using

$$u^i u_i = -1, \qquad u^i \frac{\partial u_i}{\partial x^k} = 0 \tag{2.0}$$

we get

$$-\frac{\partial(wu^k)}{\partial x^k} + u^i \frac{\partial p}{\partial x_i} = 0,$$

which is equivalent to

$$\frac{\partial}{\partial x^k}\left(\frac{w}{\delta}\delta u^k\right) - \frac{1}{\delta}\frac{\partial p}{\partial x^k}\delta u^k = 0.$$

From this and (1.2) we obtain

$$\delta u^k \left[\frac{\partial}{\partial x^k}\left(\frac{w}{\delta}\right) - \frac{1}{\delta}\frac{\partial p}{\partial x^k}\right] = 0$$

so using the thermodynamical identity, we can write

$$T\delta u^k \frac{\partial}{\partial x^k}\left(\frac{s}{\delta}\right) = 0$$

where s is the entropy.

Finally, because $u^k s\big(-\frac{1}{\delta}\big)\frac{\partial \delta}{\partial x^k} = s\frac{\partial x^k}{\partial x^k}$ from (1.2), we get $T\frac{\partial}{\partial x^k}(su^k) = 0$, so

$$\frac{\partial}{\partial x^k}(su^k) = 0$$

is a new conservation law.

We have shown that equations (1.1)–(1.3) are linearly dependent, that is, there exist functions λ^m such that

$$\lambda^i \frac{\partial T_i^k}{\partial x^k} + \lambda^4 \frac{\partial(\delta u^k)}{\partial x^k} + \lambda^5 \frac{\partial(su^k)}{\partial x^k} = 0$$

for arbitrary functions $z^j(p, u^1, u^2, u^3, \delta)$, where $\lambda^i = u^i$, $i = 0, 1, 2, 3$; $\lambda^4 = (w - sT)/\delta$, $\lambda^5 = T$ and $T = T(\delta, p)$, $s = s(\delta, p)$.

Equations (1.1)–(1.3) can be written in the form

$$\partial_{z^j} q_m^k(z)\frac{\partial z^j}{\partial x^k} = 0, \qquad m = 0,1,\dots,5,$$

and multiplying by $\partial_{z^\tau}\lambda^m$ we obtain

$$\partial_{z^\tau}\lambda^m \partial_{z^j} q_m^k(z)\frac{\partial z^j}{\partial x^k} = 0 \Leftrightarrow A^k_{\tau j}\frac{\partial z^j}{\partial x^k} = 0$$

where the matrices $A^k(z)$ are symmetric (see [1], [2]).

The matrices $\partial_z q^k(z)$ take the form

$$\partial_z q^0 = \begin{pmatrix} 1-\frac{1}{\beta^2}-\frac{1}{\beta^2}\frac{\partial e}{\partial p} & -2u^1 w & -2u^2 w & -2u^3 w & -\frac{1}{\beta^2}\frac{\partial e}{\partial \delta} \\ \frac{1}{\beta}u_1+\frac{1}{\beta}u_1\frac{\partial e}{\partial p} & \beta u_1^2 w+\frac{w}{\beta} & \beta u_1 u^2 w & \beta u_1 u^3 w & \frac{u_1}{\beta}\frac{\partial e}{\partial \delta} \\ \frac{1}{\beta}u_2+\frac{1}{\beta}u_2\frac{\partial e}{\partial p} & \beta u_2 u^1 w & \beta u_2^2 w+\frac{w}{\beta} & \beta u_2 u^3 w & \frac{u_2}{\beta}\frac{\partial e}{\partial \delta} \\ \frac{1}{\beta}u_3+\frac{1}{\beta}u_3\frac{\partial e}{\partial p} & \beta u_3 u^1 w & \beta u_3 u^2 w & \beta u_3^2 w+\frac{w}{\beta} & \frac{u_3}{\beta}\frac{\partial e}{\partial \delta} \\ 0 & \beta u^1\delta & \beta u^2\delta & \beta u^3\delta & \frac{1}{\beta} \\ -\frac{1}{\beta}\frac{\partial s}{\partial p} & \beta u^1 s & \beta u^2 s & \beta u^3 s & \frac{1}{\beta}\frac{\partial s}{\partial \delta} \end{pmatrix},$$

$$\partial_z q^1 = \begin{pmatrix} -\frac{u^1}{\beta}-\frac{u^1}{\beta}\frac{\partial e}{\partial p} & -\beta u_1^2 w-\frac{w}{\beta} & -\beta u^1 u^3 w & -\beta u^1 u^2 w & -\frac{u^1}{\beta}\frac{\partial e}{\partial \delta} \\ 1+u_1^2+u_1^2\frac{\partial e}{\partial p} & 2u_1 w & 0 & 0 & u_1^2\frac{\partial e}{\partial \delta} \\ u^1 u_2+u^1 u_2\frac{\partial e}{\partial p} & u_2 w & u^1 w & 0 & u^1 u_2\frac{\partial e}{\partial \delta} \\ u^1 u_3+u^1 u_3\frac{\partial e}{\partial p} & u_3 w & 0 & u^1 w & u^1 u_3\frac{\partial e}{\partial \delta} \\ 0 & \delta & 0 & 0 & u^1 \\ u^1\frac{\partial s}{\partial p} & s & 0 & 0 & u^1\frac{\partial s}{\partial \delta} \end{pmatrix},$$

$$\partial_z q^2 = \begin{pmatrix} -\frac{u^2}{\beta}-\frac{u^2}{\beta}\frac{\partial e}{\partial p} & -\beta u_2 u_1 w & -\beta u_2^2 w-\frac{w}{\beta} & -\beta u_3 u_2 w & -\frac{u^2}{\beta}\frac{\partial e}{\partial \delta} \\ u^2 u_1+u^2 u_1\frac{\partial e}{\partial p} & u^2 w & u_1 w & 0 & u^2 u_1\frac{\partial e}{\partial \delta} \\ 1+u_2^2+u_2^2\frac{\partial e}{\partial p} & 0 & 2u_2 w & 0 & u_2^2\frac{\partial e}{\partial \delta} \\ u^2 u_3+u^2 u_3\frac{\partial e}{\partial p} & 0 & u_3 w & u^2 w & u^2 u_3\frac{\partial e}{\partial \delta} \\ 0 & 0 & \delta & 0 & u^2 \\ u^2\frac{\partial s}{\partial p} & 0 & s & 0 & u^2\frac{\partial s}{\partial \delta} \end{pmatrix},$$

$$\partial_z q^3 = \begin{pmatrix} -\frac{u^3}{\beta}-\frac{u^3}{\beta}\frac{\partial e}{\partial p} & -\beta u_3 u_1 w & -\beta u^3 u_2 w & -\beta u_3^2 w-\frac{w}{\beta} & -\frac{u^3}{\beta}\frac{\partial e}{\partial \delta} \\ u^3 u_1+u^3 u_1\frac{\partial e}{\partial p} & u^3 w & 0 & u_1 w & u^3 u_1\frac{\partial e}{\partial \delta} \\ u^3 u_2+u^3 u_2\frac{\partial e}{\partial p} & 0 & u^3 w & u_2 w & u^3 u_2\frac{\partial e}{\partial \delta} \\ 1+u_3^2+u_3^2\frac{\partial e}{\partial p} & 0 & 0 & 2u_3 w & u_3^2\frac{\partial e}{\partial \delta} \\ 0 & 0 & 0 & \delta & u^3 \\ u^3\frac{\partial s}{\partial p} & 0 & 0 & s & u^3\frac{\partial s}{\partial \delta} \end{pmatrix}.$$

The matrix $\partial_{z^\tau}\lambda^m$ has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\partial}{\partial p}\left(\frac{w-sT}{\delta}\right) & \frac{\partial}{\partial p}T \\ \beta u_1 & 1 & 0 & 0 & 0 & 0 \\ \beta u_2 & 0 & 1 & 0 & 0 & 0 \\ \beta u_3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial \delta}\left(\frac{w-sT}{\delta}\right) & \frac{\partial}{\partial \delta}T \end{pmatrix}$$

so we get

$$A^0 = \begin{pmatrix} \frac{1}{\beta}\frac{\partial s}{\partial p}\frac{\partial T}{\partial p} & \beta u_1 & \beta u_2 & \beta u_3 & \frac{1}{\beta}\frac{\partial s}{\partial p}\frac{\partial T}{\partial \delta} \\ \beta u_1 & -\beta u_1^2 w + \frac{w}{\beta} & -\beta u_1 u^2 w & -\beta u_1 u^3 w & 0 \\ \beta u_2 & -\beta u_2 u^1 w & -\beta u_2^2 w + \frac{w}{\beta} & -\beta u_2 u^3 w & 0 \\ \beta u_3 & -\beta u_3 u^1 w & -\beta u_3 u^2 w & -\beta u_3^2 w + \frac{w}{\beta} & 0 \\ \frac{1}{\beta}\frac{\partial s}{\partial p}\frac{\partial T}{\partial \delta} & 0 & 0 & 0 & \frac{1}{\beta}\frac{\partial T}{\partial \delta}\left(\frac{\partial s}{\partial \delta} - \frac{s}{\delta}\right) \end{pmatrix},$$

$$A^1 = \begin{pmatrix} u^1\frac{\partial s}{\partial p}\frac{\partial T}{\partial p} & 1 & 0 & 0 & u^1\frac{\partial s}{\partial p}\frac{\partial T}{\partial \delta} \\ 1 & -\beta^2 u_1^3 w + u^1 w & -\beta^2 u_1^2 u^2 w & -\beta^2 u_1^2 u^3 w & 0 \\ 0 & -\beta^2 u_1^2 u_2 w & -\beta^2 u^1 u_2^2 w + u^1 w & -\beta^2 u^1 u_2 u^3 w & 0 \\ 0 & -\beta^2 u_1^2 u_3 w & -\beta^2 u^1 u^2 u_3 w & -\beta^2 u^1 u_3^2 w + u^1 w & 0 \\ u^1\frac{\partial s}{\partial p}\frac{\partial T}{\partial \delta} & 0 & 0 & 0 & u^1\frac{\partial T}{\partial \delta}\left(\frac{\partial s}{\partial \delta} - \frac{s}{\delta}\right) \end{pmatrix},$$

(2.1)
$$\begin{aligned} &A^k_{00} = u^k\frac{\partial s}{\partial p}\frac{\partial T}{\partial p}, \quad A^0_{\alpha 0} = \beta u_\alpha, \quad A^{k'}_{\alpha 0} = \delta^{k'}_\alpha, \quad \alpha, k', \gamma = 1,2,3, \\ &A^k_{40} = u^k\frac{\partial s}{\partial p}\frac{\partial T}{\partial \delta}, \quad A^k_{\alpha\gamma} = -\beta^2 u^k u_\alpha u^\gamma w + u^k w \delta^\gamma_\alpha, \\ &A^k_{4\alpha} = 0, \qquad A^k_{44} = u^k\frac{\partial T}{\partial \delta}\left(\frac{\partial s}{\partial \delta} - \frac{s}{\delta}\right). \end{aligned}$$

Now we consider the following symmetric system:

$$A^k(z)\begin{pmatrix} p_{,x^k} \\ u^1_{,x^k} \\ u^2_{,x^k} \\ u^3_{,x^k} \\ \delta_{,x^k} \end{pmatrix} = 0; \qquad \begin{aligned} &k = 0,1,2,3, \\ &z = (p, u^1, u^2, u^3, \delta), \end{aligned}$$

or, because $x_0 = ct$,

$$A^0(z)z_t + c\sum_{i=1}^{3} A^i(z)z_{x^i} = 0. \tag{2.2}$$

3. Notations. In the next sections we will use the following norms, spaces and notations.

We will consider initial-boundary value problems in $\Omega^T = \Omega \times [0,T]$ where $\Omega \subset \mathbb{R}^3$, $x \in \Omega$, $t \in [0,T]$. We write

$$D^\gamma_{t,x} = \frac{\partial^{\gamma^0}}{\partial t^{\gamma_0}}\frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}}\frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}}\frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}}, \quad |\gamma| = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3,$$

and we denote by $H^s(\Omega^T)$ the Sobolev space with the norm

$$\|u\|_{H^s(\Omega^T)} = \Big(\sum_{|\gamma|\le s}\int_0^T\int_\Omega |D^\gamma_{t,x}u|^2\,dx\,dt\Big)^{1/2} = \|u\|_{s,2,\Omega^T}.$$

Similarly, we introduce $H^s(\Omega)$ and $H^s(\partial\Omega^T)$ with norms $\|\ \|_{s,2,\Omega}$ and $\|\ \|_{s,2,\partial\Omega^T}$. We will use $L_p(\Omega^T)$ and $L_p(\Omega)$ with norms $\|\ \|_{p,\Omega^T}$ and $\|\ \|_{p,\Omega}$, respectively.

For $\alpha \in \mathbb{R}$ we denote by $H^s_\alpha(\Omega^T)$ the weighted Sobolev space, the closure of $C^s(\Omega^T)$ in the norm

$$\|u\|_{H^s_\alpha(\Omega^T)} = \|u\|_{s,\Omega^T,\alpha} = \Big(\sum_{|\gamma|\le s}\int_0^T\int_\Omega |D^\gamma_{t,x}u|^2 e^{-2\alpha t}\,dx\,dt\Big)^{1/2}$$

so we obtain $L_{2,\alpha}(\Omega^T) = H^0_\alpha(\Omega^T)$ with $\|\ \|_{L_{2,\alpha}(\Omega^T)} = \|\ \|_{\Omega^T,\alpha}$.

Let

$$u \in L^s_\infty(0,T;H^i(\Omega)) \Leftrightarrow \operatorname*{ess\,sup}_{t\in[0,T]}\Big\|\frac{\partial^s}{\partial t^s}u(t)\Big\|_{i,2,\Omega} < \infty.$$

Then we define

$$\Pi^l_k(\Omega^T) = \bigcap_{i=k}^{l} L^{l-i}_\infty(0,T;H^i(\Omega))$$

with $\|u\|_{\Pi^l_k(\Omega^T)} = \|u\|_{l,k,\infty,\Omega^T}$. Finally, we introduce $\Gamma^l_0(\Omega)$ by

$$\|u\|_{\Gamma^l_0(\Omega)} = |u|_{l,0,\Omega} = \Big(\sum_{|\gamma|\le l}\int_\Omega |D^\gamma_{t,x}u|^2\,dx\Big)^{1/2}.$$

Furthermore, $\mathring{\Gamma}^l_0$, $\mathring{H}^s_\alpha$ denote the sets of functions in the respective spaces vanishing on the boundary $\partial\Omega$; $|u|$ is the Euclidean norm.

To simplify the following considerations, in Sections 4 and 5 we will consider the mixed problem

$$
\begin{aligned}
&Lu \equiv E(t,x,u)u_t + \sum_{i=1}^{3} A_i(t,x,u)u_{x_i} = F(t,x),\\
&M(t,x',u)u|_{\partial\Omega} = g(t,x'), \qquad x' \in \partial\Omega,\\
&u|_{t=0} = u_0(x),
\end{aligned}
\tag{3.1}
$$

where u takes values in $\mathbb{R}^m$, $x \in \Omega \subset \mathbb{R}^3$, $t \in [0,T]$, E, A_i are real $m \times m$ matrices, the values of u lie in an open domain G and the values of the initial data u_0 belong to an open subset G_0 such that $\overline{G}_0 \subset G$. Next, assuming $u := z \in \mathbb{R}^5$, $z = (p, u^1, u^2, u^3, \delta)$ we will formulate results for problem (2.2) with initial and boundary conditions (1.6), (1.7).

4(a) *The existence of solutions for the linearized equations in a half-space*. In this part we shall consider the linearized problem (3.1) in the half-space $x_1 > 0$:

$$
\begin{aligned}
&Lu = E(t,x)u_t + \sum_{i=1}^{3} A_i(t,x)u_{x_i} = F(t,x),\\
&M(t,x')u|_{x_1=0} = g(t,x'),\\
&u|_{t=0} = u_0,
\end{aligned}
\tag{4.1}
$$

where $x = (x_1, x')$ and we assume that $\Omega = \{x \in \mathbb{R}^3 : x_1 > 0\}$, $\partial\Omega = \{x \in \mathbb{R}^3 : x_1 = 0\}$. In part (c) we shall obtain results for a bounded domain Ω, using a partition of unity.

LEMMA 4.1. (1) *Let E, A_i, $i = 1, \dots, 3$, be symmetric matrices and*

$$
Eu \cdot u \ge \alpha_0 u^2, \qquad \alpha_0 > 0. \tag{4.2}
$$

Let $\overline{n}$ be the unit outward vector normal to $\partial\Omega$ and assume $-A_{\bar{n}} = A_1$ has eigenvalues λ_μ, where λ_μ^+, $\mu = 1, \dots, k$, and λ_μ^-, $\mu = k+1, \dots, m$, are respectively the positive and negative ones. Suppose that

$$
\min_{\mu} \min_{\Omega^T} |\lambda_\mu| \ge c_0 > 0 \tag{4.3}
$$

and

$$
\max_{\nu \in \{1,\dots,k\}} \max_{\partial\Omega^T} \lambda_\nu^+(x',t) \le c_1, \tag{4.4}
$$

where c_0, c_1 are constants.

(2) *Let γ_μ^+, γ_μ^- be orthonormal eigenvectors of the matrix $-A_{\bar{n}}$, corresponding to the eigenvalues λ_μ^+, λ_μ^-. Assume that the matrix $M(t,x')$ has*

the form

$$M = \sum_{\mu,\nu=1}^{k} \alpha_{\mu\nu}(t,x')\gamma_\mu^+(t,x')\gamma_\nu^+(t,x') + \sum_{\mu=1}^{k}\sum_{\nu=k+1}^{m} \beta_{\mu\nu}(t,x')\gamma_\mu^+(t,x')\gamma_\nu^-(t,x') \tag{4.5}$$

where

$$\begin{aligned} &\text{(a)}\ \max_{\partial\Omega^T} |\alpha_{\mu\nu}^{-1}(t,x')| \le \delta_0^{-1},\\ &\text{(b)}\ \max_{\partial\Omega^T} |\beta_{\mu\nu}(t,x')| \le \beta_0,\\ &\text{(c)}\ (c_0+c_1)\delta_0^{-2}\beta_0^2 \le \tfrac{1}{2}c_0, \end{aligned} \tag{4.6}$$

and δ_0, β_0 are constants.

(3) *Let $\widetilde{L} = (E, A_1, A_2, A_3)$, $\widetilde{L}, M \in \Pi_0^3(\Omega^T)$ and suppose that α satisfies*

$$|\widetilde{L}|_{3,0,\infty,\Omega^T} < \alpha\alpha_0/2 \tag{4.7}$$

and

$$\sup_{\Omega^T} |E| \le c_2, \quad \text{where } c_2 \text{ is a constant.} \tag{4.8}$$

Then, for every $u \in C^\infty(\Omega^T)$ and $t \le T$ we have the estimate

$$\begin{aligned} \alpha_0 e^{-2\alpha t}\int_\Omega u^2\,dx + \frac{\alpha\alpha_0}{2}\int_{\Omega^t} u^2 e^{-2\alpha s}\,dx\,ds + \frac{\alpha_0}{2}\int_{\partial\Omega^t} u^2 e^{-2\alpha s}\,dx'\,ds \\ \le (c_0+c_1)\delta_0^{-2}\int_{\partial\Omega^t} |Mu|^2 e^{-2\alpha s}\,dx'ds \\ + \frac{2}{\alpha\alpha_0}\int_{\Omega^t} |Lu|^2 e^{-2\alpha s}\,dx\,ds + c_2\int_\Omega u^2\,dx\Big|_{t=0}. \end{aligned} \tag{4.9}$$

P r o o f. Multiplying $(4.1)_1$ by $ue^{-2\alpha t}$ and integrating by parts in Ω, we obtain

$$\begin{aligned} \frac{d}{dt}e^{-2\alpha t}\int_\Omega Eu^2\,dx + 2\alpha e^{-2\alpha t}\int_\Omega Eu^2\,dx + e^{-2\alpha t}\int_{\partial\Omega} A_n u^2 dx' \\ - e^{-2\alpha t}\int_\Omega \Big(\sum_{i=1}^3 A_{i,x_i} + E_t\Big)u^2\,dx - 2e^{-2\alpha t}\int_\Omega Lu\cdot u\,dx = 0. \end{aligned} \tag{4.10}$$

Integrating (4.10) from 0 to t, using (4.2) and (4.8) we get

$$\alpha_0 e^{-2\alpha t}\int_\Omega u^2\,dx + 2\alpha\alpha_0\int_{\Omega^t} u^2 e^{-2\alpha s}\,dx\,ds + \int_{\partial\Omega^t} A_n u^2 e^{-2\alpha s}\,dx'\,ds \tag{4.11}$$

$$\leq \int_{\Omega^t} \Big(\sum_{i=1}^{3} A_{i,x_i} + E_s\Big) u^2 e^{-2\alpha s}\,dx\,ds + 2\int_{\Omega^t} Lu\cdot ue^{-2\alpha s}\,dx\,ds + c_2 \int_{\Omega} u^2\,dx\Big|_{t=0}.$$

From (4.7) we get

$$\max_{\Omega^t}\Big(|E_t| + \sum_{i=1}^{3}|A_{i,x_i}|\Big) \leq 2c|\widetilde{L}|_{3,0,\infty,\Omega^T} \leq \alpha\alpha_0$$

so using the Young inequality (with $\varepsilon = \sqrt{2/(\alpha\alpha_0)}$) in (4.11) we have

$$\alpha_0 e^{-2\alpha t}\int_{\Omega} u^2\,dx + \frac{\alpha\alpha_0}{2}\int_{\Omega^t} u^2 e^{-2\alpha s}\,dx\,ds + \int_{\partial\Omega^t} A_n u^2 e^{-2\alpha s}\,dx'\,ds \leq \frac{2}{\alpha\alpha_0}\int_{\Omega^t}|Lu|^2 e^{-2\alpha s}\,dx\,ds + c_2\int_{\Omega} u^2\,dx\Big|_{t=0}. \tag{4.12}$$

We have to consider the boundary term. From (2),

$$u = \sum_{\mu=1}^{k} c_\mu \gamma_\mu^+ + \sum_{\mu=k+1}^{m} c_\mu \gamma_\mu^- = u' + u'' \quad \text{where } c_\mu = u\gamma_\mu,$$

$$u' = \sum_{\mu=1}^{k} c_\mu\gamma_\mu^+, \quad \text{so} \quad |u'|^2 = \sum_{\mu=1}^{k} c_\mu^2, \quad |u''|^2 = \sum_{\mu=k+1}^{m} c_\mu^2,$$

and

$$-A_{\bar n}u^2 = \sum_{\mu=1}^{k}\lambda_\mu^+ c_\mu^2 + \sum_{\mu=k+1}^{n}\lambda_\mu^- c_\mu^2.$$

Using this and (4.3), (4.4) we get, from (4.12),

$$\begin{aligned}\alpha_0 e^{-2\alpha t}\int_{\Omega} u^2\,dx &+ \frac{\alpha\alpha_0}{2}\int_{\Omega^t} u^2 e^{-2\alpha s}\,dx\,ds + c_0\int_{\partial\Omega^t}|u''|^2 e^{-2\alpha s}\,dx'\,ds \\ &\leq c_1\int_{\partial\Omega^t}|u'|^2 e^{-2\alpha s}\,dx'\,ds + \frac{2}{\alpha\alpha_0}\int_{\Omega^t}|Lu|^2 e^{-2\alpha s}\,dx\,ds + c_2\int_{\Omega} u^2\,dx\Big|_{t=0}.\end{aligned} \tag{4.13}$$

Now, to express $|u'|^2 = \sum_{\mu=1}^k c_\mu^2$ by $|Mu|^2$, we consider

$$Mu = \sum_{\mu,\nu=1}^{k} \alpha_{\mu\nu}\gamma_\mu^+ c_\nu^+ + \sum_{\mu=1}^{k}\sum_{\nu'=k+1}^{m} \beta_{\mu\nu'}\gamma_\mu^+ c_{\nu'}^- = \sum_{\mu=1}^{k} g_\mu\gamma_\mu^+$$

so

$$g_\mu = \sum_{\nu=1}^{k} \alpha_{\mu\nu} c_\nu^+ + \sum_{\nu'=k+1}^{m} \beta_{\mu\nu'} c_{\nu'}^-$$

and this implies

$$c_\nu^+ = \sum_{\mu=1}^{k} \alpha_{\mu\nu}^{-1} g_\mu - \sum_{\nu'=k+1}^{m}\sum_{\mu=1}^{k} \alpha_{\mu\nu}^{-1}\beta_{\mu\nu'} c_{\nu'}^-.$$

Adding $c_0 \int_{\partial\Omega^t} |u'|^2 e^{-2\alpha s}\,dx'\,ds$, using (4.6) and the last expression, we obtain

$$\begin{aligned}(4.14)\quad \alpha_0 e^{-2\alpha t}\int_\Omega u^2\,dx + \frac{\alpha\alpha_0}{2}\int_{\Omega^t} u^2 e^{-2\alpha s}\,dx\,ds + c_0\int_{\partial\Omega^t} u^2 e^{-2\alpha s}\,dx'\,ds \\ \le (c_0+c_1)\delta_0^{-2}\int_{\partial\Omega^t} |Mu|^2 e^{-2\alpha s}\,dx'\,ds \\ + \frac{2}{\alpha\alpha_0}\int_{\Omega^t} |Lu|^2 e^{-2\alpha s}\,dx\,ds \\ + (c_0+c_1)\delta_0^{-2}\beta_0^2\int_{\partial\Omega^t} |u''|^2 e^{-2\alpha s}\,dx'\,ds + c_2\int_\Omega u^2\,dx\Big|_{t=0}.\end{aligned}$$

Finally, from (4.14) and (4.6)(c) we have (4.9). ■

To prove the existence of solutions to (4.1) we have to split it into a Cauchy problem and a boundary value problem. Let $\chi \in C_0^\infty(-\delta,\delta)$; we assume that a solution of (4.1) has the form $u = \chi u_1 + u_2$, where

$$(4.15)\qquad Lu_1 = 0, \quad u_1|_{t=0} = u_0,$$

and

$$(4.16)\qquad Lu_2 = F - Eu_1\frac{\partial}{\partial t}\chi, \quad Mu_2|_{\partial\Omega} = g, \quad u_2|_{t=0} = 0.$$

Further, introducing $w_1 = u_1 - \widetilde{u}_0$ (where $\widetilde{u}_0$ denotes an extension of u_0 to the half-space $t > 0$) we get, from (4.15),

$$(4.17)\qquad Lw_1 = -L\widetilde{u}_0, \quad w_1|_{t=0} = 0.$$

We define the formally adjoint operator $L^{(*)}$ by

$$(4.18)\qquad L^{(*)} = -E\partial_t - \sum_{i=1}^{3} A_i\partial_{x_i} - E_t - \sum_{i=1}^{3} A_{i,x_i}$$

so we have the identity

$$(4.19)\qquad (Lw_1, v_1)_{\Omega^T} = (w_1, L^{(*)}v_1)_{\Omega^T}$$

for all $w_1, v_1 \in C_0^\infty(\Omega^T)$ with $w_1|_{t=0} = 0$ and $v_1|_{t=T} = 0$.

Next, for such w_1, v_1 we obtain by (4.9) the following estimates:

$$(4.20)\qquad \alpha_0 e^{-2\alpha t}\int\limits_{\Omega} w_1^2\,dx + \frac{\alpha\alpha_0}{2}\int\limits_{\Omega^t} w_1^2 e^{-2\alpha s}\,dx\,ds \le \frac{2}{\alpha\alpha_0}\int\limits_{\Omega^t} |Lw_1|^2 e^{-2\alpha s}\,dx\,ds$$

and (with time travelling backward)

$$(4.21)\qquad \alpha_0 e^{2\alpha t}\int\limits_{\Omega} v_1^2\,dx + \frac{\alpha\alpha_0}{2}\int\limits_{\Omega^t} v_1^2 e^{2\alpha s}\,dx\,ds \le \frac{2}{\alpha\alpha_0}\int\limits_{\Omega^t} |L^{(*)}v_1|^2 e^{2\alpha s}\,dx\,ds.$$

Now we use the following (see [3]).

THEOREM 4.1. *Let $\mathcal{L}$ denote the space of square integrable functions on Ω^T, D_L the domain of L consisting of $u \in C^\infty(\Omega^T \cup \partial\Omega^T)$ which satisfy the boundary (initial) condition, and $D_{L^{(*)}}$ the domain of $L^{(*)}$ of those $v \in C^\infty(\Omega^T \cup \partial\Omega^T)$ which satisfy the adjoint boundary (initial) condition. If there exists a constant c such that*

$$c\|u\| \le \|\bar{L}u\|, \qquad c\|v\| \le \|\bar{L}^{(*)}v\|,$$

for $u \in D_{\bar{L}}$ and $v \in D_{\bar{L}^{(*)}}$, then $\bar{L}$ and $\bar{L}^{(*)}$ map their domains one-to-one onto $\mathcal{L}$.

From this theorem and inequalities (4.20), (4.21) we get:

LEMMA 4.2. *There exists a unique weak solution u_1 of (4.15) such that $w_1 \in \mathring{L}_{2,\alpha}(\Omega^T)$.*

Now we are looking for solutions of problem (4.16). For the adjoint $L^{(*)}$ we obtain the identity

$$(4.22)\qquad (Lu, v) = (u, L^*v) + (A_{\bar{n}}u, v) = (u, L^{(*)}v) - (A_1 u, v)$$

where $u, v \in C_0^1(\Omega \times \mathbb{R})$.

We can find the boundary matrix M^* for the adjoint problem from $(A_1 u, v) = 0$ for $u \in \ker M$ and $v \in \ker M^*$ (see [10], [11]). Let $F, g = 0$ for $t < 0$ and $t > T$. Then we consider (4.16) in $\Omega \times \mathbb{R}$ and we can prove, similarly to (4.9),

$$(4.23)\qquad \frac{\alpha\alpha_0}{2}\int\limits_{\Omega\times\mathbb{R}} u_2^2 e^{-2\alpha s}\,dx\,ds + \frac{c_0}{2}\int\limits_{\partial\Omega\times\mathbb{R}} u_2^2 e^{-2\alpha s}\,dx'\,ds \le (c_0 + c_1)\delta_0^{-2}\int\limits_{\partial\Omega\times\mathbb{R}} |Mu_2|^2 e^{-2\alpha s}\,dx'\,ds + \frac{2}{\alpha\alpha_0}\int\limits_{\Omega\times\mathbb{R}} |Lu_2|^2 e^{-2\alpha t}\,dx\,ds.$$

We have to obtain an estimate for the adjoint problem. If we take $L^{(*)}$, M^* instead of L, M and we assume that the time is travelling backward, then we can prove (in the same way as Lemma 4.1):

Lemma 4.3. *Assume that* (1) *and* (3) *of Lemma* 4.1 *hold. Let*

$$M^* = \sum_{\mu,\nu=k+1}^{m} \alpha^*_{\mu\nu}(t,x')\gamma^-_\mu(t,x)\gamma^-_\nu(t,x') + \sum_{\mu=k+1}^{m}\sum_{\nu=1}^{k}\beta^*_{\mu\nu}(t,x')\gamma^-_\mu(t,x')\gamma^+_\nu(t,x'),$$

with

$$\max_{\partial\Omega^T}|\alpha^{*-1}_{\mu\nu}| \le \delta_0^{-1}, \quad \max_{\partial\Omega^T}|\beta^*_{\mu\nu}| \le \gamma_0, \quad (c_0+c_4)\delta_0^{-2}\gamma_0^2 \le c_0/2. \tag{4.24}$$

Moreover, *let*

$$\max_{\nu\in\{1,\dots,m\}}\max_{\partial\Omega\times\mathbb{R}}|\lambda^-_\nu(x',t)| \le c_4. \tag{4.25}$$

Then for $v_2 \in C_0^\infty(\Omega\times\mathbb{R}) \cap L_{2,-\alpha}(\Omega\times\mathbb{R})$ *we obtain*

$$\frac{\alpha\alpha_0}{2}\int_{\Omega\times\mathbb{R}} v_2^2 e^{2\alpha s}\,dx\,ds + \frac{c_0}{2}\int_{\partial\Omega\times\mathbb{R}} v_2^2 e^{2\alpha s}\,dx'\,ds \le (c_0+c_4)\delta_0^{-2}\int_{\partial\Omega\times\mathbb{R}}|M^*v_2|^2 e^{2\alpha s}\,dx'\,ds + \frac{2}{\alpha\alpha_0}\int_{\Omega\times\mathbb{R}}|L^*v_2|^2 e^{2\alpha s}\,dx\,ds. \tag{4.26}$$

Now, by (4.23), (4.26) and Theorem 4.1 we have

Lemma 4.4. *Let* $g \in L_{2,\alpha}(\partial\Omega\times\mathbb{R})$ *with* $g|_{t<0} = g|_{t>T} = 0$ *and* $F \in L_{2,\alpha}(\Omega\times\mathbb{R})$ *with* $F|_{t<0} = F|_{t>T} = 0$. *Let the assumptions of Lemmas* 4.1 *and* 4.3 *be satisfied. Then there exists a unique solution* $u_2 \in L_{2,\alpha}(\Omega\times\mathbb{R})$ *of* (4.16) *such that* $u_2|_{\partial\Omega} \in L_{2,\alpha}(\partial\Omega\times\mathbb{R})$.

In Lemmas 4.2 and 4.4 we can obtain strong solutions, using the technique of mollifiers (see [6], [12]) with respect to (x',t) where $x = (x_1, x')$. Then we have the sequence $u_\varepsilon = J_\varepsilon u = j_\varepsilon * u$ (the operator J_ε is the mollifier) and from the properties of J_ε we have the convergences

$$\begin{aligned} u_\varepsilon &\to u && \text{in } L_2(\Omega^T),\\ Lu_\varepsilon &\to Lu = F && \text{in } L^2(\Omega^T),\\ Mu_\varepsilon &\to Mu = g && \text{in } L^2(\partial\Omega^T), \end{aligned}$$

and u_ε is continuous up to the boundary.

Now for $u = \chi u_1 + u_2$ we formulate

Theorem 4.2. *Let* $u_0 \in H^1(\Omega)$, $u_0|_{\partial\Omega} = 0$, $\widetilde{L} \in H^1_\alpha(\Omega^T)$ *and* $F \in L_{2,\alpha}(\Omega^T)$, $g \in L_{2,\alpha}(\partial\Omega^T)$. *Let the assumptions of Lemmas* 4.1 *and* 4.3 *be* *satisfied. Then there exists a unique strong solution* u *of problem* (4.1) *in the half-space* Ω *such that* $u \in L_{2,\alpha}(\Omega^T) \cap L_{2,\alpha}(\partial\Omega^T) \cap L_\infty(0,T;\Gamma_0^1(\Omega))$ *and* (4.9) *holds.*

4(b) *Regularity of solutions.* To prove the existence of solutions of (3.1) we have to use the method of successive approximations; so we need more regular solutions of (4.1) such that $u \in H^3(\Omega^T)$. Since $u \in L_{2,\alpha}(\Omega^T)$ we have to use mollifiers to derive the regularity of u. Let $u_\delta = j_\delta * u = J_\delta u$, where $j(t,x) \in C_0^\infty(\mathbb{R}^1 \times \mathbb{R}^n)$, $j \ge 0$, $\int j(t,x)\,dx\,dt = 1$ and $j_\delta(t,x) = \delta^{-n-1} j(t/\delta, x/\delta)$. We consider the problems

$$
\begin{aligned}
&LD^s_{t,x'}u_\delta = D^s_{t,x'}Lu_\delta + (LD^s_{t,x'}u_\delta - D^s_{t,x'}Lu_\delta),\\
&MD^s_{t,x'}u_\delta|_{x_1=0} = D^s_{t,x'}Mu_\delta + (MD^s_{t,x'}u_\delta - D^s_{t,x'}Mu_\delta),\\
&D^s_{t,x'}u_\delta|_{t=0} = D^s_{t,x'}u_\delta|_{t=0},
\end{aligned}
\tag{4.27}
$$

for $s = 1,2,3$, where

$$
D^s_{t,x'}u = \sum_{|\gamma|=s} \frac{\partial^{\gamma_0}}{\partial t^{\gamma_0}} \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} \frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}} u, \qquad |\gamma| = \gamma_0 + \gamma_2 + \gamma_3,
$$

$$
\begin{aligned}
Lu_\delta &= (Lu)_\delta - [(Lu)_\delta - Lu_\delta]\\
&= (Lu)_\delta - C_\delta u \quad (C_\delta u \text{ is called the commutator}).
\end{aligned}
$$

LEMMA 4.5. *Assume that* (1)–(3) *of Lemma* 4.1 *hold*, $M \in H^3_\alpha(\partial\Omega^T)$, $g \in H^3_\alpha(\partial\Omega^T)$, $u_0 \in H^3(\Omega)$ *and* $F \in H^3_\alpha(\Omega^T)$. *Set*

$$
a = |\widetilde{L}|_{3,0,\infty,\Omega^t}, \qquad b = |M|_{3,0,\infty,\Omega^T} + \|M\|_{3,\partial\Omega^t,\alpha}
$$

for $t \le T$. *Then there exist polynomials* $p_0(a,b)$, $p_s(a,b)$, $q_s(a,b)$, $1 \le s \le 3$, *such that the solution of problem* (4.1) *satisfies the following estimate*:

$$
\begin{aligned}
\alpha_0|u|^2_{s,0,\Omega}e^{-2\alpha t} + \frac{\alpha\alpha_0}{4}\|u\|^2_{s,\Omega^t,\alpha} + \frac{c_0}{2}\|u\|^2_{s,\partial,\Omega^t,\alpha}
&\le p_s(a,b)[|Lu|^2_{s,\Omega^t,\alpha} + |Lu|^2_{s-1,0,\Omega}|_{t=0}]\\
&\quad + q_s(a,b)\|Mu\|^2_{s,\partial\Omega^t,\alpha} + c_2|u|^2_{s,0,\Omega}|_{t=0}
\end{aligned}
\tag{4.28}
$$

where

$$
\alpha \text{ satisfies } p_0(a,b) \le \alpha\alpha_0. \tag{4.29}
$$

Moreover, *there exist polynomials* r *such that*

$$
|u|^2_{s,0,\Omega}|_{t=0} \le r(|\widetilde{L}|_{s-1,0,\Omega}|_{t=0}, |Lu|_{s-1,0,\Omega}|_{t=0}, \|u_0\|_{s,2,\Omega}). \tag{4.30}
$$

Proof. For $|s| = 1$ and problem (4.27) we have by (4.9),

$$\begin{aligned}(4.31)\qquad \alpha_0 e^{-2\alpha t}\int\limits_{\Omega}|D^1_{t,x'}u_\delta|^2\,dx + \frac{\alpha\alpha_0}{2}\int\limits_{\Omega^t}|D^1_{t,x'}u_\delta|^2e^{-2\alpha s}\,dx\,ds\\ + \frac{c_0}{2}\int\limits_{\partial\Omega^t}|D^1_{t,x'}u_\delta|e^{-2\alpha s}\,dx'\,ds\\ \le \frac{2}{\alpha\alpha_0}\int\limits_{\Omega^t}|D^1_{t,x'}Lu_\delta|^2e^{-2\alpha s}\,dx\,ds\\ + \frac{2}{\alpha\alpha_0}\int\limits_{\Omega^t}|LD^1_{t,x'}u_\delta - D^1_{t,x'}Lu_\delta|^2e^{-2\alpha s}\,dx\,ds\\ + c_3\int\limits_{\partial\Omega^t}|D^1_{t,x'}g'_\delta|^2e^{-2\alpha s}\,dx'\,ds\\ + c_3\int\limits_{\partial\Omega^t}|MD^1_{t,x'}u_\delta - D^1_{t,x'}Mu_\delta|^2e^{-2\alpha s}\,dx'\,ds\\ + c_2\int\limits_{\Omega}|D^1_{t,x'}u_\delta|^2\,dx\Big|_{t=0}\end{aligned}$$

where

$$c_3 = (c_0+c_1)\delta_0^{-2},\qquad g'_\delta = Mu_\delta.$$

We have to estimate the second and fourth terms on the right-hand side of (4.31). We can write

$$\begin{aligned}\int\limits_{\Omega^t}(LD^1_{t,x'}u_\delta - D^1_{t,x'}Lu_\delta)^2e^{-2\alpha s}\,dx\,ds &\le \int\limits_{\Omega^t}|D^1_{t,x'}\widetilde{L}|^2|D^1_{t,x}u_\delta|^2e^{-2\alpha s}\,dx\,ds\\ &\le ca^2\int\limits_{\Omega^t}|D^1_{t,x}u_\delta|^2e^{-2\alpha s}\,dx\,ds.\end{aligned}$$

Because

$$(4.32)\qquad D^1_{x_1}u_\delta = A_1^{-1}[Lu_\delta - Eu_{\delta t} - A'u_{\delta x'}],\qquad A'u_{\delta x'} = \sum_{i=2}^3 A_iu_{\delta x_i}$$

and $\det A_1 \ge c_0^m$, $|A_1^{-1}| \le cc_0^{-m}a^{m-1}$ we get

$$(4.33)\qquad \int\limits_{\Omega^t}|D^1_{x_1}u_\delta|^2e^{-2\alpha s}\,dx\,ds \le ca^{2(m-1)}(\|Lu_\delta\|^2_{0,\Omega^t,\alpha} + a^2\|u_\delta\|'^2_{1,\Omega^t,\alpha})$$

(the prime denotes that the derivative D_{x_1} does not appear), so finally

$$\begin{aligned}(4.34)\qquad &\int\limits_{\Omega^t}|LD^1_{t,x'}u_\delta - D^1_{t,x'}Lu_\delta|^2e^{-2\alpha s}\,dx\,ds\\ &\qquad\le ca^{2m}\|Lu_\delta\|^2_{0,\Omega^t,\alpha} + ca^2(a^{2m}+1)\|u_\delta\|'^2_{1,\Omega^t,\alpha}.\end{aligned}$$

For the boundary term we have

$$(4.35)\qquad \int\limits_{\partial\Omega^t} |MD^1_{t,x'}u_\delta - D^1_{t,x'}Mu_\delta| e^{-2\alpha s}\,dx'\,ds$$
$$\le \int\limits_{\partial\Omega^t} |D^1_{t,x'}M|^2 u_\delta^2 e^{-2\alpha s}\,dx'\,ds$$
$$\le cb^2 \int\limits_{\partial\Omega^t} |u_\delta|^2 e^{-2\alpha s}\,dx\,ds \le cb^2 \int\limits_0^t e^{-2\alpha s} \int\limits_\Omega |D^1_x u_\delta|^2\,dx\,ds$$
$$\le cb^2 \int\limits_0^t e^{-2\alpha s} \int\limits_\Omega (|D^1_{x_1}u_\delta|^2 + |D^1_{x'}u_\delta|^2 + |D^1_t u_\delta|^2)\,dx\,ds$$
$$\le cb^2[\|u_\delta\|'^2_{1,\Omega^t,\alpha} + \|D^1_{x_1}u_\delta\|^2_{0,\Omega^t,\alpha}]$$
$$\le cb^2\|Lu_\delta\|^2_{0,\Omega^t,\alpha} + cb^2(a^{2m}+1)\|u_\delta\|'^2_{1,\Omega^t,\alpha}.$$

Assuming

$$\frac{\alpha\alpha_0}{4} \ge c\left[\frac{2}{\alpha\alpha_0}a^2(a^{2m}+1) + b^2 c_3(a^{2m}+1)\right]$$

we obtain from (4.9), (4.31), (4.34) and (4.35),

$$(4.36)\qquad \alpha_0|u_\delta|'^2_{1,0,\Omega}e^{-2\alpha t} + \frac{\alpha\alpha_0}{4}\|u_\delta\|'^2_{1,\Omega^t,\alpha} + \frac{c_0}{2}\|u_\delta\|'^2_{1,\partial\Omega^t,\alpha}$$
$$\le c\widetilde{p}_1(a,b)\|Lu_\delta\|^2_{1,\Omega^t,\alpha} + c\widetilde{q}_1(a,b)\|Mu_\delta\|^2_{1,\partial\Omega^t,\alpha} + c_2|u_\delta|'^2_{1,0,\Omega}|_{t=0}$$

where $\widetilde{p}_1$, $\widetilde{q}_1$ are polynomials.

Using (4.32) we have

$$(4.37)\qquad \|u_{\delta x_1}\|^2_{0,2,\Omega} \le ca^{2(m-1)}(\|Lu_\delta\|^2_{0,2,\Omega} + a^2\|u_\delta\|'^2_{1,0,\Omega})$$

so finally from (4.33), (4.37) and (4.36),

$$(4.38)\qquad \alpha_0|u_\delta|^2_{1,0,\Omega}e^{-2\alpha t} + \frac{\alpha\alpha_0}{4}\|u_\delta\|^2_{1,\Omega^t,\alpha} + \frac{c_0}{2}\|u_\delta\|^2_{1,\partial\Omega^t,\alpha}$$
$$\le p_1(a,b)(\|Lu_\delta\|^2_{1,\Omega^t,\alpha} + \|Lu_\delta\|^2_{0,2,\Omega})$$
$$+ q_1(a,b)\|Mu_\delta\|^2_{1,\partial\Omega^t,\alpha} + c_2|u_\delta|^2_{1,0,\Omega}|_{t=0}$$

where p_1, q_1 are polynomials.

Using the convergence $u_\delta \to u$ in H^1, and $Lu_\delta = F_\delta - C_\delta u \to F = Lu$ in H^1 (because $C_\delta u \to 0$ in L^2 and H^1 for sufficiently regular $\widetilde{L}$) we obtain

$$(4.39)\qquad \alpha_0|u|^2_{1,0,\Omega}e^{-2\alpha t} + \frac{\alpha\alpha_0}{4}\|u\|^2_{1,\Omega^t,\alpha} + \frac{c_0}{2}\|u\|^2_{1,\partial\Omega^t,\alpha}$$
$$\le p_1(a,b)(\|Lu\|^2_{1,\Omega^t,\alpha} + \|Lu\|^2_{0,2,\Omega})$$
$$+ q_1(a,b)\|Mu\|^2_{1,\partial\Omega^t,\alpha} + c_2|u|^2_{1,0,\Omega}|_{t=0}$$

so we have (4.28) for $s = 1$.

Let us consider the case $s=2$. We have, using (4.9) to (4.27),

$$
\begin{aligned}
(4.40)\qquad \alpha_0 e^{-2\alpha t}\int_{\Omega} |D^2_{t,x'}u_\delta|^2\,dx &+ \frac{\alpha\alpha_0}{2}\int_{\Omega^t} |D^2_{t,x'}u_\delta|^2 e^{-2\alpha s}\,dx\,ds\\
&+ \frac{c_0}{2}\int_{\partial\Omega^t} |D^2_{t,x'}u_\delta|^2 e^{-2\alpha s}\,dx'\,ds\\
&\le \frac{2}{\alpha\alpha_0}\int_{\Omega^t} |D^2_{t,x'}Lu_\delta|^2 e^{-2\alpha s}\,dx\,ds\\
&+ \frac{2}{\alpha\alpha_0}\int_{\Omega^t} |LD^2_{t,x'}u_\delta - D^2_{t,x'}Lu_\delta|^2 e^{-2\alpha s}\,dx\,ds\\
&+ c_3\int_{\partial\Omega^t} |D^2_{t,x'}g'_\delta|^2 e^{-2\alpha s}\,dx'\,ds\\
&+ c_3\int_{\partial\Omega^t} |MD^2_{t,x'}u_\delta - D^2_{t,x'}Mu_\delta|^2 e^{-2\alpha s}\,dx'\,ds\\
&+ c_2\int_{\Omega} |D^2_{t,x'}u_\delta|^2\,dx\Big|_{t=0}.
\end{aligned}
$$

As before, we estimate

$$
\begin{aligned}
\int_{\Omega^t} &|LD^2_{t,x'}u_\delta - D^2_{t,x'}Lu_\delta|^2 e^{-2\alpha s}\,dx\,ds\\
&\le \int_{\Omega^t} (|D^2_{t,x'}\widetilde{L}|^2|D^1_{t,x'}u_\delta|^2 + |D^1_{t,x'}\widetilde{L}|^2|D^1_{t,x'}D^1_{t,x}u_\delta|^2)e^{-2\alpha s}\,dx\,ds\\
&\le ca^2\Big(\|u_\delta\|^2_{1,\Omega^t,\alpha} + \int_{\Omega^t} |D^1_{t,x'}D^1_{t,x'}u_\delta|^2 e^{-2\alpha s}\,dx\,ds\Big),
\end{aligned}
$$

$$
\begin{aligned}
(4.41)\qquad \int_{\Omega^t} &|D^1_{x_1}D^1_{t,x'}u_\delta|^2 e^{-2\alpha s}\,dx\,ds\\
&\le ca^{2(m-1)}[\|Lu_\delta\|^2_{1,\Omega^t,\alpha} + a^2(\|u_\delta\|'^2_{1,\Omega^t,\alpha} + \|D^2_{t,x'}u_\delta\|^2_{0,\Omega^t,\alpha})],
\end{aligned}
$$

$$
\begin{aligned}
(4.42)\qquad \int_{\Omega^t} &|D^1_{x_1}D^1_{x_1}u_\delta|^2 e^{-2\alpha s}\,dx\,ds\\
&\le ca^{2(m-1)}[\|Lu_\delta\|^2_{1,\Omega^t,\alpha} + a^2(\|u_\delta\|^2_{1,\Omega^t,\alpha} + \|D^1_{x_1}D^1_{t,x'}u_\delta\|^2_{0,\Omega^t,\alpha})]\\
&\le c[(a^{2(m-1)} + a^{4(m-1)})\|Lu_\delta\|^2_{1,\Omega^t,\alpha}\\
&\quad + (a^{2m} + a^{4m})\|u_\delta\|^2_{1,\Omega^t,\alpha} + a^{4m}\|D^2_{t,x'}u_\delta\|^2_{0,\Omega^t,\alpha}]
\end{aligned}
$$

where (4.41), (4.42) are obtained by differentiating (4.32) with respect to

(t,x') and x_1, respectively. Hence

$$(4.43)\qquad \int_{\Omega^t} |LD^2_{t,x'}u_\delta - D^2_{t,x'}Lu_\delta|^2 e^{-2\alpha s}\,dx\,ds$$
$$\le ca^2[(1+a^{2m})(\|u_\delta\|^2_{1,\Omega^t,\alpha} + \|D^2_{t,x'}u_\delta\|^2_{0,\Omega^t,\alpha}) + a^{2(m-1)}\|Lu_\delta\|^2_{1,\Omega^t,\alpha}].$$

For the boundary term we have

$$\int_{\partial\Omega^t} |MD^2_{t,x'}u_\delta - D^2_{t,x'}Mu_\delta|^2 e^{-2\alpha s}\,dx'\,ds$$
$$\le \int_{\partial\Omega^t} (|D^2_{t,x'}Mu_\delta|^2 + |D^1_{t,x'}M|^2|D^1_{t,x'}u_\delta|^2)e^{-2\alpha s}\,dx'\,ds$$
$$\le cb^2 \int_{\partial\Omega^t} (|u_\delta|^2 + |D^1_{t,x'}u_\delta|^2)e^{-2\alpha s}\,dx'\,ds.$$

From the Sobolev embedding

$$\left(\frac{n}{2} - \frac{n-1}{2q}\right)\frac{1}{\mu} \le 1 \Rightarrow W^\mu_2(\Omega) \hookrightarrow L_{2q}(\partial\Omega)$$

for $n=3$, $\mu=1$, $q=1$ we have

$$\|D^1_{t,x'}u_\delta\|_{2,\partial\Omega} \le \|D^1_{t,x'}u_\delta\|_{1,2,\Omega}, \qquad \|u_\delta\|_{2,\partial\Omega} \le \|u_\delta\|_{1,2,\Omega}.$$

Using this and (4.41) we get

$$(4.44)\qquad \int_{\partial\Omega^t} |MD^2_{t,x'}u_\delta - D^2_{t,x'}Mu_\delta|^2 e^{-2\alpha s}\,dx'\,ds$$
$$\le cb^2[(a^{2m}+1)(\|u_\delta\|^2_{1,\Omega^t,\alpha} + \|D^2_{t,x'}u_\delta\|^2_{0,\Omega^t,\alpha}) + a^{2(m-1)}\|Lu_\delta\|^2_{1,\Omega^t,\alpha}].$$

If we take α such that

$$(4.45)\qquad c\left[\frac{2}{\alpha\alpha_0}a^2(a^{2m}+1) + \frac{\alpha\alpha_0}{2}(a^{4m}+a^{2m}) + c_3b^2(a^{2m}+1)\right] \le \frac{\alpha\alpha_0}{4}$$

and use the inequality (following from (4.32))

$$(4.46)\qquad \|D^1_{t,x'}D^1_{x_1}u_\delta\|^2_{0,2,\Omega} + \|D^2_{x_1}u_\delta\|^2_{0,2,\Omega}$$
$$\le c(a^{4(m-1)} + 2a^{2(m-1)})\|Lu_\delta\|^2_{1,0,\Omega}$$
$$+ (2a^{2m}+a^{4m})\|u_\delta\|^2_{1,0,\Omega} + (a^{2m}+a^{4m})\|D^2_{t,x'}u_\delta\|^2_{0,2,\Omega}$$

we conclude (combining (4.38), (4.40)–(4.42), (4.46) and using (4.43)–(4.45)) that

$$(4.47)\qquad \alpha_0|u_\delta|^2_{2,0,\Omega}e^{-2\alpha t} + \frac{\alpha\alpha_0}{4}\|u_\delta\|^2_{2,\Omega^t,\alpha} + \frac{c_0}{2}\|u_\delta\|^2_{2,\partial\Omega^t,\alpha}$$
$$\le p_2(a,b)(\|Lu_\delta\|^2_{2,\Omega^t,\alpha} + \|Lu_\delta\|^2_{1,0,\Omega}e^{-2\alpha t})$$
$$+ q_2(a,b)\|Mu_\delta\|^2_{2,\partial\Omega^t,\alpha} + c_2|u_\delta|^2_{2,0,\Omega}|_{t=0}$$

where p_2, q_2 are polynomials.

Moreover, using

$$|F|^2_{\nu,0,\Omega}e^{-2\alpha t} \le \frac{c}{\alpha}|F|^2_{\nu+1,\Omega^t,\alpha} + |F|^2_{\nu,0,\Omega}|_{t=0} \tag{4.48}$$

for $\nu = 1$ and taking $\delta \to 0$, for $u = \lim_{\delta\to 0} u_\delta$ we obtain estimate (4.28) for $s = 2$.

Finally, we consider $s = 3$; like before, by (4.9) we get

$$\begin{aligned}
\alpha_0 e^{-2\alpha t}\int_\Omega |D^3_{t,x'}u_\delta|^2\,dx &+ \frac{\alpha\alpha_0}{2}\int_{\Omega^t}|D^3_{t,x'}u_\delta|^2e^{-2\alpha s}\,dx\,ds \\
&+ \frac{c_0}{2}\int_{\partial\Omega^t}|D^3_{t,x'}u_\delta|^2e^{-2\alpha s}\,dx'\,ds \\
&\le \frac{2}{\alpha\alpha_0}\int_{\Omega^t}|D^3_{t,x'}Lu_\delta - LD^3_{t,x'}u_\delta|^2e^{-2\alpha s}\,dx\,ds \\
&+ \frac{2}{\alpha\alpha_0}\int_{\Omega^t}|D^3_{t,x'}Lu_\delta|^2e^{-2\alpha s}\,dx\,ds \\
&+ c_3\int_{\partial\Omega^t}|D^3_{t,x'}Mu_\delta - MD^3_{t,x'}u_\delta|^2e^{-2\alpha s}\,dx'\,ds \\
&+ c_3\int_{\partial\Omega^t}|D^3_{t,x'}Mu_\delta|^2e^{-2\alpha s}\,dx'\,ds \\
&+ c_2\int_\Omega |D^3_{t,x'}u_\delta|^2\,dx\Big|_{t=0}.
\end{aligned} \tag{4.49}$$

Because by (4.32),

$$\begin{aligned}
&\int_{\Omega^t}|D^2_{t,x'}D^1_{x_1}u_\delta|^2e^{-2\alpha s}\,dx\,ds \\
&\quad\le ca^{2(m-1)}[\|Lu_\delta\|^2_{2,\Omega^t,\alpha} + a^2(\|u_\delta\|'^2_{2,\Omega^t,\alpha} + \|D^3_{t,x'}u_\delta\|^2_{0,\Omega^t,\alpha})]
\end{aligned} \tag{4.50}$$

we can estimate

$$\begin{aligned}
&\int_{\Omega^t}|LD^3_{t,x'}u_\delta - D^3_{t,x'}Lu_\delta|^2e^{-2\alpha s}\,dx\,ds \\
&\le \int_{\Omega^t}|D^3_{t,x'}\widetilde{L}\cdot D^1_{t,x}u_\delta \\
&\quad + D^2_{t,x'}\widetilde{L}\cdot D^1_{t,x'}D^1_{t,x}u_\delta + D^1_{t,x'}\widetilde{L}\cdot D^2_{t,x'}D^1_{t,x}u_\delta|^2e^{-2\alpha s}\,dx\,ds \\
&\le ca^2(\|u_\delta\|^2_{2,\Omega^t,\alpha} + \|D^2_{t,x'}D^1_{t,x}u_\delta\|^2_{0,\Omega^t,\alpha}) \\
&\le ca^{2m}\|Lu_\delta\|^2_{2,\Omega^t,\alpha} + a^2(a^{2m}+1)(\|u_\delta\|^2_{2,\Omega^t,\alpha} + \|D^3_{t,x'}u_\delta\|^2_{0,\Omega^t,\alpha}).
\end{aligned} \tag{4.51}$$

Let us estimate

$$
\begin{aligned}
(4.52)\qquad & \int\limits_{\Omega^t} |D^1_{t,x'} D^2_{x_1} u_\delta|^2 e^{-2\alpha s}\, dx\, ds \\
&\le ca^{2(m-1)}[\|Lu_\delta\|^2_{2,\Omega^t,\alpha} + a^2(\|u_\delta\|^2_{2,\Omega^t,\alpha} + \|D^2_{t,x'} D^1_{x_1} u_\delta\|^2_{0,\Omega^t,\alpha})] \\
&\le c[(a^{2(m-1)} + a^{4m-2})\|Lu_\delta\|^2_{2,\Omega^t,\alpha} \\
&\quad + (a^{2m} + a^{4m})\|u_\delta\|^2_{2,\Omega^t,\alpha} + a^{4m}\|D^3_{t,x'} u_\delta\|^2_{0,\Omega^t,\alpha}],
\end{aligned}
$$

$$
\begin{aligned}
(4.53)\qquad & \int\limits_{\Omega^t} |D^3_{x_1} u_\delta|^2 e^{-2\alpha s}\, dx\, ds \\
&\le ca^{2(m-1)}[\|Lu_\delta\|^2_{2,\Omega^t,\alpha} + a^2(\|u_\delta\|^2_{2,\Omega^t,\alpha} + \|D^1_{t,x} D^2_{x_1} u_\delta\|^2_{0,\Omega^t,\alpha})] \\
&\le c[(a^{2(m-1)} + a^{4m-2} + a^{6m-2})\|Lu_\delta\|^2_{2,\Omega^t,\alpha} \\
&\quad + (a^{2m} + a^{4m} + a^{6m})\|u_\delta\|^2_{2,\Omega^t,\alpha} + a^{6m}\|D^3_{t,x'} u_\delta\|^2_{0,\Omega^t,\alpha}].
\end{aligned}
$$

We have to consider

$$
\begin{aligned}
& \int\limits_{\partial\Omega^t} |MD^3_{t,x'} u_\delta - D^3_{t,x'} M u_\delta|^2 e^{-2\alpha s}\, dx'\, ds \\
&= \int\limits_{\partial\Omega^t} |(D^3_{t,x'} M) u_\delta + D^2_{t,x'} M \cdot D^1_{t,x'} u_\delta + D^1_{t,x'} M \cdot D^2_{t,x'} u_\delta|^2 e^{-2\alpha s}\, dx'\, ds \\
&\le cb^2 \int_0^t (|u_\delta|^2_{2,\partial\Omega} + |D^1_{t,x'} u_\delta|^2_{2,\partial\Omega} + |D^2_{t,x'} u_\delta|^2_{2,\partial\Omega}) e^{-2\alpha s}\, ds.
\end{aligned}
$$

Using again the Sobolev embeddings

$$
\begin{aligned}
\|u_\delta\|_{2,\partial\Omega} &\le c\|u_\delta\|_{1,2,\Omega}, \\
\|D^1_{t,x'} u_\delta\|_{2,\partial\Omega} &\le c\|D^1_{t,x'} u_\delta\|_{1,2,\Omega}, \\
\|D^2_{t,x'} u_\delta\|_{2,\partial\Omega} &\le c\|D^2_{t,x'} u_\delta\|_{1,2,\Omega},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(4.54)\qquad & \int\limits_{\partial\Omega^t} |MD^3_{t,x'} u_\delta - D^3_{t,x'} M u_\delta|^2 e^{-2\alpha s}\, dx'\, ds \\
&\le cb^2(\|u_\delta\|^2_{2,\Omega^t,\alpha} + \|D^3_{t,x'} u_\delta\|^2_{0,\Omega^t,\alpha} + \|D^1_{x_1} D^2_{t,x'} u_\delta\|^2_{0,\Omega^t,\alpha}) \\
&\le cb^2 a^{2(m-1)}\|Lu_\delta\|^2_{2,\Omega^t,\alpha} + b^2(a^{2m} + 1)(\|u_\delta\|^2_{2,\Omega^t,\alpha} + \|D^3_{t,x'} u_\delta\|^2_{0,\Omega^t,\alpha}).
\end{aligned}
$$

Let us assume that

$$
(4.55)\qquad c\left[\frac{2}{\alpha\alpha_0} a^2(a^{2m}+1) + \frac{\alpha\alpha_0}{2}(a^{2m}+a^{4m}+a^{6m}) + c_3 b^2(a^{2m}+1)\right] \le \frac{\alpha\alpha_0}{4}.
$$

Then, adding to (4.49) inequalities (4.50), (4.52)–(4.53) and

$$(4.56)\quad \|D^2_{t,x'}D^1_{x_1}u_\delta\|^2_{2,\Omega}+\|D^1_{t,x'}D^2_{x_1}u_\delta\|^2_{2,\Omega}+\|D^3_{x_1}u_\delta\|^2_{2,\Omega}$$
$$\le c[(3a^{2(m-1)}+2a^{4m-2}+a^{6m-2})\|Lu_\delta\|^2_{2,0,\Omega}$$
$$+(3a^{2m}+2a^{4m}+a^{6m})\|u_\delta\|^2_{2,0,\Omega}$$
$$+(a^{2m}+a^{4m}+a^{6m})\|D^3_{t,x'}u_\delta\|^2_{2,\Omega}]$$

and using estimates (4.51), (4.52), by inequality (4.47), and the energy inequality (4.48) for $\nu=2$, we finally obtain

$$(4.57)\quad \alpha_0|u_\delta|^2_{3,0,\Omega}e^{-2\alpha t}+\frac{\alpha\alpha_0}{4}\|u_\delta\|^2_{3,\Omega^t,\alpha}+\frac{c_0}{2}\|u_\delta\|^2_{3,\partial\Omega^t,\alpha}$$
$$\le p_3(a,b)(\|Lu_\delta\|^2_{3,\Omega^t,\alpha}+\|Lu_\delta\|^2_{2,0,\Omega}|_{t=0})$$
$$+c_2\|u_\delta\|^2_{3,0,\Omega}|_{t=0}+q_3(a,b)\|Mu_\delta\|^2_{3,\partial\Omega^t,\alpha}$$

where $p_3,\ q_3$ are polynomials.

Moreover, by convergence in suitable spaces, after passing with δ to zero, we obtain estimate (4.28) for $s=3$. This concludes the proof. ∎

Theorem 4.2 and Lemma 4.5 imply:

Theorem 4.3. *Suppose the following assumptions are satisfied*:

(1) *Ω is a half-space*, $\widetilde{L}\in \Pi^3_0(\Omega^T)$, $M\in \Pi^3_0(\Omega^T)\cap H^3_\alpha(\partial\Omega^t)$, $F\in H^3_\alpha(\Omega^T)$, $g\in H^3_\alpha(\partial\Omega^T)$, $u_0\in H^3(\Omega)$ *and* $u_0|_{\partial\Omega}=0$.

(2) *We have*

$$\min_\mu\min_{\Omega^T}|\lambda_\mu|\ge c_0>0\quad \textit{so}\quad |\det A_{\bar n}|\ge c_0^m,\quad \textit{and}$$
$$\max_{\Omega^T}|\lambda_\mu|\le \frac{\max_{\Omega^T}|\det A_{\bar n}|}{c_0^{m-1}}\le c\frac{1}{c_0^{m-1}}|\widetilde{L}|^m_{3,0,\infty,\Omega^T}.$$

(3) $Eu\cdot u\ge\alpha_0u^2$.

Then there exists a unique solution of problem (4.1) *such that* $u\in\Pi^3_0(\Omega^T)\cap H^3_\alpha(\Omega^T)\cap H^3_\alpha(\partial\Omega^T)$ *and estimate* (4.28) *holds under assumption* (4.29).

4(c) *Existence of solutions for the linearized equations in a bounded domain.* Now we want to prove Theorem 4.3 for a bounded domain Ω. Since (4.9) holds in Ω, we have the existence of solutions to the linearized problem (4.1) in $L_{2,\alpha}(\Omega^T)$ (by Lemma 4.1 and Theorem 4.1). For higher regularity we introduce a suitable partition of unity.

Take a system of $\xi_i(x)\in C^\infty(\Omega)$, $\xi_i\in[0,1]$, $i\in\mathcal{M}\cup\mathcal{N}$, $\Omega_i=\operatorname{supp}\xi_i(x)\cap\Omega$, $w_i=\{x:\xi_i(x)=1\}$,

$$i\in\mathcal{M}\Leftrightarrow\Omega_i\cap\partial\Omega=\emptyset,\qquad i\in\mathcal{N}\Leftrightarrow\Omega_i\cap\partial\Omega\ne\emptyset,$$

$\bigcup \Omega_i = \bigcup w_i = \Omega$, $\operatorname{diam} \Omega_i \le \lambda$, only finitely many Ω_i's are nonempty. Next, let $\eta_i(x) = \xi_i(x)/\sum \xi_i^2(x)$ (so $\sum \eta_i(x)\xi_i(x) = 1$). We define $f_i(x,t) = f(x,t)\eta_i(x)$; from (4.1) we get

$$\begin{aligned} Lu_i = L(u\eta_i) &= E(u\eta_i)_t + \sum_{\gamma=1}^3 A_\gamma (u\eta_i)_{x_\gamma} \\ &= E\frac{\partial}{\partial t}u\cdot\eta_i + \sum_{\gamma=1}^3 \Big(A_\gamma \frac{\partial}{\partial x_\gamma}u\cdot\eta_i + A_\gamma u\cdot\frac{\partial}{\partial x_\gamma}\eta_i\Big) \\ &= (Lu)\eta_i + \sum_{\gamma=1}^3 A_\gamma u\cdot\eta_{i,x_\gamma} \end{aligned}$$

so

$$\begin{aligned} &Lu_i = F_i + [L,\eta_i]u \quad \text{in } \Omega_i, \\ &Mu_i|_{\partial\Omega} = g_i, \qquad\qquad\qquad\qquad\qquad\qquad (4.58) \\ &u_i|_{t=0} = u_{0i}, \quad \text{where} \quad [L,\eta_i]u = \sum_{\alpha=1}^3 A_\alpha u \eta_{i,x_\alpha}. \end{aligned}$$

We consider two cases. For $i \in \mathcal{M}$ we have only the Cauchy problem $(4.58)_{1,3}$, so we obtain an estimate of type (4.28), but without the boundary term and expressions with M; denote it by $(4.28)'$. In the case $i \in \mathcal{N}$, we take a local coordinate system centred in the middle of $\partial\Omega \cap \Omega_i$ such that $x_1 > 0$ belong to Ω_i and x_1 is a coordinate along the axis generated by $\overline{n}(\widetilde{x}_i)$ and $x' = (x_2, x_3)$ are directions perpendicular to $\overline{n}$. Then, if $\partial\Omega \cap \Omega_i$ is described by $x_1 - \varphi(x') = 0$, by the transformation $y' = x'$, $y_1 = x_1 - \varphi(x')$ we get the half-space $y_1 > 0$. We can write our problem in the form

$$\begin{aligned} &\widehat{L}\widehat{u}_i = \widehat{F}_i + [\widehat{L}, \widehat{\eta}_i]u, \\ &\widehat{M}\widehat{u}_i|_{\partial\Omega} = \widehat{g}_i, \qquad\qquad\qquad\qquad (4.59) \\ &\widehat{u}_i|_{t=0} = \widehat{u}_{0i}, \quad \text{where} \quad \widehat{f}(y) = f(x)|_{x=x(y)}, \end{aligned}$$

and $\overline{n} = (-1, \varphi_{x'})(1 + \varphi_{x'}^2)^{-1/2} = -\big(\frac{\partial y_1}{\partial x_1}, \frac{\partial y_1}{\partial x_2}, \frac{\partial y_1}{\partial x_3}\big)\big(\sum_{i=1}^3 y_{1,x_i}^2\big)^{-1/2}$ implies

$$\sum_{s=1}^3 \widehat{A}_s\Big(-\frac{\partial y_1}{\partial x_s}\Big)\Big(\sum_{i=1}^3 y_{1,x_i}^2\Big)^{-1/2} = \widehat{A}\cdot\overline{n}$$

so we have new matrices $A_1' = -\widehat{A}\cdot\overline{n}$, $A_2' = \widehat{A}_2$, $A_3' = \widehat{A}_3$ (symmetric).

We can apply the considerations of part (b) to obtain an estimate for system (4.59) of type (4.28). Notice that in both cases $i \in \mathcal{M}$ and $i \in \mathcal{N}$ we must additionally consider the second term on the right-hand side of $(4.58)_1$ and $(4.59)_1$, respectively. We can write

$$\|[L,\eta_i]u\|_{\mu,\Omega^t,\alpha} = \Big\|\sum_{k=1}^3 A_k u\eta_{i,x_k}\Big\|_{\mu,\Omega^t,\alpha} = \Big\|\sum_{k=1}^3 A_k u_i\xi_i(x)\eta_{i,x_k}(x)\Big\|_{\mu,\Omega^t,\alpha} \le c\Big\|\sum_{k=1}^3 A_k u_i\Big\|_{3,\Omega^t,\alpha} \le ca\|u_i\|_{3,\Omega^t,\alpha} \tag{4.60}$$

because $\xi_i \le 1$, $\eta_i \in C^\infty(\Omega)$.

We obtain, for (4.59) and $i \in \mathcal{N}$,

$$\begin{aligned}\alpha_0|\widehat{u}_i|^2_{\mu,0,\widehat{\Omega}_i}e^{-2\alpha t} + \frac{\alpha\alpha_0}{4}\|\widehat{u}_i\|^2_{\mu,\widehat{\Omega}_i^t,\alpha}e^{-2\alpha t} &+ \frac{\alpha\alpha_0}{4}\|\widehat{u}_i\|^2_{\mu,\widehat{\Omega}_i^t,\alpha} + \frac{c_0}{2}\|\widehat{u}_i\|^2_{\mu,\partial\widehat{\Omega}_i^t,\alpha} \\ &\le \overline{p}_\mu(a,b)[\|\widehat{F}_i\|^2_{\mu,\widehat{\Omega}_i^t,\alpha} + |\widehat{F}_i|^2_{\mu-1,0,\widehat{\Omega}_i}|_{t=0}] \\ &\quad + \overline{q}_\mu(a,b)\|\widehat{g}_i\|^2_{\mu,\partial\widehat{\Omega}_i^t,\alpha} + \overline{r}_\mu(a,b)|\widehat{u}_i|^2_{\mu,0,\widehat{\Omega}_i}|_{t=0}.\end{aligned} \tag{4.61}$$

where

$$\overline{p}_\mu(a,b) + ca^2 \le \alpha\alpha_0, \tag{4.62}$$

$\overline{p}_\mu$, $\overline{r}_\mu$, $\overline{q}_\mu$, $\overline{p}_0$ are polynomials, $\widehat{\Omega}_i = T\Omega_i$, $\widehat{F}_i = (\widehat{L}u)_i$, $\widehat{g}_i = (\widehat{M}u)_i$ and T is the transformation defined by $y = y(x)$.

By summing inequalities (4.61) over $i \in \mathcal{N}$ and (4.28)$'$ over $i \in \mathcal{M}$, using $u = \sum_{i\in\mathcal{M}\cup\mathcal{N}} u_i(x)\xi_i(x)$, we obtain an estimate of the form (4.28) for a bounded domain Ω. Assuming

$$\alpha\alpha_0 \ge p_0(a,b) + ca^2 \tag{4.63}$$

for a bounded domain, we formulate:

Theorem 4.4. *Let Ω be a bounded domain with $\partial\Omega \in C^3$. Let the assumptions of Theorem* 4.3 *and* (4.63) *be satisfied for Ω. Then there exists a unique solution w of the linearized problem* (3.1), *where $u \in \Pi_0^3(\Omega^t) \cap H^3_\alpha(\Omega^t) \cap H^3_\alpha(\Omega^t) \cap H^3_\alpha(\partial\Omega^t)$ and* (4.28) *holds.*

5. The existence and uniqueness of solution of problem (3.1). We will consider the following iteration scheme:

$$\begin{aligned}&L(u_m)u_{m+1} \equiv E(t,x,u_m)u_{m+1,t} + \sum_{i=1}^3 A_i(x,t,u_m)u_{m+1,x_i} = 0,\\ &M(t,x,u_m)u_{m+1} = g(t,x) \quad \text{on } \partial\Omega^t,\\ &u_{m+1}|_{t=0} = u_0(x) \quad \text{in } \Omega,\end{aligned} \tag{5.1}$$

for $m = 0,1,2,\ldots$

Define $Q(G_0,\delta)=\{u:\Omega^t\to\mathbb{R}^m:\sup_{\Omega^t}|u(x,s)-u_0(x)|\le\delta$ for some u_0 with values in $G_0\}$. Recall that we assume that $\overline{G}\subset G$. If the values of all functions in $Q(G_0,\delta)$ lie in G (where δ depends on t and the system of equations), we assume for $u\in Q(G_0,\delta)$:

(a) The matrices E,A_i are symmetric, E is uniformly positive definite, and the matrix $-A_{\bar n}=-A\cdot\overline{n}$, where $\overline{n}$ is the unit outward vector normal to the boundary $\partial\Omega$, has eigenvalues separated from zero and positive eigenvalues bounded in a neighbourhood of the boundary.

(b) The matrix M has the following form:

$$M=\sum_{\mu,\nu=1}^{k}\alpha_{\mu\nu}(t,x',u)\gamma_\mu^+(t,x',u)\gamma_\nu^+(t,x',u)+\sum_{\mu=1}^{k}\sum_{\nu=k+1}^{m}\beta_{\mu\nu}(t,x',u)\gamma_\mu^+(t,x',u)\gamma_\nu^-(t,x',u)$$

and

$$\max_{\partial\Omega^t}|\alpha_{\mu\nu}^{-1}|\le\delta_0^{-1},\qquad\max_{\partial\Omega^t}|\beta_{\mu\nu}|\le\beta_0\qquad\forall\mu,\nu.$$

(c) $\det A_{\bar n}(t,x,u(x,t))\ne0$ in a neighbourhood of the boundary.

(d) The matrices $E(x,t,u(x,t))$, $A_i(x,t,u(x,t))$ are 3-times differentiable functions with respect to t,x,u, and belong to $L_2(\Omega)$ for each t.

We can guarantee that conditions (a)–(d) are satisfied for $u\in Q(G_0,\delta)$ in the following way. By Theorem 4.4 every solution of system (5.1) belongs to $C^\beta(\Omega^t)$, $\beta\in(0,1)$. Using continuity of u with respect to t, condition (d) and the assumption that conditions (a)–(d) are satisfied for $u_0(x)=u|_{t=0}\in G_0$ we have these properties for $u\in Q(G_0,\delta)$ for sufficiently small t; so let t^* be a time such that for $t<t^*$ we can use Theorem 4.4 for each u_m.

Let us assume $u_0|_{\partial\Omega}=0$ and consider, for $v_m=u_m-u_0$, the following system:

$$\begin{aligned}Lv_{m+1}&=E(t,x,u_m)v_{m+1,t}+\sum_{i=1}^{3}A_i(t,x,u_m)v_{m+1,x_i}\\&=-\sum_{i=1}^{3}A_i(t,x,u_m)u_{0,x_i},\\Mv_{m+1}|_{\partial\Omega}&=g,\\v_{m+1}|_{t=0}&=0.\end{aligned}\tag{5.2}$$

We get, by Theorem 4.4,

$$\begin{aligned}|||v_{m+1}|||_3^2\le{}&\widehat{p}(Q,|||u_m|||_3)[\|Au_{0,x}\|_{3,\Omega^t,\alpha}^2+|Au_{0,x}|_{2,0,\Omega}^2|_{t=0}]\\&+\widehat{q}(Q,|||u_m|||_3)\|g\|_{3,\partial\Omega^t,\alpha}^2\\&+\widehat{p}_0(Q,|||u_m|||_3)|v_{m+1}|_{3,0,\Omega}^2|_{t=0}\end{aligned}\tag{5.3}$$

where $|||v|||_3^2 \equiv |v|^2_{3,0,\infty,\Omega^t} + \|v\|^2_{3,\Omega^t,\alpha}$, $\widehat{p}_0(Q,|||u_m|||_3) \ge |E(t,x,u_m)|$, $\widehat{p}(Q,|||u|||_3) > p(a,b)$, $\widehat{q}(Q,|||u|||_3) > q(a,b)$ and a, b are defined as before. By the definition of v, we have

$$|||u|||_3 \le |||v|||_3 + \|u_0\|_{3,2,\Omega}. \tag{5.4}$$

To prove convergence of $\{u_m\}$ we have to know that if $|||v_m|||_3 \le d$ then $|||v_{m+1}|||_3 \le d^2$; using (5.4) in (5.3) we have

$$\begin{aligned}|||v_{m+1}|||_3^2 &\le \widehat{p}(Q, d + |||u_0|||_{3,2,\Omega})[\|Au_{0,x}\|^2_{3,\Omega^t,\alpha} + |Au_{0,x}|^2_{2,0,\Omega}|_{t=0}] \\ &\quad + \widehat{q}(Q, d + \|u_0\|_{3,2,\Omega})\|g\|^2_{3,\partial\Omega^t,\alpha} \\ &\quad + \widehat{p}_0(Q, d + \|u_0\|_{3,2,\Omega})|v_{m+1}|^2_{3,0,\Omega}|_{t=0}.\end{aligned} \tag{5.5}$$

Remark. We have used $\widehat{p}(Q,|||u|||_3)$ and $\widehat{q}(Q,|||u|||_3)$ by Lemma 6.1 of [10].

We see that $|||v_{m+1}|||_3^2 \le d^2$ for sufficiently small norms of $u_{0,x}$ and g, where $\|g\|_{3,\partial\Omega^t,\alpha}$ depends on the time t^*. This guarantees the convergence of the sequence $\{u_m\}$. Introducing $U_m = u_m - u_{m-1} = v_m - v_{m-1}$ we have the problem

$$\begin{aligned} L(u_m)U_{m+1} &= -[L(u_m) - L(u_{m-1})]v_m \\ &\qquad - \sum_{i=1}^{3}[A_i(u_m) - A_i(u_{m-1})]u_{0,x_i}, \\ M(u_m)U_{m+1} &= -[M(u_m) - M(u_{m-1})]v_m, \\ U_{m+1}|_{t=0} &= 0, \quad m \ge 0, \quad U_0 = u_0(x). \end{aligned} \tag{5.6}$$

By (d) we can write

$$\begin{aligned} |||\widetilde{L}(u_m) - \widetilde{L}(u_{m-1})|||_3 &\le |\widetilde{L}|_{3,0,\infty,\Omega^T}|||u_m - u_{m-1}|||_3, \\ |||M(u_m) - M(u_{m-1})|||_3 &\le |M|_{3,0,\infty,\Omega^T}|||u_m - u_{m-1}|||_3, \end{aligned} \tag{5.7}$$

therefore by Theorem 4.4 for problem (5.6) we have

$$|||U_{m+1}|||_2^2 \le h(Q,|||u_m|||_3,|||u_{m-1}|||_3)(|||v_m|||_3^2 + |||u_{0,x}|||_3^2)|||U_m|||_3^2. \tag{5.8}$$

By the smallness of $|||v_m|||_3$, adding the assumption that $\|u_{0,x}\|_{3,2,\Omega}$ is sufficiently small, we have the convergence of $\{u_m\}$ to u in $L_\infty(0,t;L_2(\Omega)) \cap L_{2,\alpha}(\Omega^t) \cap L_{2,\alpha}(\partial\Omega^t)$ and by (5.4), (5.5),

$$u \in \Pi_0^3(\Omega^t) \cap H^3_\alpha(\Omega^t) \cap H^3_\alpha(\partial\Omega^t).$$

Moreover, u is a unique solution. Assume, on the contrary, that u_1, u_2 are two solutions of the problem and $U = u_1 - u_2$. Then

$$\begin{aligned} L(u_2)U &= -[L(u_1) - L(u_2)]u_1, \\ M(u_2)U &= -[M(u_1) - M(u_2)]u_1, \\ U|_{t=0} &= 0. \end{aligned}$$

Lemma 4.1 implies

$$\alpha_0\|U\|^2_{2,\Omega}e^{-2\alpha t}+\frac{\alpha\alpha_0}{2}\|U\|^2_{0,\Omega^t,\alpha}+\frac{c_0}{2}\|U\|^2_{0,\partial\Omega^t,\alpha}$$
$$\le\frac{2}{\alpha\alpha_0}\|[L(u_1)-L(u_2)]u_1\|^2_{0,\Omega^t,\alpha}$$
$$+(c_0+c_1)\delta_0^{-2}\|[M(u_1)-M(u_2)]u_1\|_{0,\partial\Omega^t,\alpha}.$$

We can estimate

$$\|[L(u_1)-L(u_2)]u_1\|_{0,\Omega^t,\alpha}\le\sup_G\sup_{\Omega^t}|\overline{L}'(\widetilde{u})|\sup_{\Omega^t}(|u_1|+|D^1_{t,x}u_1|)\|U\|_{0,\Omega^t,\alpha},$$
$$\|[M(u_1)-M(u_2)]u_1\|_{0,\partial\Omega^t,\alpha}\le\sup_G\sup_{\partial\Omega^t}|M'(\widetilde{u})|\sup_{\Omega^t}|u_1|\cdot\|U\|_{0,\partial\Omega^t,\alpha}.$$

It is enough to assume that

$$\sup_G\sup_{\Omega^t}|\widetilde{L}'(\widetilde{u})|\sup_{\Omega^t}(|u_1|+|D^1_{t,x}u_1|)\le(\alpha\alpha_0)^2/8,\tag{5.9}$$

$$(c_0+c_1)\delta_0^{-2}\sup_G\sup_{\partial\Omega^t}|M'(\widetilde{u})|\sup_{\Omega^t}|u_1|\le c_0/4\tag{5.10}$$

to obtain

$$\alpha_0\|U\|^2_{2,\Omega}e^{-2\alpha t}+\frac{\alpha\alpha_0}{4}\|U\|^2_{0,\Omega^t,\alpha}+\frac{c_0}{4}\|U\|^2_{0,\partial\Omega^t,\alpha}\le 0$$

and this implies uniqueness.

Thus, we have proved:

Theorem 5.1. *Suppose the following assumptions are satisfied*:

(1) $g\in H^3_\alpha(\partial\Omega^t)$, $u_0|_{\partial\Omega}=0$, $u_0\in H^4(\Omega)$.

(2) $\partial\Omega\in C^3$.

(3) *The assumptions* (a)–(d) *are satisfied*, *and* $\|u_{0,x}\|_{3,2,\Omega}$ *is sufficiently small.*

(4) $t\le t^*$.

(5) α *satisfies* $\alpha\alpha_0\ge\widehat{p}_0(Q,d+\|u_0\|_{3,2,\Omega})$, $\widehat{p}_0(Q,d+\|u_0\|_{3,2,\Omega})\ge p_0(a,b)+ca^2$, *where* $p_0,\widehat{p}_0$ *are polynomials.*

(6) (5.9), (5.10) *are satisfied for some solution* $u_1\in C^1(\Omega^t)$.

Then there exists a unique solution u *of* (3.1) *such that* $u\in\Pi^3_0(\Omega^t)\cap H^3_\alpha(\Omega^t)\cap H^3_\alpha(\partial\Omega^t)$ *and we have uniqueness in* $C^1(\overline{\Omega}^t)$.

6. Equations of relativistic hydrodynamics—existence and uniqueness of solutions for the mixed problem. We have proved existence and uniqueness of solutions for the initial-boundary problem (3.1) using assumptions (a)–(d) (see Section 5). To apply these results to problem (2.2) (that is, the symmetric system of relativistic hydrodynamics), we have to check the assumptions of Theorem 5.1.

We have

$$A^0 z\cdot z=\begin{pmatrix}\frac{p}{\beta}T_p s_p+\beta u_1^2+\beta u_2^2+\beta u_3^2+\frac{\delta}{\beta}s_pT_\delta\\ p\beta u_1-\beta u_1^3 w+\frac{w}{\beta}u_1-\beta u_1u_2^2w-\beta u_1u_3^2w\\ p\beta u_2-\beta u_1^2u_2w-\beta u_2^3w+\frac{w}{\beta}u_2-\beta u_2u_3^2w\\ p\beta u_3-\beta u_3u_1^2w-\beta u_3u_2^2w-\beta u_3^3w+\frac{w}{\beta}u_3\\ \frac{p}{\beta}s_pT_\delta+\frac{\delta}{\beta}T_\delta(s_\delta-s/\delta)\end{pmatrix}^T\begin{pmatrix}p\\u^1\\u^2\\u^3\\\delta\end{pmatrix}$$

$$\begin{aligned}&=p^2\frac{s_pT_p}{\beta}+2p\delta\frac{s_pT_\delta}{\beta}+\delta^2\frac{T_\delta}{\beta}(s_\delta-s/\delta)\\&\quad+(u_1^2+u_2^2+u_3^2)(2p\beta+w/\beta)\\&\quad-(u_1^2+u_2^2+u_3^2)\beta w\end{aligned}$$

so from $u_1^2+u_2^2+u_3^2=1/\beta^2-1$ we get

$$A^0z\cdot z=p^2\frac{s_pT_p}{\beta}+2p\delta\frac{s_pT_\delta}{\beta}+\delta^2\frac{T_\delta(s_\delta-s/\delta)}{\beta}+(u_1^2+u_2^2+u_3^2)(2p\beta+w\beta).\tag{6.1}$$

Lemma 6.1. *Assume that there exists a constant* $\varrho\in(0,1)$ *such that for the initial data* $z_0=(p_0,u_{01},u_{02},u_{03},\delta_0)$, $u_\alpha=v_\alpha/(c\beta)$ (v_α *is the velocity*) *we have*

$$v_0^2\le(1-\varrho^2)c^2,\qquad p_0>0,\qquad \delta_0>0,\tag{6.2}$$

and, *for some* $\varepsilon>0$,

$$\begin{aligned}&\delta_0>c_1(p_0+\varepsilon)+\varepsilon,\\&\delta_0<c_1^{-1}(p_0-\varepsilon)\bigg(\gamma+\log\bigg\{\frac{p_0-\varepsilon}{\gamma-1}(\delta_0+\varepsilon)^\gamma\bigg\}\bigg)-\varepsilon\end{aligned}\tag{6.3}$$

where $c_1=2/\varrho^2-1+6\varepsilon^2$ *and* γ *is the adiabatic exponent. Then for* $z\in Q(G_0,\varepsilon)$ *there exists* $\alpha_0>0$ *such that* $Ez\cdot z\ge\alpha_0z^2$ *where* $E=A^0$.

Proof. By definition of $Q(G_0,\varepsilon)$, $|z(t)-z_0|\le\varepsilon$, so $p>p_0-\varepsilon$, $\delta\ge\delta_0-\varepsilon$, $u_\alpha\le u_{0\alpha}+\varepsilon$. (6.2) implies

$$\beta|_{t=0}=\sqrt{1-v_0^2/c^2}\ge\varrho^2$$

so we have

$$\begin{aligned}1/\beta^2-1&=\sum_{\alpha=1}^3u_\alpha^2\le\sum_{\alpha=1}^3(u_{0\alpha}+\varepsilon)^2\le2\sum_{\alpha=1}^3u_{0\alpha}^2+6\varepsilon^2\\&=2(1/\beta^2|_{t=0}-1)+6\varepsilon^2\le2/\varrho^2-2+6\varepsilon^2\end{aligned}\tag{6.4}$$

hence

$$\beta\ge(2/\varrho^2-1+6\varepsilon^2)^{-1}.\tag{6.5}$$

From the state equation $p/\delta = RT$ we calculate

(i) $$T_p = \frac{1}{R}\cdot\frac{1}{\delta}, \qquad T_\delta = -\frac{1}{R}\cdot\frac{p}{\delta^2}.$$

Taking entropy in the form $s - s_0 = c_v \log\{p/((\gamma-1)\delta^\gamma)\}$ where c_v is the specific heat at constant volume, we get

(ii) $$s_p = c_v/p, \qquad s_\delta = -\gamma c_v/\delta.$$

Assuming $\varepsilon < \min\{p_0, \delta_0\}$ and using (i), (ii) and (6.4), by the inequality $2p\delta < p^2 + \delta^2$, we can estimate

$$\begin{aligned} Ez\cdot z &\ge p^2 s_p T_p - (p^2+\delta^2)s_p|T_\delta|c_1 + \delta^2(s_\delta - s/\delta)T_\delta \\ &\quad + 3(p_0-\varepsilon)(2/\varrho^2 - 1 + 6\varepsilon^2)^{-1}u^2 \\ &= p^2 s_p(T_p - c_1|T_\delta|) + \delta^2|T_\delta|(|s_\delta| + s/\delta - c_1 s_p) \\ &\quad + 3(p_0-\varepsilon)(2/\varrho^2 - 1 + 6\varepsilon^2)^{-1}u^2. \end{aligned}$$

Assumptions (6.3) guarantee that for $z \in Q(G_0,\varepsilon)$, $T_p - c_1|T_\delta| > 0$ and $|s_\delta| + s/\delta - c_1 s_p > 0$, so we can estimate $Ez\cdot z \ge \alpha_0 z^2$, where

$$\alpha_0 = \min\{3c_1^{-1}(p_0-\varepsilon), s_p(T_p - c_1|T_\delta|), |T_\delta|(|s_\delta| + s/\delta - c_1 s_p)\},$$

which concludes the proof. ■

Remark 6.1. To satisfy (6.3) we need

$$\left(\frac{1+v_0^2/c^2}{1-v_0^2/c^2}\right)^2 < \log\frac{e^\gamma R T_0}{(\gamma-1)\delta_0^{\gamma-1}},$$

where we used the fact that $\varepsilon > 0$ and is small. To have v_0 close to c we have to assume either T_0 large, or δ_0 small or γ close to 1.

Let us consider $A_{\bar n} = c\sum_{i=1}^3 A^i(z)n_i$, where c is the speed of light, and $\overline{n}$ the unit outward normal vector to $\partial\Omega$. The matrix $A_{\bar n}$ has the form

$$A_{\bar n} = c\begin{pmatrix} u_n s_p T_p & n_1 & n_2 & n_3 & u_n s_p T_\delta \\ n_1 & u_n w(1-\beta^2 u_1^2) & -\beta^2 u_n w u_1 u_2 & -\beta^2 u_n w u_1 u_3 & 0 \\ n_2 & -\beta^2 u_n w u_1 u_2 & u_n w(1-\beta^2 u_2^2) & -\beta^2 u_n w u_2 u_3 & 0 \\ n_3 & -\beta^2 u_n w u_1 u_3 & -\beta^2 u_n w u_2 u_3 & u_n w(1-\beta^2 u_3^2) & 0 \\ u_n s_p T_\delta & 0 & 0 & 0 & u_n T_\delta(s_\delta - s/\delta) \end{pmatrix}.$$

From (6.5) we have

(6.6) $\det(-A_{\bar n} - \lambda I)$

$= -c^5(u_n w + \lambda')\{(u_n T_\delta(s_\delta - s/\delta) + \lambda')[(u_n s_p T_p + \lambda')(u_n w + \lambda')(u_n w\beta^2 + \lambda')$

$$-(u_n w\beta^2(u_n^2+1)+\lambda')]-(u_n s_p T_\delta)^2(u_n w+\lambda')(u_n w\beta^2+\lambda')\}$$

where $\lambda'=\lambda/c$ so $\lambda_1=-cu_n w$.

By local straightening of the boundary (given by the transformation $T\colon y=y(x)$, see 4(c)), we can assume

$$u_n^2=(u_1n_1+u_2n_2+u_3n_3)^2=u_1^2+u_2^2+u_3^2=1/\beta^2-1.$$

Therefore we get

$$\det(-A_{\bar n}-\lambda I)=(cu_n w+\lambda)^2(\lambda^3+cb\lambda^2-c^2a\lambda+c^3d)$$

where

$$\begin{aligned} a&=(u_n s_p T_\delta)^2-u_n^2T_\delta(s_\delta-s/\delta)s_pT_p\\ &\quad-u_n^2T_\delta(s_\delta-s/\delta)w\beta^2-u_n^2s_pT_pw\beta^2+1,\\ b&=u_nT_\delta(s_\delta-s/\delta)+u_ns_pT_p+u_nw\beta^2,\\ d&=-(u_ns_pT_\delta)^2u_nw\beta^2-u_nT_\delta(s_\delta-s/\delta)\\ &\quad+u_n^3T_\delta(s_\delta-s/\delta)s_pT_pw\beta^2. \end{aligned} \tag{6.7}$$

We examine the polynomial

$$f(\lambda)=\lambda^3+cb\lambda^2-c^2a\lambda+c^3d$$

with derivative

$$f'(\lambda)=3\lambda^2+2cb\lambda-c^2a.$$

Using the solutions of $f'(\lambda)=0$:

$$x_1=-c\frac{b+\sqrt{b^2+3a}}{3},\quad x_2=c\frac{-b+\sqrt{b^2+3a}}{3},$$

we can calculate the local maximum $f(x_1)$ and minimum $f(x_2)$ of $f(\lambda)$; next, solving $f(x)-f(x_1)=0$ and $f(x)-f(x_2)=0$ we find x_r and x_l, respectively, such that

$$x_l<\lambda_3\le x_1\le\lambda_4\le x_2\le\lambda_5<x_r \tag{6.8}$$

where $\lambda_3,\ \lambda_4,\ \lambda_5$ are the roots of the second term of the characteristic polynomial (6.6).

Moreover, for $\lambda_1=\lambda_2=-cu_nw$ we have

$$u_n\le 3(2/\varrho^2-1+6\varepsilon^2),\qquad w\le w_0+\varepsilon. \tag{6.9}$$

Hence we formulate

Lemma 6.2. *Let the assumptions of Lemma* 6.1 *be satisfied and additionally suppose that*

(a) $s_p,\ T_p,\ T_\delta,\ s_\delta-s/\delta$ *are bounded*,
(b) $d=-(u_ns_pT_\delta)^2u_nw\beta^2-u_nT_\delta(s_\delta-s/\delta)+u_n^3T_ps_pT_\delta(s_\delta-s/\delta)w\beta^2\ne0$.

Then the eigenvalues λ_i *of the matrix* $-A_{\overline{n}} = -c\sum_{i=1}^{3} A^i(z)\overline{n}_i$ *and the matrix* $E = A^0(z)$ *satisfy conditions* (a) *and* (c) *of Section* 5 *and Theorem* 5.1.

Now we are finally prepared to formulate the result:

THEOREM 6.1. *Suppose the following assumptions are satisfied*:

(1) $g \in H^3_\alpha(\partial\Omega^t)$, $z_0 \in H^4(\Omega)$, $z_0|_{\partial\Omega} = 0$.
(2) $\partial\Omega \in C^3$.
(3) *For* $z_0 = (p_0, u_{01}, u_{02}, u_{03}, \delta_0)$ *we have*

(a) $p_0 > 0$, $\delta_0 > 0$,
(b) $v_0^2 = v_{01}^2 + v_{02}^2 + v_{03}^2 \le (1-\varrho^2)c^2$, *where* $\varrho \in (0,1)$.

(4) s_p, T_p, T_δ, $s_\delta - s/\delta$ *are bounded and of the same sign.*
(5) $d \ne 0$ (see (6.7) or Lemma 6.2).
(6) *The matrix* $M(x,t,z)$ *has the form described in* 5.1(b),
(7) *The matrices* $E(t,x,z(x,t))$, $A_i(t,x,z(x,t)) = cA^i(t,x,z(x,t))$ *are* 3-*times differentiable functions with respect to* t, x, z.
(8) $\|g\|_{3,\partial\Omega^t,\alpha}$ *and* $\|z_{0,x}\|_{3,2,\Omega}$ *are sufficiently small*, *and* $t \le t^*$ (see Section 5).
(9) α *satisfies* $\alpha\alpha_0 \ge \widehat{p}_0(Q, d + \|z_0\|_{3,2,\Omega}) \ge p_0(a,b) + ca^2$ *for some polynomials* $\widehat{p}_0$, p_0 *and* $Q = Q(G_0, \varepsilon)$ *is defined in Section* 5.

Then there exists a solution of (2.2) *such that*

$$z \in \Pi^3_0(\Omega^t) \cap H^3_\alpha(\Omega^t) \cap H^3_\alpha(\partial\Omega^t).$$

Moreover, *under the assumptions*

$$\sup_{Q}\sup_{\Omega^t} |\widetilde{L}'(z)| \sup_{\Omega^t}(|z_1| + |D^1_{t,x} z_1|) \le (\alpha\alpha_0)^2/8 \tag{6.10}$$

$$(c_0 + c_1)\delta_0^{-2} \sup_{Q}\sup_{\partial\Omega^t} |M'(z)| \sup_{\Omega^t} |z_1| \le c_0/4 \tag{6.11}$$

for some solutions $z_1 \in C^1(\Omega^t)$ *we have uniqueness.*

REMARK. Introducing the quantity $z - z_0$ in the method of successive approximations, we have avoided the assumption that z_0 is small. We need, in fact, the smallness of $z_{0,x}$ in H^3 and of $g(t,x)$ in $H^3_\alpha(\partial\Omega^t)$. That is very important in the relativistic case, where the condition $|z_0| < 1$ means that $v_0^2 < c^2/2$, which is very restrictive.

7. Barotropic case. We additionally consider the problem (1.1)–(1.2) in the barotropic case (that means, the pressure p is an explicit function of the density δ). As before, p and δ denote variables as measured in the reference frame moving with the fluid.

We assume that

$$w = \delta c^2 + \delta e_0 + p, \tag{7.1}$$

$$p=\delta^2\frac{\partial e_0}{\partial\delta}, \tag{7.2}$$

where e_0 is the specific internal energy $e_0=e_0(\delta)$.

We can write equations (1.1)–(1.2) in the form

$$\frac{\partial}{\partial x^k}[\delta(c^2+e_0)+p]u_iu^k+[\delta(c^2+e_0)+p]\frac{\partial}{\partial x^k}(u_iu^k)+\frac{\partial}{\partial x_i}p=0, \tag{7.3}$$

$$\frac{\partial}{\partial x^i}(\delta u^i)=0. \tag{7.4}$$

Notice that we now have 5 equations and 4 unknowns (because p is given by (7.2)). Moreover, it is easier to find $\lambda^i(\underline{z})$, where $\underline{z}=(u^1,u^2,u^3,\delta)$, such that λ^i are the coefficients of linear dependence for equations (7.3)–(7.4). By multiplying (7.3) by u^i and summing over i we get

$$-\frac{\partial}{\partial x^k}[\delta(c^2+e_0)+p]u^k-[\delta(c^2+e_0)+p]\frac{\partial u^k}{\partial x^k}+\frac{\partial p}{\partial x^i}u^i=0.$$

This implies

$$-(c^2+e_0)\frac{\partial\delta}{\partial x^k}u^k-(c^2+e_0)\delta\frac{\partial u^k}{\partial x^k}-\delta\frac{\partial e_0}{\partial x^k}u^k-p\frac{\partial u^k}{\partial x^k}=0.$$

Using (7.2) we get

$$-(c^2+e_0)\frac{\partial}{\partial x^k}(\delta u^k)-\delta\frac{\partial e_0}{\partial\delta}\left(\frac{\partial\delta}{\partial x^k}u^k+\delta\frac{\partial u^k}{\partial x^k}\right)=0 \tag{7.5}$$

so adding the equation of continuity (7.4) with multiplier $\lambda^4=c^2+e_0+\delta\partial e_0/\partial\delta\equiv c^2+e_0+p/\delta\equiv w/\delta$ to (7.5) we obtain zero.

In this way we have found $\lambda^m=(u^0,u^1,u^2,u^3,w/\delta)$ for system (7.3)–(7.4). We calculate

$$\partial_{\underline{z}^\tau}\lambda^m=\begin{pmatrix}\beta u_1 & 1 & 0 & 0 & 0\\ \beta u_2 & 0 & 1 & 0 & 0\\ \beta u_3 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\delta}\frac{\partial p}{\partial\delta}\end{pmatrix}$$

because

$$\frac{\partial}{\partial\delta}\left(\frac{w}{\delta}\right)=\frac{\partial e_0}{\partial\delta}+\frac{\partial}{\partial\delta}\left(\delta\frac{\partial e_0}{\partial\delta}\right)=2\frac{\partial e_0}{\partial\delta}+\delta\frac{\partial^2e_0}{\partial\delta^2}=\frac{1}{\delta}\frac{\partial p}{\partial\delta}.$$

Rewriting (7.3)–(7.4) in the form

$$\partial_{\underline{z}^j}q_m^k(\underline{z})\frac{\partial\underline{z}^j}{\partial x^k}=0,\qquad m=0,\dots,4,$$

we obtain

$$B^k_{\tau_j}\frac{\partial\underline{z}^j}{\partial x^k}=0\quad\text{where}\quad B^k_{\tau_j}=\partial_{\underline{z}^\tau}\lambda^m\partial_{\underline{z}^j}q_m^k(\underline{z}),\ \ i,k,j=0,\dots,3,$$

and $B^k(\underline{z})$ are symmetric.

Let us consider $B^0(\underline{z})$ and the condition $B^0\underline{z}\cdot\underline{z} > \alpha_0\underline{z}^2$. We calculate

$$\partial_{\underline{z}^j} q^0_m(\underline{z}) = \begin{pmatrix} -2wu_1 & -2wu_2 & -2wu_3 & p_\delta - \frac{1}{\beta^2}\frac{w}{\delta} \\ \frac{1}{\beta}w + \beta wu_1^2 & \beta wu_1u_2 & \beta wu_1u_3 & \frac{1}{\beta}\frac{w}{\delta}u_1 \\ \beta wu_1u_2 & \frac{1}{\beta}w + \beta wu_2^2 & \beta wu_2u_3 & \frac{1}{\beta}\frac{w}{\delta}u_2 \\ \beta wu_1u_3 & \beta wu_2u_3 & \frac{1}{\beta}w + \beta wu_3^2 & \frac{1}{\beta}\frac{w}{\delta}u_3 \\ \beta\delta u_1 & \beta\delta u_2 & \beta\delta u_3 & \frac{1}{\beta} \end{pmatrix}$$

and multiplying by $\partial_{\underline{z}^\tau}\lambda^m$ gives

$$B^0(\underline{z}) = \begin{pmatrix} -\beta wu_1u_1^2 + \frac{w}{\beta} & -\beta wu_1u_2 & -\beta wu_1u_3 & \beta p_\delta u_1 \\ -\beta wu_1u_2 & -\beta wu_2^2 + \frac{w}{\beta} & -\beta wu_2u_3 & \beta p_\delta u_2 \\ -\beta wu_1u_3 & -\beta wu_2u_3 & -\beta wu_3^2 + \frac{w}{\beta} & \beta p_\delta u_3 \\ \beta p_\delta u_1 & \beta p_\delta u_2 & \beta p_\delta u_3 & \frac{1}{\beta}\frac{p_\delta}{\delta} \end{pmatrix}$$

so we find

$$\begin{aligned} B^0\underline{z}\cdot\underline{z} &= \begin{pmatrix} \beta wu_1\big(\frac{1}{\beta^2} - u_1^2 - u_2^2 - u_3^2\big) + \beta p_\delta\delta u_1 \\ \beta wu_2\big(\frac{1}{\beta^2} - u_1^2 - u_2^2 - u_3^2\big) + \beta p_\delta\delta u_2 \\ \beta wu_3\big(\frac{1}{\beta^2} - u_1^2 - u_2^2 - u_3^2\big) + \beta p_\delta\delta u_3 \\ \beta p_\delta(u_1^2 + u_2^2 + u_3^2 + \frac{1}{\beta^2}) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \delta \end{pmatrix} \\ &= \beta w(u_1^2 + u_2^2 + u_3^3) + 2\beta p_\delta\delta(u_1^2 + u_2^2 + u_3^2) + \frac{p_\delta}{\beta}\delta. \end{aligned}$$

Using (see Lemma 6.1) $\beta \geq (2/\varrho^2 - 1 + 6\varepsilon^2)^{-1}$, $w \geq p_0 - \varepsilon$ and $\partial p/\partial\delta > 0$, $\delta > 0$ we have

$$B^0\underline{z}\cdot\underline{z} \geq (2/\varrho^2 - 1 + 6\varepsilon^2)^{-1}(p_0 - \varepsilon)(u_1^2 + u_2^2 + u_3^2) + \frac{p_\delta}{\delta}\delta^2.$$

Because $\partial p/\partial\delta > 0$ we can find some constant $\widetilde{c}$ such that $\partial p/\partial\delta > \widetilde{c} > 0$; as $\delta \leq \delta_0 + \varepsilon$ we hence obtain

$$\begin{aligned} B^0\underline{z}\cdot\underline{z} &\geq (2/\varrho^2 - 1 + 6\varepsilon^2)^{-1}(p_0 - \varepsilon)(u_1^2 + u_2^2 + u_3^2) \\ &\quad + \frac{\widetilde{c}}{\delta_0 + \varepsilon}\delta^2 \leq \underline{\alpha}_0\underline{z}^2, \end{aligned}$$

where

$$\underline{\alpha}_0 = \min\{(2/\varrho^2 - 1 + 6\varepsilon^2)^{-1}(p_0 - \varepsilon), \widetilde{c}/(\delta_0 + \varepsilon)\}.$$

References

[1] K. O. Friedrichs, *Conservation equations and laws of motion in classical physics*, Comm. Pure Appl. Math. 31 (1978), 123–131.

[2] —, *On the laws of relativistic electromagnetofluid dynamics*, ibid. 27 (1974), 749–808.

[3] K. O. Friedrichs and P. D. Lax, *Boundary value problem for the first order operators*, ibid. 18 (1965), 355–388.
[4] L. Landau and E. Lifschitz, *Hydrodynamics*, Nauka, Moscow, 1986 (in Russian); English transl.: *Fluid Mechanics*, Pergamon Press, Oxford, 1987.
[5] P. D. Lax and R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure Appl. Math. 13 (1960), 427–455.
[6] S. Mizohata, *Theory of Partial Differential Equations*, Mir, Moscow, 1977 (in Russian).
[7] M. Nagumo, *Lectures on Modern Theory of Partial Differential Equations*, Moscow, 1967 (in Russian).
[8] J. Smoller and B. Temple, *Global solutions of the relativistic Euler equations*, Comm. Math. Phys. 156 (1993), 67–99.
[9] W. M. Zajączkowski, *Non-characteristic mixed problems for non-linear symmetric hyperbolic systems*, Math. Meth. Appl. Sci. 11 (1989), 139–168.
[10] —, *Non-characteristic mixed problem for ideal incompressible magnetohydrodynamics*, Arch. Mech. 39 (1987), 461–483.

Joanna Rencławowicz and Wojciech M. Zajączkowski
Institute of Mathematics
'Sniadeckich 8
00-950 Warszawa, Poland
E-mail: jr@impan.gov.pl
wmzajacz@impan.gov.pl

Received on 14.4.1997;
revised version on 17.10.1997