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ON THE CONNECTIVITY OF EFFICIENT POINT SETS

Abstract. The connectivity of the efficient point set and of some proper efficient point sets in locally convex spaces is investigated.

1. Introduction. An important problem in vector optimization is the connectivity of efficient point sets. Several authors have discussed this topic in finite-dimensional spaces and normed spaces (cf. [3], [6], [7], [10], [12], [15]–[16], [18], [19], [21] and [23]). Recently, Gong [8] and Song [22] studied the connectivity of the efficient point set and some proper efficient point sets with respect to the weak topology of a normed space. As an application, the connectivity of the efficient solution set and of some proper efficient solution sets for a vector optimization problem with a set-valued mapping which has weakly compact images was also discussed.

In this note, we first study the connectivity of the efficient point set and of some proper efficient point sets of a weakly compact, cone-convex subset of a locally convex space. As an application, we study the connectivity of efficient solution sets for set-valued vector optimization. Our results unify the corresponding results of [21] and [22].

2. Preliminaries. Let Y be a locally convex space with topological dual space Y^*. Let $S \subset Y$ be a convex cone (i.e., $S + S \subset S$, and $\lambda S \subset S$ for every $\lambda \geq 0$). Denote by S^+ the dual cone of S:

$$S^+ = \{h \in Y^* \mid h(y) \geq 0 \text{ for all } y \in S\}.$$

Denote by S^{+i} the set of all strictly positive linear functionals in S^+, i.e.,

$$S^{+i} = \{h \in Y^* \mid h(y) > 0 \text{ for all } y \in S \setminus \{0\}\}.$$

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Let C be a subset of Y. By $\overline{C}$ (resp. $\overline{C}^w$) and $\operatorname{int} C$ we denote the closure (resp. the closure with respect to the weak topology) and interior of C, respectively.

A nonempty convex subset B of the convex cone S is said to be a *base* for S if $\overline{B}$ does not contain the origin and $S = \operatorname{cone}(B) = \{\lambda b \mid b \in B,\ \lambda \geq 0\}$.

It is clear that if S^{+i} is nonempty, then S is *pointed*, i.e., $S \cap (-S) = \{0\}$. Moreover, if S is a nonempty convex cone in Y, then $S^{+i} \neq \emptyset$ if and only if S has a base (see [13]).

Let C be a subset of Y and $S \subset Y$ be a closed convex pointed cone. A point $y_0 \in C$ is said to be a *positive proper efficient point* of C, $y_0 \in \operatorname{Pos}(C, S)$, if there exists an $h \in S^{+i}$ such that

$$h(y_0) \leq h(y) \quad \text{for all } y \in C.$$

A point $y_0 \in C$ is said to be:

• an *efficient point* of C with respect to S, $y_0 \in \mathrm{E}(C, S)$, if

$$(C - y_0) \cap (-S) = \{0\};$$

• a *Borwein proper efficient point* of C with respect to S (cf. [4]), $y_0 \in \operatorname{Bo}(C, S)$, if

$$\overline{\operatorname{cone}}(C - y_0) \cap (-S) = \{0\},$$

where $\overline{\operatorname{cone}}(C - y_0)$ denotes the closure of the set $\operatorname{cone}(C - y_0)$;

• a *Benson proper efficient point* of C with respect to S (cf. [2]), $Y_0 \in \operatorname{Be}(C, S)$, if

$$\overline{\operatorname{cone}}(C + S - y_0) \cap (-S) = \{0\}.$$

It is easy to show that $\operatorname{Pos}(C, S) \subset \operatorname{Bo}(C + S, S) = \operatorname{Be}(C, S) \subset \operatorname{Bo}(C, S) \subset \mathrm{E}(C, S)$ (cf. [9]).

Let Y be a normed space. A point $y_0 \in C$ is said to be a *super efficient point* of C with respect to S (cf. [5]) if there exists a number $M > 0$ such that

$$\overline{\operatorname{cone}}(C - y_0) \cap (B - S) \subset MB,$$

where B is the closed unit ball of Y. By $\operatorname{SE}(C, S)$ we denote the set of all super efficient points of C with respect to the cone S.

A subset C of Y is said to be S-*convex* if $C + S$ is convex (cf. Yu [24]).

Let A be a subset of a topological vector space X, and let $F : A \to Y$ be a set-valued mapping.

We say that F is *upper semicontinuous* at $x_0 \in A$ (see [1]) if for every open set U such that $F(x_0) \subset U$, there exists a neighborhood $N(x_0)$ of x_0 such that $F(x) \subset U$ for every $x \in N(x_0)$. F is said to be upper semicontinuous on A if F is upper semicontinuous at every point $x \in A$.

3. Connectivity of efficient point sets. In this section we shall discuss the connectivity of efficient point sets and some proper efficient point sets of a weakly compact and cone-convex subset in a locally convex space.

The following lemma will be used.

LEMMA 1. *Let Y be a locally convex space and let C be a weakly bounded subset of Y. Then the function $\phi : (C, \sigma(Y, Y^*)|_C) \times (Y^*, \beta(Y^*, Y)) \to \mathbb{R}$ defined by*

$$\phi(y, h) = h(y),$$

where $\sigma(Y, Y^*)$ is the weak topology on Y and $\beta(Y^*, Y)$ is the strong topology on Y^* (see [20]), *is continuous.*

P r o o f. Let $\{(y_\alpha, h_\alpha)\}$ be a net converging to (y, h) in $C \times Y^*$. Then $\{y_\alpha\}$ tends weakly to y in C and $\{h_\alpha\}$ tends to h in $(Y^*, \beta(Y^*, Y))$. Consider the equality

$$h_\alpha(y_\alpha) - h(y) = (h_\alpha - h)(y_\alpha) + h(y_\alpha - y).$$

Since C is weakly bounded and $\{y_\alpha\} \in C$, it is clear that $h_\alpha(y_\alpha) \to h(y)$. Hence ϕ is continuous. ∎

We recall that $y_0 \in C$ is a *point of continuity* (PC point) for C (see [14]) if for any 0-neighborhood V,

$$y_0 \notin \overline{[C \setminus (y_0 + V)]}^w.$$

Equivalently, $y_0 \in C$ is a PC point for C if a net in C converges (strongly) to y_0 whenever it converges weakly to y_0. We denote by $\mathrm{PC}(C)$ the set of all PC points of C.

If C is a compact subset of Y, then $C = \mathrm{PC}(C)$.

THEOREM 3.1. *Let Y be a locally convex space and let S be a closed convex cone in Y such that the set S^{+i} is nonempty. Let C be a weakly compact, S-convex subset of Y and let $\mathrm{E}(C, S) \subset \mathrm{PC}(C)$. Then $\mathrm{Pos}(C, S)$, $\mathrm{Be}(C, S)$, $\mathrm{Bo}(C, S)$, and $\mathrm{E}(C, S)$ are connected.*

P r o o f. The first part of the proof follows the lines of the proof of Theorem 1 of [21] or Theorem 3.1 of [22]. We present it here for completeness.

Define a set-valued mapping $G : Y^* \to Y$ by

$$G(h) = \{y \in C \mid h(y) = \min\{h(z) \mid z \in C\}\}.$$

We first show that $G(h)$ is convex for all $h \in S^{+i}$. Indeed, if $y_1, y_2 \in G(h)$ then $y_1, y_2 \in C$, and $r = h(y_1) = h(y_2) = \min\{h(y) \mid y \in C\}$. Since h is a linear functional, we have $h(\lambda y_1 + (1-\lambda)y_2) = \lambda h(y_1) + (1-\lambda)h(y_2) = r$ for all $\lambda \in [0, 1]$. If $\lambda y_1 + (1-\lambda)y_2 \in C$, then $\lambda y_1 + (1-\lambda)y_2 \in G(h)$, i.e. $G(h)$ is convex. Assume that $\lambda y_1 + (1-\lambda)y_2 \notin C$ for some $\lambda \in [0, 1]$. The convexity of $C + S$ implies that $\lambda y_1 + (1-\lambda)y_2 \in C + S$. Then there exists $y_0 \in C$ satisfying $y_0 \neq \lambda y_1 + (1-\lambda)y_2$ such that $\lambda y_1 + (1-\lambda)y_2 \in y_0 + S$. Since

$h \in S^{+i}$, this implies that $r = h(\lambda y_1 + (1-\lambda)y_2) > h(y_0)$, a contradiction. Thus $\lambda y_1 + (1-\lambda)y_2 \in C$ for all $\lambda \in [0,1]$.

Next we show that $G(\cdot)$ is an upper semicontinuous set-valued map from Y^* into Y, where Y^* is supplied with the strong topology $\beta(Y^*, Y)$.

Assume that $G(\cdot)$ is not upper semicontinuous at some $h_0 \in Y^*$. Then there exist an open set V in Y such that $G(h_0) \subset V$ and a net $\{h_\alpha\}_{\alpha \in I}$ in $(Y^*, \beta(Y^*, Y))$ converging to $h_0 \in Y^*$ such that

$$G(h_\alpha) \not\subset V \quad \text{for all } \alpha \in I.$$

Hence, there exist some $y_\alpha \in G(h_\alpha) \setminus V$ for $\alpha \in I$, i.e., $y_\alpha \in C \setminus V$ and

$$h_\alpha(y_\alpha) \le h_\alpha(y) \quad \text{for all } y \in C.$$

Since C is weakly compact, without loss of generality, we may assume that $\{y_\alpha\}$ converges weakly to a point $y_0 \in C$. Let $\phi : (C, \sigma(Y, Y^*)|_C) \times (Y^*, \beta(Y^*, Y)) \to \mathbb{R}$ be the function defined by

$$\phi(y, h) = h(y).$$

Note that C is weakly bounded. By Lemma 1, ϕ is continuous. Hence $h_\alpha(y_\alpha) \to h_0(y_0)$ and $h_\alpha(y) \to h_0(y)$ for all $y \in C$. Therefore,

$$h_0(y_0) \le h_0(y) \quad \text{for all } y \in C.$$

This means that $y_0 \in G(h_0) \subset V$ and hence $y_0 \in \mathrm{E}(C, S)$. Since $y_0 \in \mathrm{PC}(C)$, we infer that $\{y_\alpha\}$ converges to y_0, a contradiction.

It is obvious that S^{+i} is a connected subset of $(Y^*, \beta(Y^*, Y))$. By Theorem 3.1 of [11], we know that $\mathrm{Pos}(C, S) = \bigcup_{h \in S^{+i}} G(h)$ is a connected set. Moreover, $\mathrm{Pos}(C, S) \subset \mathrm{Be}(C, S) \subset \overline{\mathrm{Bo}(C, S)} \subset \mathrm{E}(C, S)$. By Theorems 4.4 and 3.2 of [17], we have $\mathrm{E}(C, S) \subset \overline{\mathrm{Pos}(C, S)}$. Therefore, the sets $\mathrm{Be}(C, S)$, $\mathrm{Bo}(C, S)$ and $\mathrm{E}(C, S)$ are connected. ■

Remark 1. If C is compact, then it is also weakly compact and $\mathrm{E}(C, S) \subset \mathrm{PC}(C)$, but not conversely. For instance, the unit ball B_Y in an infinite-dimensional and uniformly convex Banach space Y is weakly compact and each point of its sphere is a PC point, but the ball is not compact. Theorem 3.1 unifies and improves Theorem 1 of [21] on the connectivity of the efficient point set with respect to the norm topology of a normed space and Theorems 3.1 and 3.2 of [22] for the weak topology of a normed space.

Remark 2. When Y is a normed space, if S has a closed bounded base, then the norm-interior $\operatorname{int} S^+$ is nonempty and it is also connected with respect to the norm topology. By the proof of Theorem 2.2 of [8], we have $\mathrm{SE}(C, S) = \bigcup_{h \in \operatorname{int} S^+} G(h)$. From the proof of Theorem 3.1, it is clear that $\mathrm{SE}(C, S)$ is also a connected set. In this case, $\operatorname{int} S^+ \subset S^{+i}$. Equality holds if, in addition, Y is a reflexive Banach space, and hence $\mathrm{SE}(C, S) = \mathrm{Pos}(C, S)$.

In a nonreflexive space, equality above does not hold in general. Therefore, in the final remark of [22], one has to add the assumption that Y is reflexive.

Let A be a subset of a topological vector space X, and let $F : A \to Y$ be a set-valued mapping. We consider the following vector optimization problem:

$$\text{(P)} \qquad \min F(x), \quad \text{subject to } x \in A.$$

A point $x_0 \in A$ is said to be an *efficient* (resp. *Borwein proper efficient*, *Benson proper efficient*, *positive proper efficient*) *solution* of the problem (P) if there exists $y_0 \in F(x_0)$ such that y_0 is in $\mathrm{E}(F(A), S)$ (resp. $\mathrm{Bo}(F(A), S)$, $\mathrm{Be}(F(A), S)$, $\mathrm{Pos}(F(A), S)$).

Let us denote by $\mathrm{E}(A, S, F)$, $\mathrm{Bo}(A, S, F)$, $\mathrm{Be}(A, S, F)$, and $\mathrm{Pos}(A, S, F)$ the sets of all efficient, Borwein proper efficient, Benson proper efficient, and positive proper efficient solutions of the problem (P), respectively.

When Y is a normed space, the set of all super efficient solutions of (P) will be denoted by SE(A, F, S), i.e.

$$\mathrm{SE}(A, F, S) = \{x \in A \mid \exists y \in F(x) \text{ such that } y \in \mathrm{SE}(F(A), S)\}.$$

Let A be a convex subset of X. A set-valued mapping $F : A \to Y$ is said to be S-*convex* if for all $x_1, x_2 \in A$ and $t \in [0, 1]$ we have

$$tF(x_1) + (1 - t)F(x_2) \subset F(tx_1 + (1 - t)x_2) + S.$$

F is said to be S-*quasiconvex* if $x_1, x_2 \in A$, $y \in Y$,

$$F(x_1) \cap (y - S) \neq \emptyset \quad \text{and} \quad F(x_2) \cap (y - S) \neq \emptyset$$

imply

$$F(tx_1 + (1 - t)x_2) \cap (y - S) \neq \emptyset \qquad \text{for all } t \in [0, 1].$$

It is easy to show that F is S-quasiconvex if and only if

$$F^{-1}(y - S) = \{x \in A \mid F(x) \cap (y - S) \neq \emptyset\}$$

is convex for all $y \in Y$.

Clearly, if $F(\cdot)$ is S-convex, then F is S-quasiconvex and $F(A)$ is S-convex. The converse is not true as was shown by an example of [21].

THEOREM 3.2. *Let A be a compact subset of a Hausdorff topological vector space X, let Y be a locally convex space and let S be a closed convex cone in Y such that the set S^{+i} is nonempty. If $F : A \to Y$ is an upper semicontinuous set-valued mapping with compact images*, *$F(\cdot)$ is S-quasiconvex and $F(A)$ is S-convex*, *then* $\mathrm{Pos}(A, S, F)$, $\mathrm{Bo}(A, S, F)$, $\mathrm{Be}(A, S, F)$, *and* $\mathrm{E}(A, S, F)$ *are connected.*

Proof. The proof is exactly the same as those in [21] or [22], so we omit it.

REMARK 3. Under the assumptions of Theorem 3.2, if, in addition, Y is a normed space and S has a closed bounded base (cf. Remark 2), by using a similar argument, we can prove that $\mathrm{SE}(A,F,S)=F^{-1}(E(F(A),S))$ is connected. This result generalizes Theorem 2.2 of [8] where the set-valued mapping F was assumed to be S-convex.

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