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OPTIMALITY OF REPLICATION IN THE CRR MODEL WITH TRANSACTION COSTS

Abstract. Recently, there has been a growing interest in optimization problems associated with the arbitrage pricing of derivative securities in imperfect markets (in particular, in models with transaction costs). In this paper, we examine the valuation and hedging of European claims in the multiplicative binomial model proposed by Cox, Ross and Rubinstein [5] (the CRR model), in the presence of proportional transaction costs. We focus on the optimality of replication; in particular, we provide sufficient conditions for the optimality of the replicating strategy in the case of long and short positions in European options. This work can be seen as a continuation of studies by Bensaid *et al.* [2] and Edirisinghe *et al.* [13]. We put, however, more emphasis on the martingale approach to the claims valuation in the presence of transaction costs, focusing on call and put options. The problem of optimality of replication in the CRR model under proportional transaction costs was recently solved in all generality by Stettner [30].

1. Introduction. The CRR model is a discrete-time model of financial market, with two primary securities, a risky stock and a riskless bond, and with a finite set of dates $\{0, 1, \ldots, T\}$. A riskless bond is assumed to yield a constant return $r \geq 0$ over each time period $[t, t+1]$. This means that its price process, B, equals (by convention $B_0 = 1$)

$$B_t = (1+r)^t, \quad \forall t = 0, \ldots, T. \tag{1}$$

The stock price, S, is assumed to satisfy

$$\xi_{t+1} = S_{t+1}/S_t \in \{u, d\}, \quad \forall t = 0, \ldots, T-1, \tag{2}$$

with $S_0 \in \mathbb{R}_+$, where u and d are real numbers. We assume throughout that

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$0 < d < 1 + r < u$. In order to construct a probabilistic model of the stock price, we assume that ξ_t, $t = 1, \dots, T$, are mutually independent random variables, given on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with identical probability law: $\mathbf{P}(\xi_t = u) = p = 1 - \mathbf{P}(\xi_t = d)$ for $t = 1, \dots, T$, where $p \in (0, 1)$. It follows from (2) that

$$S_t = S_0 \prod_{j=1}^{t} \xi_j, \qquad \forall t = 0, \dots, T. \tag{3}$$

The process given by (3) is frequently referred to as the *multiplicative* (or *geometric*) random walk. Note that the assumption of independence of the random variables ξ_t, $t = 1, \dots, T$, is not essential; we can make this assumption, without loss of generality, for the sake of convenience. Also the specific value of the probability p plays no role whatsoever in what follows. It should be stressed that we do not examine here the no-arbitrage condition in the presence of transaction costs. It is intuitively clear, however, that under the standard assumption (i.e., when $0 < d < 1+r < u$) the CRR model with transaction costs is still arbitrage-free, meaning that self-financing strategies cannot result in riskless profits. Typically, papers devoted to the CRR model with transaction costs deal either with the exact *replication* of contingent claims, or with the *perfect hedging* ([1]) of contingent claims (other proposed approaches are: the *mean-variance* hedging, the *risk-minimizing* hedging, and the *expected utility* maximization approach; in contrast to the perfect hedging, these methodologies are not preference-free, however). In the latter case, it is not required that a portfolio matches exactly the value of the claim at the terminal date and in each state. One assumes instead that the terminal wealth is sufficient to cover the liabilities, with possibly a surplus of funds in some states. It is important to acknowledge that if the transaction costs are large enough, the perfect hedging of a contingent claim is more efficient a strategy than the replication of a claim. Hedging of contingent claims in the CRR binomial model (or in its multinomial generalization) with proportional transaction costs was examined by, among others, Boyle and Vorst [3], Bensaid *et al.* [2], Edirisinghe *et al.* [13], Mercurio and Vorst [24]. The aim of the present paper is twofold: first, we intend to examine the class of those European contingent claims for which the exact replication is the cheapest way of hedging the risk exposure (this question is addressed and solved in Stettner [30]). Second, we analyse a probabilistic approach to the valuation of options under proportional transaction costs

([1]) In existing literature, a strategy which is termed here a *perfect hedging* is frequently referred to as a *super-hedging* or a *super-replication.* The term *perfect hedging* is sometimes informally identified with *replication.* It is thus essential to stress that we adopt here a different terminological convention.

based on the notion of a *quasi-martingale measure*. It should be stressed that several papers deal, using various methodologies, with hedging and valuation under transaction costs in a continuous-time Black–Scholes framework; to mention a few: Leland [22], Hodges and Neuberger [18], Davis *et al.* [11], Davis and Clark [8], Cvitanić and Karatzas [6]. However, the practical conclusions deriving from these works are sometimes rather disappointing. In particular, it was recently shown by Soner *et al.* [29] (confirming the conjecture of Davis and Clark [8]) that in the Black–Scholes framework with proportional transaction costs, the unique *perfect hedging* strategy for the writer of a European call option is the trivial one: "buy one share of the stock at time 0 and hold it until the option's expiry". Therefore, we feel there is still a motivation to investigate discrete-time financial models in the presence of transaction costs. For related portfolio optimization problems under transaction costs, see [7]–[9], [23], [27]–[28], [32]–[33].

1.1. *Self-financing trading strategies.* In the existing literature, it is common to assume that proportional transaction costs are incurred when shares of a risky asset are traded. On the other hand, it is usually postulated that the trading in riskless bonds is exempted from transaction costs. We find it convenient to assume that the total cost of buying one share at time t is $(1+\lambda_t)S_t$ and the amount received for the sale of one share at time t equals $(1-\mu_t)S_t$, where $\lambda_t \in [0,+\infty)$ and $\mu_t \in [0,1)$ are real numbers. Conventions adopted in [2] and [3] correspond to the following values of the cost coefficients:

$$\lambda_t = \kappa, \quad \mu_t = \frac{\kappa}{1+\kappa}, \qquad \forall t = 1,\dots,T, \tag{4}$$

and

$$\lambda_t = \kappa, \quad \mu_t = \kappa, \qquad \forall t = 1,\dots,T, \tag{5}$$

respectively. A reader should thus be advised that the results established in various papers are not necessarily directly comparable. Denote by $\phi_t = (\alpha_t, \beta_t)$, $t = 0,\dots,T$, a trading strategy, where α_t stands for the number of shares and β_t denotes the amount of funds invested in riskless bonds at time t (after the portfolio rebalancement). We assume that a trading strategy ϕ is self-financing inclusive of transaction costs. Therefore, the self-financing condition has the following form ([2]):

$$\beta_t = \beta_{t-1}(1+r) - (1+\lambda_t)S_t|\Delta_t\alpha|\mathrm{I}_{\{\Delta_t\alpha>0\}} + (1-\mu_t)S_t|\Delta_t\alpha|\mathrm{I}_{\{\Delta_t\alpha<0\}} \tag{6}$$

for every $t = 0,\dots,T$, where $\Delta_t\alpha = \alpha_t - \alpha_{t-1}$. Equivalently, we have

([2]) We could have alternatively assumed that β_t is less than or equal to the right-hand side of (6)—such a modification corresponds to the allowance for the intertemporal consumption. This is not relevant from the viewpoint of our further results, however.

$$\alpha_{t-1}S_t + \beta_{t-1}(1+r) = \alpha_t S_t + \beta_t + \lambda_t(\alpha_t - \alpha_{t-1})S_t \mathrm{I}_{\{\alpha_t > \alpha_{t-1}\}} + \mu_t(\alpha_{t-1} - \alpha_t)S_t \mathrm{I}_{\{\alpha_{t-1} > \alpha_t\}}$$

for every $t = 0, \ldots, T$. In view of (6) it is clear that the transaction cost is charged on the change in the net stock position. Notice also that if trading at time 0 is not cost-free, we need to introduce also the notion of a pre-trading portfolio at time 0, $(\widetilde{\alpha}_0, \widetilde{\beta}_0)$ say. It will soon become clear that the relevant quantities, such as the *replication cost* or the *seller's price*, depend on the pre-trading composition of the portfolio (more exactly, on the individual's endowment in shares). It is useful to observe that the numbers α_t, which represent the number of shares held at time t, can be chosen with no restrictions. Given the values of α's for all dates, the corresponding β's are uniquely determined by the self-financing condition. In general, it is essential to assume that a portfolio can be rebalanced at the terminal date T (a revision of a portfolio at the terminal date is irrelevant when trading at time T is cost-free). It will be sometimes convenient to make the following additional assumptions, which refer to the absence of transaction costs at the initial (or terminal) date.

Assumption (TC.1). No transaction costs are incurred when a portfolio is established at time 0 (i.e., $\lambda_0 = \mu_0 = 0$). Consequently, the initial wealth of any self-financing trading strategy ϕ equals $V_0(\phi) = \alpha_0 S_0 + \beta_0$.

Under (TC.1), the concept of a pre-trading portfolio at time 0 has no relevance. Indeed, since the portfolio can be revised at no cost, the only quantity that really matters is the initial wealth $V_0(\phi)$, and not, for instance, the initial endowment in shares. For the sake of expositional clarity we make this assumption throughout this section. Some authors also make the following assumption, which has even more important consequences for the valuation of derivative securities.

Assumption (TC.2). No transaction costs are incurred when a portfolio is liquidated at the terminal date T (i.e., $\lambda_T = \mu_T = 0$). In this case, the self-financing condition takes the following form for $t = T$:

$$\alpha_{T-1}S_T + \beta_{T-1}(1+r) = \alpha_T S_T + \beta_T. \tag{7}$$

Assumption (TC.2) allows to assume, as in the case of a model without transaction costs, that all claims are settled in cash and all portfolios are liquidated at time T. Let us explain this point in some detail. Generally speaking, in the framework of models with transaction costs, it is essential to distinguish between various ways in which a claim is settled at time T. Hence, it appears convenient to define a contingent claim as a two-dimensional random variable, $X = (g_T, h_T)$ say, where g_T and h_T represent the number of shares and the amount of cash transferred at time T from the writer of the claim X to its holder (or conversely, depending on the actual sign of g_T and h_T). We say that a trading strategy ϕ replicates the claim X that is *settled by delivery* if $\alpha_T = g_T$ and $\beta_T = h_T$. It is clear that the (minimal) initial value of the replicating portfolio of an arbitrary contingent claim X settled by delivery, referred to as the *seller's replicating cost*, can be computed using the standard recursive procedure, provided, of course, that the class of replicating strategies for X is non-empty. On the other hand, under assumption (TC.2), one may assume, without loss of generality, that all claims are settled in cash, meaning that the claim $X = (g_T, h_T)$ is identified with the claim $\widetilde{X} = (0, g_T S_T + h_T)$. We shall assume throughout that a claim is settled by delivery.

1.2. *Replication of options.* For simplicity, let us first consider replication of a European call option in the CRR model under assumption (TC.1). A long call option *settled by delivery* corresponds, by definition, to the claim $C_T = (\mathrm{I}_{\{S_T > K\}}, -K\mathrm{I}_{\{S_T > K\}})$. It is thus clear that the replicating portfolio is composed, at the expiry date T, of one share of the stock, combined with a short position in riskless bonds equal to the strike price if $S_T > K$ (otherwise, it contains no assets at all). A short call option settled by delivery is represented by the claim $-C_T = (-\mathrm{I}_{\{S_T > K\}}, K\mathrm{I}_{\{S_T > K\}})$. Hence, the replicating portfolio of short call involves, at the option's expiry, a short position in one share and a long position in riskless bonds when $S_T > K$. In the presence of transaction costs, one needs to distinguish replicating costs of short and long positions. Basically, a seller (a buyer, resp.) of a given claim should replicate the long position (the short position, resp.) in order to hedge the risk. Therefore the *seller's replication cost* of X is given by the formula

$$p_0^{\mathrm{s}}(X) := \inf\{V_0(\phi) \mid \exists \phi \in \Phi : \alpha_T = g_T \text{ and } \beta_T = h_T\}, \tag{8}$$

where Φ is the class of all self-financing trading strategies. It is clear that $p_0^{\mathrm{s}}(X)$ represents the minimal cost of a replicating strategy (if it exists). It is natural to introduce the *buyer's replication cost* $p_0^{\mathrm{b}}(X)$ by setting $p_0^{\mathrm{b}}(X) = -p_0^{\mathrm{s}}(-X)$, or more explicitly,

$$p_0^{\mathrm{b}}(X) := -\inf\{V_0(\phi) \mid \exists \phi \in \Phi : \alpha_T = -g_T \text{ and } \beta_T = -h_T\}. \tag{9}$$

The first minus sign in (9) allows us to directly compare both quantities. In the absence of transaction costs, we have $p_0^{\mathrm{s}}(X) = p_0^{\mathrm{b}}(X) = \pi_0(X)$; that is, the replicating costs coincide with the arbitrage price of the claim X. As already mentioned, it is reasonable to assume that in some circumstances, a writer of a call option would like to hedge against his short position using the replicating strategy of a long call (though such a strategy could appear to be sub-optimal, in the sense made precise in what follows). Similarly, a buyer of the option may find it useful to hedge against his long call position (or rather against the associated short position in bonds ([3])) by forming a portfolio which dynamically replicates the short call option. Note that under transaction costs, the existence of a replicating strategy is no longer a trivial matter. It can be checked, however, that in the CRR model with proportional transaction costs, a European contingent claim admits a replicating (self-financing) strategy.

1.3. *Perfect hedging of contingent claims.* As pointed out, among others, by Bensaid *et al.* [2] and Edirisinghe *et al.* [13], the perfect replication of a contingent claim is not necessarily the optimal (i.e., cheapest) way of hedging the risk exposure. In some circumstances, it is possible to find a dynamic portfolio which ultimately dominates a given contingent claim for any state of nature, and requires less wealth at the initial date than the dynamic portfolio that replicates the claim. Such a trading strategy, which is sometimes referred to as a *super-hedging* strategy for X, is termed here a *perfect hedging* against a short position in a claim X. If a claim is settled by physical delivery of an underlying asset, we have the following definition (notice that the possibility of a revision of the portfolio ϕ at the terminal date T is essential here).

Definition 1.1. We say that a self-financing trading strategy ϕ is a *perfect hedging* against a short position in a contingent claim $X = (g_T, h_T)$ settled by delivery at time T if, at the terminal date, we have $\alpha_T \geq g_T$ and $\beta_T \geq h_T$.

Our aim is to determine a trading strategy with the minimal initial wealth among all perfect hedging strategies. It is rather obvious that we may restrict our attention ([4]) to these strategies for which $\alpha_T = g_T$ and $\beta_T \geq h_T$. Notice that for any claim X a perfect hedging against a short position in X leads to a terminal portfolio which is sufficient to meet the liability represented by X, with possibly some excess in certain states. First consider a party who has sold at time 0 a claim X for the price $c^{\mathrm{s}}(X)$. Formally, we assume that the pre-trading composition of the seller's portfolio at time 0 is $(0, c^{\mathrm{s}}(X), -1)$, where the last component represents the short position in X. The post-trading portfolio of the seller at time 0 is $(\alpha_0^{\mathrm{s}}, \beta_0^{\mathrm{s}}, -1)$, and the post-trading portfolio at the terminal date T needs to satisfy (it is implicitly

([3]) It is implicitly assumed that the option was purchased on margin, that is, with borrowed cash. Therefore, an option buyer is exposed to the risk of insolvency at the terminal date T.

([4]) Essentially, this follows from the fact that the stock price is strictly positive, and from our current assumption that transaction costs are proportional to the turnover. This wouldn't be true if, for instance, a constant (i.e., independent of the transaction's size) transaction cost were considered.

assumed that the claim X has already been settled)

$$(\alpha_T^{\mathrm{s}} - g_T, \beta_T^{\mathrm{s}} - h_T, 0) = (\alpha_T^{\mathrm{s}} - g_T, \beta_T^{\mathrm{s}} - h_T, 0) \geq (0, 0, 0),$$

where the last inequality is component-wise. The minimal price $c^{\mathrm{s}}(X)$ for which there exists a self-financing strategy with the above properties is called the *seller's price* of X, and is denoted by $\pi_0^{\mathrm{s}}(X)$. Now consider a party who contemplates the purchase of a claim X. Generally speaking, a buyer of the claim X can always be seen as a seller of the claim $-X$. The buyer's portfolio at time 0 is $(0, c^{\mathrm{s}}(-X), 1)$, and the post-trading portfolio at time 0 equals $(\alpha_0^{\mathrm{b}}, \beta_0^{\mathrm{b}}, 1)$. At the terminal date we require that

$$(\alpha_T^{\mathrm{b}} + g_T, \beta_T^{\mathrm{b}} + h_T, 0) = (\alpha_T^{\mathrm{b}} + g_T, \beta_T^{\mathrm{b}} + h_T, 0) \geq (0, 0, 0).$$

The least real number $c^{\mathrm{s}}(-X)$ for which there exists a trading strategy with these properties is, of course, the seller's price of $-X$, that is, $\pi_0^{\mathrm{s}}(-X)$. Notice that the number $-\pi_0^{\mathrm{s}}(-X)$ determines the maximal amount of cash the buyer can borrow in order to purchase the claim X, and still be able to repay his debts at the terminal date after the claim X is settled. This latter value is denoted by $\pi_0^{\mathrm{b}}(X)$, and is referred to as the *buyer's price* of X. Typically, if X is a non-negative claim then both the seller's price $\pi_0^{\mathrm{s}}(X)$ and the buyer's price $\pi_0^{\mathrm{b}}(X)$ are non-negative numbers; in addition, the inequality $\pi_0^{\mathrm{b}}(X) \leq \pi_0^{\mathrm{s}}(X)$ is valid. Formally, we have the following definition (the *seller's price* is sometimes referred to as the *super-hedging price*).

Definition 1.2. The *seller's price* of X at time 0 of a claim $X = (g_T, h_T)$, denoted by $\pi_0^{\mathrm{s}}(X)$, is the minimal initial cost of a perfect hedging strategy against the short position in X, that is,

$$\pi_0^{\mathrm{s}}(X) = \inf\{V_0(\phi) \mid \exists \phi \in \Phi : \alpha_T \geq g_T \text{ and } \beta_T \geq h_T\}. \tag{10}$$

The *buyer's price* of X at time 0, denoted by $\pi_0^{\mathrm{b}}(X)$, is set to be equal to $-\pi_0^{\mathrm{s}}(-X)$. More explicitly,

$$\pi_0^{\mathrm{b}}(X) = -\inf\{V_0(\phi) \mid \exists \phi \in \Phi : \alpha_T \geq -g_T \text{ and } \beta_T \geq -h_T\}. \tag{11}$$

Observe that the seller's and buyer's prices do not depend on an investor's preferences and probability beliefs. Moreover, it should be stressed that if an individual were able to sell the claim X at the price $\pi_0^{\mathrm{s}}(X)$, such a transaction would lead to an arbitrage opportunity in the market, in general (by symmetry, a similar remark applies to the buyer's price). For this reason, the seller's and buyer's prices can hardly be seen as arbitrage prices.

Remarks. In contrast to the option's replicating strategy in the CRR model with no transaction costs, the perfect hedging under transaction costs is a path-dependent strategy, in general. This may be explained by the fact that the optimal trading policy depends not only on the current stock price, but also on the shares portfolio inherited from the preceding date.

2. Example. The following example of the two-period CRR model was considered in Bensaid *et al.* [2] (see also Edirisinghe *et al.* [13]). For the reader's convenience we preserve their assumptions concerning the cost coefficients (see (4)). We consider a call option with exercise price $K = 100$, assuming the following binomial lattice describing the stock price: $S_0 = 100$, $S_1^u = 130$, $S_1^d = 90$ at time $t = 1$, and finally

$$S_2^{uu} = 169, \qquad S_2^{ud} = S_2^{du} = 117, \qquad S_2^{dd} = 81$$

at the terminal date $T = 2$ (this means that $u = 1.3$ and $d = 0.9$). Furthermore, we assume here that conditions (TC.1)–(TC.2) are met, so that transactions made at time 0 and T are exempted from costs. Take $r = 0$, and define $C_T = (\mathrm{I}_{\{S_T > K\}}, -K\mathrm{I}_{\{S_T > K\}})$, or equivalently (in view of (TC.2)), $C_T = (0, (S_T - K)^+)$. It is easy to check that in the absence of transaction costs, the arbitrage price of a European call option with strike $K = 100$ equals $\pi_0(C_T) = \$10.69$. We assume from now on that $\lambda_1 = 0.2$ and $\mu_1 = 0.2(1 + 0.2)^{-1}$ (that is, $\kappa = 0.2$ in (4) for $t = 1$).

2.1. *Seller's costs.* It is not hard to check that the initial cost of the unique replicating strategy of the long call option equals $p^{\mathrm{s}}(C_T) = \$15.33$. In particular, the unique replicating strategy involves at time 0 the purchase of 0.7263 shares and the borrowing of \$57.30; the portfolio is then revised at time 1 in an appropriate way. [2] noted the existence a trading strategy, $\widehat{\phi}$ say, which involves at time 0 the purchase of 0.8 shares of stock, combined with the borrowing of \$64.80, and such that the terminal value of $\widehat{\phi}$ dominates X. To completely specify the strategy $\widehat{\phi}$, it is sufficient to assume that it involves no trading at time 1. The initial cost of the strategy $\widehat{\phi}$ amounts to \$15.2, so that it is less than the replicating cost of the long call. This shows that under transaction costs, the dynamic portfolio that matches the claim X at the terminal is not necessarily the cheapest way of hedging the risk exposure. It should be stressed, however, that the strategy $\widehat{\phi}$ is not the optimal perfect hedging of the call option. It can be shown by solving a simple minimization problem that the minimal cost of hedging against the short position in the option equals (approximately) $\pi_0^{\mathrm{s}}(C_T) = \$14.19$. The trading strategy $\widetilde{\phi}$ which realizes this initial cost involves the buying at time 0 of 0.7467 shares (this requires, of course, \$74.67 cash) combined with borrowing \$60.48. If the stock price declines during the first period, the portfolio is not modified at time 1. On the other hand, if the stock price rises during this period, we buy an additional 0.2533 shares of stock. In this case, the portfolio is composed at time 1, after the rebalancement, of one share of stock, combined with borrowing \$100. The latter number is found from the following equality:

$$60.48 + 0.2533 \times 130 \times 1.2 = 100.$$

It is thus clear that if the stock price rises in the first period, the rebalancement at time 1 leads to the perfect replication of the option at expiry. Furthermore, if the stock price declines twice in a row, the terminal wealth is

$$-60.48 + 74.67 \times 0.9 \times 0.9 = 0,$$

so that the payoff from the option is matched exactly. Finally, if the stock price falls in the first period and then goes up, the terminal value of our portfolio is

$$-60.48 + 74.67 \times 0.9 \times 1.3 = 26.88 > 17,$$

hence, after meeting the liability, we end up with a surplus of cash. We conclude that the seller's price of the option equals $\pi_0^{\mathrm{s}}(C_T) = \$14.19$.

2.2. *Buyer's costs.* Let us now examine the buyer's price—that is, the maximal amount of cash one may borrow against the call option. By reasoning in a similar way to that above, one finds that the optimal trading strategy now involves selling short of 0.4722 shares of the stock and the long position in riskless bonds at \$38.25. In contrast to the previous case, no trading takes place at time 1. If the price rises during both periods, the terminal wealth amounts to

$$38.25 - 47.22 \times (1.3)^2 = -41.55 > -69.$$

In all other cases the portfolio's value at the terminal date exactly matches the claim $-X$, since

$$38.25 - 47.22 \times 1.3 \times 0.9 = -17 \quad \text{and} \quad 38.25 - 47.22 \times (0.9)^2 = 0.$$

This shows that the buyer's price equals (approximately) $\pi_0^{\mathrm{b}}(C_T) = \$8.97$. As already mentioned, the buyer's price may be interpreted as the maximal amount of cash the owner of the call option may borrow from the bank, and still be sure that he will be able to repay his loan at time T in all circumstances. For completeness, we shall now find the replication cost of a short call option. It appears that the unique replicating strategy for the short position in a call option involves selling short 0.3353 shares of stock and investing \$27.98 in bonds (this generates \$5.55 of cash). The portfolio is then revised at time 1 by shorting, in addition, 0.6647 shares in the up-state, and shorting 0.1396 shares in the down-state. This means that in the up-state we have $-\$130$ in shorted shares and \$100 in riskless bonds. In the down-state, the corresponding numbers are $-\$42.498$ and \$38.25, respectively. One can easily check that this portfolio replicates the short call option. The amount \$5.55 can be seen as the maximal amount of cash one can borrow from the bank against the call option, if one wishes to repay the debt exactly at every state at time T. Summarizing, we obtain the following chain of inequalities:

$$\begin{aligned} p_0^{\rm b}(C_T) &= 5.55 < \pi_0^{\rm b}(C_T) \\ &= 8.97 < \pi_0(C_T) = 10.69 < \pi_0^{\rm s}(C_T) = 14.19 < p_0^{\rm s}(C_T) = 15.33, \end{aligned}$$

where $p_0^{\rm s}(C_T)$ and $p_0^{\rm b}(C_T)$ denote the seller's and buyer's replicating cost, respectively. It is worthwhile to point out that any price belonging to the open interval $(8.97, 15.33)$ would be consistent with the absence of arbitrage in the market model. More precisely, if the option was sold at some price from this interval, neither the seller of the option nor its buyer would be able to make riskless profit. On the other hand, if someone was able to sell the option at the seller's price $\pi_0^{\rm s}(C_T)$ (or buy it at the buyer's price $\pi_0^{\rm b}(C_T)$), the market model would no longer be arbitrage-free. In this sense, the buyer's and seller's prices provide the lower and upper bounds for the values of the option's price consistent with no-arbitrage. Recall that we assume here that conditions (TC.1)–(TC.2) are satisfied. Somewhat surprisingly, assumption (TC.2), which was imposed to make the calculations simpler, is in fact responsible for the non-trivial form of the optimal strategy (we say that a perfect hedging strategy with the minimal cost is *trivial* if it replicates a given claim).

3. Optimality of replicating strategies. In this section, we shall frequently assume that the cost coefficients are constant over time: $\lambda_t = \lambda$ and $\mu_t = \mu$ for every date $t = 0, \ldots, T$. Our main goal is to show that for a large class of contingent claims the optimal perfect hedging strategy coincides with the replicating strategy. For convenience, we restrict our attention to the *path-independent* European claims; that is, claims of the form $X = (g_T, h_T)$, where $g_T = g(S_T)$ and $h_T = h(S_T)$ are functions $g, h : \mathbb{R} \to \mathbb{R}$. Instead of analysing the number of long or short positions in shares, we shall focus on the amount of funds invested in shares and bonds at any date t. We write $x_t = \alpha_t S_t$ and $y_t = \beta_t$ to denote the post-trading amounts of funds which are invested at time t in shares and bonds respectively. In other words, a post-trading portfolio at time t is identified with a vector (x_t, y_t), with both components expressed in units of cash. Similarly, we write $(\widetilde{x}_t, \widetilde{y}_t)$ to denote the pre-trading portfolio at time t. In particular, $(\widetilde{x}_0, \widetilde{y}_0)$ represents the pre-trading portfolio at time 0, that is, the initial endowment in shares and cash. We denote by M_t and L_t the amounts of funds transferred at time t from shares to bonds and from bonds to shares, respectively. For any t, the non-negative random variables M_t and L_t are assumed to be measurable with respect to the σ-field $\mathcal{F}_t = \sigma(S_0, \ldots, S_t)$ generated by the observations of stock price up to time t. We postulate, without loss of generality, that the equality $M_t L_t = 0$ holds for any t. The portfolio's dynamics are subject to the following rules:

$$\widetilde{x}_{t+1} = \xi_{t+1}(\widetilde{x}_t + L_t - M_t), \qquad \widetilde{y}_{t+1} = (1+r)(\widetilde{y}_t + (1-\mu_t)M_t - (1+\lambda_t)L_t)$$

for every $t \leq T-1$, where $\xi_{t+1} = S_{t+1}/S_t$ is a random variable which takes values in the set $\{u, d\}$ (see (2)) and the initial condition $(\widetilde{x}_0, \widetilde{y}_0)$ is represented by an arbitrary vector in $\mathbb{R}^2$. We find it convenient to decouple the portfolio's evolution in the following way:

$$x_t = \widetilde{x}_t + L_t - M_t, \qquad y_t = \widetilde{y}_t + (1-\mu_t)M_t - (1+\lambda_t)L_t$$

for $t = 0, \ldots, T$, and

$$\widetilde{x}_{t+1} = \xi_{t+1}x_t, \qquad \widetilde{y}_{t+1} = (1+r)y_t$$

for $t = 0, \ldots, T-1$. The first pair of equations reflects transactions which occur at time $t = 0, \ldots, T$; the second governs the dynamics of the portfolio over each period $[t, t+1]$ for $t = 0, \ldots, T-1$. Denote by $p_0^{\mathrm{s}}(X|\widetilde{x}_0)$ the seller's replication cost of X, given that a seller is endowed with $\widetilde{x}_0$ units of cash invested in shares (i.e., owns $\widetilde{x}_0/S_0$ shares of the stock) before the trade at time 0 begins. Formally, the *seller's replication cost* $p_0^{\mathrm{s}}(X|\widetilde{x}_0)$ equals (cf. (8))

$$p_0^{\mathrm{s}}(X|\widetilde{x}_0) := \inf\{\widetilde{y}_0 \in \mathbb{R} \mid \exists \phi \in \Phi_{(\widetilde{x}_0,\widetilde{y}_0)} : x_T = S_T g_T \text{ and } y_T = h_T\},$$

where $\Phi_{(\widetilde{x}_0,\widetilde{y}_0)}$ stands for the class of those self-financing trading strategies which start from the initial endowment $(\widetilde{x}_0, \widetilde{y}_0)$. The *buyer's replication cost* is defined by the equality $p_0^{\mathrm{b}}(X|\widetilde{x}_0) = -p_0^{\mathrm{s}}(-X|\widetilde{x}_0)$, or explicitly (cf. (9)) by

$$p_0^{\mathrm{b}}(X|\widetilde{x}_0) := -\inf\{\widetilde{y}_0 \in \mathbb{R} \mid \exists \phi \in \Phi_{(\widetilde{x}_0,\widetilde{y}_0)} : x_T = -S_T g_T \text{ and } y_T = -h_T\}.$$

One can easily check that under (TC.1) we have $p_0^{\mathrm{s}}(X|\widetilde{x}_0) = p_0^{\mathrm{s}}(X|0) - \widetilde{x}_0$, so that, in this case, it is enough to search for the replication cost $p_0^{\mathrm{s}}(X|0)$ which corresponds to zero initial endowment. The *seller's price,* i.e., the minimal cost of perfect hedging, is given by

$$\pi_0^{\mathrm{s}}(X|\widetilde{x}_0) := \inf\{\widetilde{y}_0 \in \mathbb{R} \mid \exists \phi \in \Phi_{(\widetilde{x}_0,\widetilde{y}_0)} : x_T \geq S_T g_T \text{ and } y_T \geq h_T\} \tag{12}$$

and the *buyer's price* equals

$$\pi_0^{\mathrm{b}}(X|\widetilde{x}_0) := -\inf\{\widetilde{y}_0 \in \mathbb{R} \mid \exists \phi \in \Phi_{(\widetilde{x}_0,\widetilde{y}_0)} : x_T \geq -S_T g_T \text{ and } y_T \geq -h_T\}. \tag{13}$$

From the definitions above it is obvious that, for any contingent claim X and any initial endowment $\widetilde{x}_0$, we have $\pi_0^{\mathrm{s}}(X|\widetilde{x}_0) \leq p_0^{\mathrm{s}}(X|\widetilde{x}_0)$ and $p_0^{\mathrm{b}}(X|\widetilde{x}_0) \leq \pi_0^{\mathrm{b}}(X|\widetilde{x}_0)$. A trading strategy which realizes the infimum in (12) (in (13), resp.) is referred to as the *optimal strategy* for a seller (for a buyer, resp.) of X. Our aim is now to show that for some classes of contingent claims, replicating strategies are optimal for both the seller and the buyer (independently of the initial endowment in shares). In other words, we wish to prove that for any $x_0 \in \mathbb{R}$ we have $\pi_0^{\mathrm{s}}(X|\widetilde{x}_0) = p_0^{\mathrm{s}}(X|\widetilde{x}_0)$ and $\pi_0^{\mathrm{b}}(X|\widetilde{x}_0) = p_0^{\mathrm{b}}(X|\widetilde{x}_0)$, provided that specific assumptions are imposed on the claim X and/or on the model's coefficients. For the sake of convenience, we write $s_1 > \ldots > s_{T+1}$ to denote the stock price values at the terminal date. Also, we write $g_i = s_i g(s_i)$ and $h_i = h(s_i)$, and we identify a claim X with a finite subset of points in the plane: (g_i, h_i), $i = 1, \ldots, T+1$. Since the case of general claims is rather cumbersome, we restrict out attention, unless otherwise specified, to those claims which satisfy

$$(g_i/u - g_{i+1}/d)(h_{i+1} - h_i) \geq 0, \quad \forall i = 1, \ldots, T+1. \tag{14}$$

It is easily seen that claims corresponding to the long and short positions in European options settled by delivery satisfy condition (14).

3.1. *Long call and put options.* We are now in a position to formulate conditions which will appear sufficient for the optimality of replication.

ASSUMPTION (TC.3). We say that a European contingent claim $X = (g(S_T), h(S_T))$, which settles at time T, satisfies condition (TC.3) if $h_{i+1} \geq h_i$ for all $i = 1, \ldots, T$, and the following implications are valid:

$$h_{i+1} - h_i = 0 \Rightarrow \frac{g_i}{u} - \frac{g_{i+1}}{d} = 0, \tag{15}$$

$$h_{i+1} - h_i > 0 \Rightarrow \frac{h_{i+1} - h_i}{u(1+\lambda)} < \frac{g_i}{u} - \frac{g_{i+1}}{d} < \frac{h_{i+1} - h_i}{d(1-\mu)}. \tag{16}$$

Let us examine some special, but important, cases of contingent claims which satisfy assumption (TC.3). For instance, for long call and put options we have $h_{i+1} \geq h_i$ for every i; in addition, the implication (15) is easily seen to be satisfied. Finally, in both cases, it is enough to verify the validity of (16) only at the node corresponding to the exercise price K, that is, for i such that $h_{i+1} \geq K > h_i$. Consider first a long call option. Denote by s the unique value of the stock price at time $T-1$ which satisfies $su > K$ and $sd \leq K$. Condition (16) is easily seen to hold, as it takes the following form:

$$\frac{K}{u(1+\lambda)} < \frac{su}{u} - \frac{0}{d} < \frac{K}{d(1-\mu)},$$

or equivalently, $su(1+\lambda) > K$ and $sd(1-\mu) < K$. Similarly, for a long put option, we consider the value of s for which $su \geq K$ and $sd < K$. We need to show that

$$\frac{K}{u(1+\lambda)} < \frac{0}{u} - \frac{-sd}{d} < \frac{K}{d(1-\mu)},$$

but this condition is, of course, identical to the one for the long call option.

3.2. *Direct approach.* By the direct approach, as opposed to the martingale approach, we mean a straightforward analysis of the class of feasible strategies. A trading strategy is called *feasible* if it is self-financing, inclusive of transaction costs, and satisfies the terminal constraints as in (12)–(13). We start this section with simple, but useful, auxiliary results.

For any two points (g_i, h_i) and (g_{i+1}, h_{i+1}) which satisfy (16), we denote by (G_i, H_i) the intersection of the following lines:

$$x = -\frac{(1+r)y - h_i}{u(1+\lambda)} + \frac{g_i}{u}, \qquad x = -\frac{(1+r)y - h_{i+1}}{d(1-\mu)} + \frac{g_{i+1}}{d}.$$

For convenience, we write $\widetilde{u} = 1/u$, $\widetilde{d} = 1/d$, $\widetilde{\mu} = (1+r)/(1-\mu)$, $\widetilde{\lambda} = (1+r)/(1+\lambda)$, $\widetilde{h}_i = h_i/(1+r)$, and $\widetilde{h}_{i+1} = h_{i+1}/(1+r)$. Then the equations above become

$$x = -\widetilde{\lambda}\widetilde{u}(y - \widetilde{h}_i) + \widetilde{u}g_i, \qquad x = -\widetilde{\mu}\widetilde{d}(y - \widetilde{h}_{i+1}) + \widetilde{d}g_{i+1}.$$

It is easily seen (since $\widetilde{d}\widetilde{\mu} - \widetilde{u}\widetilde{\lambda} > 0$) that

$$H_i = \frac{\widetilde{d}\widetilde{\mu}\widetilde{h}_{i+1} - \widetilde{u}\widetilde{\lambda}\widetilde{h}_i + \widetilde{d}g_{i+1} - \widetilde{u}g_i}{\widetilde{d}\widetilde{\mu} - \widetilde{u}\widetilde{\lambda}},$$

or equivalently,

$$H_i = \frac{(1+\lambda)uh_{i+1} - (1-\mu)dh_i + (1+\lambda)(1-\mu)(ug_{i+1} - dg_i)}{(1+r)((1+\lambda)u - (1-\mu)d)}. \tag{17}$$

Furthermore, we have

$$\begin{aligned} G_i &= \frac{\widetilde{u}\widetilde{d}(\widetilde{\mu}\widetilde{\lambda}(\widetilde{h}_i - \widetilde{h}_{i+1}) + \widetilde{\mu}g_i - \widetilde{\lambda}g_{i+1})}{\widetilde{d}\widetilde{\mu} - \widetilde{u}\widetilde{\lambda}} \\ &= \frac{h_i - h_{i+1} + (1+\lambda)g_i - (1-\mu)g_{i+1}}{(1+\lambda)u - (1-\mu)d}. \end{aligned} \tag{18}$$

The next lemma will prove useful in the martingale approach (both formulae easily follow by straightforward calculations).

LEMMA 3.1. *Let*

$$\widehat{p} = \frac{(1+\lambda)(1+r) - (1-\mu)d}{(1+\lambda)u - (1-\mu)d}, \qquad \widetilde{p} = \frac{(1-\mu)(1+r) - (1-\mu)d}{(1+\lambda)u - (1-\mu)d}.$$

Then

$$(1+r)(H_i + (1+\lambda)G_i) = \widehat{p}(h_i + (1+\lambda)g_i) + (1-\widehat{p})(h_{i+1} + (1-\mu)g_{i+1})$$

and

$$(1+r)(H_i + (1-\mu)G_i) = \widetilde{p}(h_i + (1+\lambda)g_i) + (1-\widetilde{p})(h_{i+1} + (1-\mu)g_{i+1}).$$

Since in what follows we shall apply the backward induction, the following elementary lemma will prove useful.

LEMMA 3.2. *Consider the points (g_i, h_i), $i = j-1, j, j+1$, which correspond to the three consecutive values $s_{j-1} > s_j > s_{j+1}$ of the terminal stock price. Suppose that for $i = j-1$ and $i = j$ the pair of points (g_i, h_i) and*

(g_{i+1},h_{i+1}) *satisfies* (15)–(16). *Then for the pair of the corresponding intersection points*, (G_{j-1},H_{j-1}) *and* (G_j,H_j), *conditions* (15)–(16) *are also met.*

Proof. Suppose that the points (g_i,h_i), $i=j-1,j,j+1$, satisfy (TC.3). We wish to show that the intersection points (G_{j-1},H_{j-1}) and (G_j,H_j) also satisfy (TC.3)—that is, implication (15) holds (this is rather trivial) and

$$\frac{H_j-H_{j-1}}{u(1+\lambda)}<\frac{G_{j-1}}{u}-\frac{G_j}{d}<\frac{H_j-H_{j-1}}{d(1-\mu)}$$

provided that $H_j>H_{j-1}$ (it will soon become clear that H_j-H_{j-1} is never negative). For simplicity of notation, we write $j-1=1$ and $j=2$. We need to show that $H_2>H_1$ and

$$\frac{\widetilde{u}\widetilde{\lambda}}{1+r}<\frac{\widetilde{u}G_1-\widetilde{d}G_2}{H_2-H_1}<\frac{\widetilde{d}\widetilde{\mu}}{1+r}.$$

By straightforward calculations we find that

$$\widetilde{u}G_1-\widetilde{d}G_2 = \frac{\widetilde{u}\widetilde{d}(\widetilde{u}\widetilde{\mu}\widetilde{\lambda}(\widetilde{h}_1-\widetilde{h}_2)+\widetilde{\mu}(\widetilde{u}g_1-\widetilde{d}g_2)+\widetilde{d}\widetilde{\mu}\widetilde{\lambda}(\widetilde{h}_3-\widetilde{h}_2)-\widetilde{\lambda}(\widetilde{u}g_2-\widetilde{d}g_3))}{\widetilde{d}\widetilde{\mu}-\widetilde{u}\widetilde{\lambda}}$$

and

$$H_2-H_1=\frac{\widetilde{u}\widetilde{\lambda}(\widetilde{h}_1-\widetilde{h}_2)+\widetilde{u}g_1-\widetilde{d}g_2+\widetilde{d}\widetilde{\mu}(\widetilde{h}_3-\widetilde{h}_2)-\widetilde{u}g_2+\widetilde{d}g_3}{\widetilde{d}\widetilde{\mu}-\widetilde{u}\widetilde{\lambda}}.$$

We define $\gamma=\widetilde{u}\widetilde{\lambda}(\widetilde{h}_1-\widetilde{h}_2)+\widetilde{u}g_1-\widetilde{d}g_2$ and $\delta=\widetilde{d}\widetilde{\mu}(\widetilde{h}_3-\widetilde{h}_2)-\widetilde{u}g_2+\widetilde{d}g_3$. Under (TC.3), $\gamma+\delta$ is a strictly positive number if $h_3>h_1$. Indeed, it follows from (15)–(16) that

$$\widetilde{u}g_1-\widetilde{d}g_2\ge\widetilde{u}\widetilde{\lambda}(\widetilde{h}_2-\widetilde{h}_1),\qquad \widetilde{u}g_2-\widetilde{d}g_3\le\widetilde{d}\widetilde{\mu}(\widetilde{h}_3-\widetilde{h}_2).$$

and the first (the second, resp.) inequality is strict if $h_2>h_1$ (if $h_3>h_2$, resp.). Since manifestly

$$H_2-H_1=\frac{\gamma+\delta}{\widetilde{d}\widetilde{\mu}-\widetilde{u}\widetilde{\lambda}},\qquad \widetilde{u}G_1-\widetilde{d}G_2=\frac{\widetilde{u}\widetilde{d}(\widetilde{\mu}\gamma+\widetilde{\lambda}\delta)}{\widetilde{d}\widetilde{\mu}-\widetilde{u}\widetilde{\lambda}},$$

it is clear that $H_2-H_1>0$, and

$$\frac{\widetilde{u}\widetilde{\lambda}}{1+r}<\widetilde{u}\widetilde{d}\widetilde{\lambda}\le\frac{\widetilde{u}G_1-\widetilde{d}G_2}{H_2-H_1}=\frac{\widetilde{u}\widetilde{d}(\widetilde{\mu}\gamma+\widetilde{\lambda}\delta)}{\gamma+\delta}\le\widetilde{u}\widetilde{d}\widetilde{\mu}<\frac{\widetilde{d}\widetilde{\mu}}{1+r},$$

since $\widetilde{d}>1/(1+r)$ and $\widetilde{u}<1/(1+r)$. If $h_3=h_2=h_1$ then, of course, $\gamma=\delta=0$, so that $H_1=H_2$ and $\widetilde{u}G_1=\widetilde{d}G_2$. ∎

We are now in a position to prove the main result of this section (it is a straightforward generalization of Theorem 3.3 in [2]).

PROPOSITION 3.1. *Let a European contingent claim* $X = (g(S_T), h(S_T))$ *satisfy condition* (TC.3). *Then the optimal hedging strategy for a seller of* X *coincides with the unique self-financing replicating strategy.*

Proof. The proof is based on a direct analysis of the set of feasible portfolios. Consider the class $\mathcal{A}_{s_i}$ of post-trading feasible portfolios (x, y) at time T for a fixed (but otherwise arbitrary) level s_i of the stock price. It is clear that $\mathcal{A}_{s_i} = \{(x, y) \in \mathbb{R}^2 \mid x \geq s_i g(s_i),\ y \geq h(s_i)\}$. The class $\widetilde{\mathcal{A}}_{s_i}$ of pre-trading feasible portfolios at time T equals

$$\begin{aligned}\widetilde{\mathcal{A}}_{s_i} = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2 \mid \\ \exists l, m \geq 0,\ \exists (x, y) \in \mathcal{A}_{s_i} : (\widetilde{x}, \widetilde{y}) = (x + m - l, y + (1 + \lambda)l - (1 - \mu)m)\}.\end{aligned}$$

In other words, the pre-trading feasible set $\widetilde{\mathcal{A}}_{s_i}$ is the image of the mapping $\Theta : \mathcal{A}_{s_i} \times \mathbb{R}^2_+ \to \mathbb{R}^2$ which is given by

$$\Theta(x, y, m, l) = (x+m-l,\ y-(1-\mu)m+(1+\lambda)l), \qquad \forall (x, y, m, l) \in \mathcal{A}_{s_i} \times \mathbb{R}^2_+,$$

that is, $\widetilde{\mathcal{A}}_{s_i} = \Theta(\mathcal{A}_{s_i} \times \mathbb{R}^2_+)$. Now we make the inductive step. For any generic value s of the stock price at time $T - 1$, the class of post-trading feasible portfolios equals

$$\begin{aligned}\mathcal{A}_s = \{(x, y) \in \mathbb{R}^2 \mid (ux, (1 + r)y) \in \widetilde{\mathcal{A}}_{su}\} \\ \cap \{(x, y) \in \mathbb{R}^2 \mid (dx, (1 + r)y) \in \widetilde{\mathcal{A}}_{sd}\}\end{aligned}$$

and, as before, $\widetilde{\mathcal{A}}_s = \Theta(\mathcal{A}_s \times \mathbb{R}^2_+)$. Using Lemma 3.2, it is not difficult to check that the post-trading feasible set and, more importantly, the pre-trading feasible set at any time and at any node of the binomial lattice is a (shifted) convex cone. Moreover, the slopes of the half-lines which determine the pre-trading cone are always $-(1-\mu)$ and $-(1+\lambda)$. Therefore, at any node of the binomial lattice the corresponding pre-trading cone is uniquely determined by its vertex. In order to find the seller's price (and the associated trading strategy) it is thus sufficient to determine at each stage the set of vertices of pre-trading cones. This can be easily done using the backward induction, and taking into account, in particular, formulae (17)–(18). ∎

The following corollary to Proposition 3.1 summarizes the properties of the optimal hedging of European options from the perspective of the option writer.

COROLLARY 3.1. *The seller's price of the long call and put options coincides with the seller's replication cost of the long call and put options. The initial portfolio of the replicating strategy for the long call option* (*for the long put option*, *resp.*) *involves purchase of shares* (*short-selling of shares*, *resp.*). *In both cases any upward movement of the stock price is associ-*

ated with a purchase of additional shares of stock. On the other hand, an additional amount of shares is shorted after every decline of the stock price.

Proof. All assertions easily follow from an analysis of the set of feasible portfolios. ■

3.2.1. *Martingale approach.* We shall now examine the martingale approach to the perfect hedging against short positions in options. Let us start by analysing the one-period case, i.e., $T = 1$. We assume also that $\widetilde{x}_0 = 0$ and we put $c = \widetilde{y}_0$. Generally speaking, we search for the minimal value of c such that

$$\alpha_0 = l_0 - m_0, \qquad \beta_0 = c + (1-\mu_0)m_0S_0 - (1+\lambda_0)l_0S_0,$$

for some real numbers $m_0 \geq 0$, $l_0 \geq 0$ such that $l_0m_0 = 0$, and

$$\alpha_1 = \alpha_0 + l_1 - m_1 = g(S_1),$$
$$\beta_1 = \beta_0(1+r) + (1-\mu_1)m_1S_1 - (1+\lambda_1)l_1S_1 \geq h(S_1),$$

where l_1 and m_1 are non-negative random variables such that $l_1m_1 = 0$. Consider first a call option. Since the replicating strategy of the long call involves purchasing shares at time 0, we have

$$\alpha_0 = l_0, \qquad \beta_0 = c - (1+\lambda_0)l_0S_0,$$

for some real number $l_0 \geq 0$. At time 1, in the up-state we have

$$\alpha_1^u = \alpha_0 + l_1^u = g(S_1^u), \qquad \beta_1^u = \beta_0(1+r) - (1+\lambda_1)l_1^uS_1^u \geq h(S_1^u),$$

for some $l_1^u \geq 0$, and in the down-state

$$\alpha_1^d = \alpha_0 - m_1^d = g(S_1^d), \qquad \beta_1^d = \beta_0(1+r) + (1-\mu_1)m_1^dS_1^d \geq h(S_1^d),$$

for some $m_1^d \geq 0$. As already mentioned, we wish to minimize the initial cost c subject to the above set of constraints. This optimization problem can be easily solved, yielding the following explicit formula for the seller's price of the call option (we use the general notation, in order to emphasise that the result holds for any claim X which satisfies (TC.3), and such that the replicating strategy involves purchasing shares at time 0 and at time 1 in the up-state, and selling shares at time 1 in the down-state)

$$\begin{aligned}(19)\qquad &\pi_0^{\mathrm{s}}(X|0)\\ &= \frac{\widehat{p}_{\mathrm{s}}(h(S_1^u) + (1+\lambda_1)S_1^ug(S_1^u)) + (1-\widehat{p}_{\mathrm{s}})(h(S_1^d) + (1-\mu_1)S_1^dg(S_1^d))}{1+r},\end{aligned}$$

where

$$\widehat{p}_{\mathrm{s}} := \frac{(1+\lambda_0)(1+r)S_0 - (1-\mu_1)S_1^d}{(1+\lambda_1)S_1^u - (1-\mu_1)S_1^d} = \frac{(1+\lambda_0)(1+r) - (1-\mu_1)d}{(1+\lambda_1)u - (1-\mu_1)d}.$$

It is easily seen that $\widehat{p}_{\mathrm{s}} \in (0,1)$ and the pair $\{\widehat{p}_{\mathrm{s}}, 1-\widehat{p}_{\mathrm{s}}\}$ defines the martingale measure $\widehat{\mathbf{P}}_{\mathrm{s}}$ for the process $\widehat{S}^{\mathrm{s}}$, which equals $\widehat{S}_0^{\mathrm{s}} = (1+\lambda_0)S_0$ at time 0, and satisfies

$$\widehat{S}^{\mathrm{s}}_1 = \begin{cases} (1+\lambda_1)(1+r)^{-1}S^u_1 & \text{in the up-state,} \\ (1-\mu_1)(1+r)^{-1}S^d_1 & \text{in the down-state,} \end{cases}$$

at time 1. In particular, the seller's price of a call option admits the following representation (we assume that $dS_0 \le K < uS_0$):

$$\pi^{\mathrm{s}}_0(C_T|0) = (1+r)^{-1}\widehat{p}_{\mathrm{s}}((1+\lambda_1)uS_0 - K) = \mathbf{E}_{\widehat{\mathbf{P}}_{\mathrm{s}}}((1+r)^{-1}(\widehat{S}^{\mathrm{s}}_T - K)^+).$$

The last equality is, of course, a straightforward generalization of the standard risk-neutral valuation formula.

Now consider the seller's price of a put option. Since the replicating strategy of the long put involves shorting shares at time 0 (recall that we consider an individual who does not own shares at time 0 before trading) we need to modify the first pair of equations. We now have

$$\alpha_0 = -m_0, \qquad \beta_0 = c + (1-\mu_0)m_0S_0,$$

for some $m_0 \ge 0$. On the other hand, the equations associated with the portfolio's revision at time 1 are the same as in the case of a call option. The seller's price of a put option is easily seen to be

$$\pi^{\mathrm{s}}_0(X|0) = \frac{\widetilde{p}_{\mathrm{s}}(h(S^u_1) + (1+\lambda_1)S^u_1 g(S^u_1)) + (1-\widetilde{p}_{\mathrm{s}})(h(S^d_1) + (1-\mu_1)S^d_1 g(S^d_1))}{1+r},$$

where

$$\widetilde{p}_{\mathrm{s}} := \frac{(1-\mu_0)(1+r)S_0 - (1-\mu)S^d_1}{(1+\lambda_1)S^u_1 - (1-\mu_1)S^d_1} = \frac{(1-\mu_0)(1+r) - (1-\mu)d}{(1+\lambda_1)u - (1-\mu_1)d}.$$

Note that the pair $\{\widetilde{p}_{\mathrm{s}}, 1-\widetilde{p}_{\mathrm{s}}\}$ is the unique martingale measure $\widetilde{\mathbf{P}}_{\mathrm{s}}$ for the process $\widetilde{S}^{\mathrm{s}}$, which equals $\widetilde{S}^{\mathrm{s}}_0 = (1-\mu_0)S_0$ at time 0, and

$$\widetilde{S}^{\mathrm{s}}_1 = \begin{cases} (1+\lambda_1)(1+r)^{-1}S^u_1 & \text{in the up-state,} \\ (1-\mu_1)(1+r)^{-1}S^d_1 & \text{in the down-state,} \end{cases}$$

at time 1. In particular, the seller's price of a put option satisfies (we assume that $uS_0 \ge K$)

$$\pi^{\mathrm{s}}_0(P_T|0) = (1+r)^{-1}(1-\widetilde{p}_{\mathrm{s}})(K - (1-\mu_1)dS_0) = \mathbf{E}_{\widetilde{\mathbf{P}}_{\mathrm{s}}}((1+r)^{-1}(K - \widetilde{S}_T)^+).$$

Now consider a multi-period case. We shall focus on the case of a call option. Assume that the cost coefficients are constant over time. In this case, we may proceed by working backwards in time from the terminal date T. First, we associate with any generic terminal value s_i of the stock price two values, $f_1(s_i)$ and $f_2(s_i)$ say, where

$$f_1(s_i) = f^u(g_i, h_i) := h_i + (1+\lambda)g_i, \qquad f_2(s_i) = f^d(g_i, h_i) := h_i + (1-\mu)g_i.$$

Then, at any node of the lattice corresponding to the date $T-1$ we introduce the two values

$$f_1(S_{T-1}) = (1+r)^{-1}(\widehat{p}_{\mathrm{s}} f_1(uS_{T-1}) + (1-\widehat{p}_{\mathrm{s}}) f_2(dS_{T-1})),$$
$$f_2(S_{T-1}) = (1+r)^{-1}(\widetilde{p}_{\mathrm{s}} f_1(uS_{T-1}) + (1-\widetilde{p}_{\mathrm{s}}) f_2(dS_{T-1})).$$

From Lemma 3.1, it is clear that

$$f_1(S_{T-1}) = H_i + (1+\lambda)G_i = f^u(G_i, H_i),$$
$$f_2(S_{T-1}) = H_i + (1-\mu)G_i = f^d(G_i, H_i),$$

where, by convention, $uS_{T-1} = s_i$ (so that $dS_{T-1} = s_{i+1}$). From Proposition 3.1, we know already that the point (G_i, H_i) represents the option's replicating portfolio at time $T-1$, for the value S_{T-1} of the stock price. This shows that we may consider $T-1$ as the terminal date in what follows. By repeating this procedure $T-2$ times, we find the values $f^u(uS_0)$ and $f^d(dS_0)$. In view of (19), we have

$$\pi_0^{\mathrm{s}}(X|0) = (1+r)^{-1}(\widehat{p}_{\mathrm{s}}(H_1^u + (1+\lambda)G_1^u) + (1-\widehat{p}_{\mathrm{s}})(H_2^d + (1-\mu)G_2^d)),$$

where (G_1^u, H_1^u) and (G_1^d, H_1^d) represent the option's replicating portfolio at time 1, in the up-state and in the down-state, respectively. Put another way, the seller's price of the option equals

$$\pi_0^{\mathrm{s}}(X|0) = (1+r)^{-1}(\widehat{p}_{\mathrm{s}} f^u(G_1^u, H_1^u) + (1-\widehat{p}_{\mathrm{s}}) f^d(G_2^d, H_2^d)).$$

The considerations above may be used to construct a simple recursive procedure for finding the seller's price of the option and the option's replicating strategy.

3.3. *Short call and put option.* We shall now examine the optimality of the replicating strategy from the perspective of the option buyer—that is, for the short call and put options. Note first that we now have $h_{i+1} \le h_i$ for all i. We assume throughout that $d(1+\lambda) < u(1-\mu)$, that is, $\widetilde{u}\widetilde{\mu} < \widetilde{d}\widetilde{\lambda}$. Actually, we shall need an even stronger assumption, namely

$$\widetilde{u}\widetilde{\mu} = \frac{1+r}{u(1-\mu)} < \frac{1}{1+\lambda} = \frac{\widetilde{\lambda}}{1+r},$$
$$\frac{\widetilde{\mu}}{1+r} = \frac{1}{(1-\mu)} < \frac{1+r}{d(1+\lambda)} = \widetilde{d}\widetilde{\lambda}, \tag{20}$$

that is,

$$\frac{1+\lambda}{1-\mu} < \min\left(\frac{u}{1+r}, \frac{1+r}{d}\right). \tag{21}$$

It is obvious that, for fixed u and d, inequality (21) is satisfied provided that the cost coefficients λ and μ are small enough. We will check later that if condition (21) holds, the set of pre-trading feasible portfolios is, at every node of the lattice, a convex cone with the same properties as in the case of long options. This will imply (see Proposition 3.3) that condition (21) is sufficient for the optimality of replicating strategy for any contingent claim. However, since the replicating strategy of a short option has some interesting features, we shall first consider this specific class of contingent claims. The following condition is a counterpart of assumption (TC.3).

ASSUMPTION (TC.4). We say that a claim X satisfies (TC.4) if $h_{i+1} \le h_i$ for all $i = 1, \dots, T$, condition (15) holds, and the following implication is valid:

$$h_i - h_{i+1} > 0 \Rightarrow \frac{h_i - h_{i+1}}{u(1-\mu)} < \frac{g_{i+1}}{d} - \frac{g_i}{u} < \frac{h_i - h_{i+1}}{d(1+\lambda)}. \tag{22}$$

In order to show that the short call option settled by delivery satisfies (22) we need to verify that if $su > K$ and $sd \le K$ then

$$\frac{K}{u(1-\mu)} < \frac{0}{d} - \frac{-su}{u} < \frac{K}{d(1+\lambda)},$$

or equivalently, that $su(1-\mu) > K$ and $sd(1+\lambda) < K$. It is thus clear that condition (TC.4) is satisfied by a short call option provided that the cost coefficients are small enough. For the short put option, we need to check that if a generic stock price s satisfies $su \ge K$ and $sd < K$, then

$$\frac{K}{u(1-\mu)} < \frac{sd}{d} - \frac{0}{u} < \frac{K}{d(1+\lambda)}.$$

We obtain the same inequalities as in the case of a short call.

3.4. *Direct approach.* Once again we start by a straightforward analysis of the class of feasible trading strategies. Since we shall proceed along similar lines to the case of long options, we do not go into details. For any two points (g_i, h_i) and (g_{i+1}, h_{i+1}) which satisfy (22), we write $(\widehat{G}_i, \widehat{H}_i)$ to denote the intersection of the following lines:

$$x = -\frac{(1+r)y - h_i}{u(1-\mu)} + \frac{g_i}{u}, \qquad x = -\frac{(1+r)y - h_{i+1}}{d(1+\lambda)} + \frac{g_{i+1}}{d}.$$

Using the same notation as before, we find that

$$\widehat{H}_i = \frac{\widetilde{d}\widetilde{\lambda}\widetilde{h}_{i+1} - \widetilde{u}\widetilde{\mu}\widetilde{h}_i + \widetilde{d}g_{i+1} - \widetilde{u}g_i}{\widetilde{d}\widetilde{\lambda} - \widetilde{u}\widetilde{\mu}},$$

$$\widehat{G}_i = \frac{\widetilde{u}\widetilde{d}(\widetilde{\mu}\widetilde{\lambda}(\widetilde{h}_i - \widetilde{h}_{i+1}) + \widetilde{\lambda}g_i - \widetilde{\mu}g_{i+1})}{\widetilde{d}\widetilde{\lambda} - \widetilde{u}\widetilde{\mu}}$$

and thus

$$\widehat{H}_1 - \widehat{H}_2 = \frac{\widetilde{\gamma} + \widetilde{\delta}}{\widetilde{d}\widetilde{\lambda} - \widetilde{u}\widetilde{\mu}}, \qquad \widetilde{d}\widehat{G}_2 - \widetilde{u}\widehat{G}_1 = \frac{\widetilde{u}\widetilde{d}\,(\widetilde{\lambda}\widetilde{\gamma} + \widetilde{\mu}\widetilde{\delta})}{\widetilde{d}\widetilde{\lambda} - \widetilde{u}\widetilde{\mu}},$$

where $\widetilde{\gamma} = \widetilde{u}\widetilde{\mu}(\widetilde{h}_2 - \widetilde{h}_1) - \widetilde{u}g_1 + \widetilde{d}\,g_2$ and $\widetilde{\delta} = \widetilde{d}\,\widetilde{\lambda}(\widetilde{h}_2 - \widetilde{h}_3) + \widetilde{u}g_2 - \widetilde{d}\,g_3$ are non-negative numbers ($\widetilde{\gamma} + \widetilde{\delta}$ is strictly positive if $h_1 > h_3$). It remains to check that (cf. (22))

$$\frac{\widetilde{u}\widetilde{\mu}}{1+r} < \frac{\widetilde{d}\,\widehat{G}_2 - \widetilde{u}\widehat{G}_1}{\widehat{H}_1 - \widehat{H}_2} < \frac{\widetilde{d}\,\widetilde{\lambda}}{1+r}$$

if $\widehat{H}_1 - \widehat{H}_2 > 0$. Indeed, using (21), we obtain

$$\frac{\widetilde{d}\,\widehat{G}_2 - \widetilde{u}\widehat{G}_1}{\widehat{H}_1 - \widehat{H}_2} \le \widetilde{u}\widetilde{d}\,\widetilde{\mu} < \frac{\widetilde{d}\,\widetilde{\lambda}}{1+r}, \qquad \frac{\widetilde{d}\,\widehat{G}_2 - \widetilde{u}\widehat{G}_1}{\widehat{H}_1 - \widehat{H}_2} \ge \widetilde{u}\widetilde{d}\,\widetilde{\lambda} > \frac{\widetilde{u}\widetilde{\mu}}{1+r}.$$

We are in a position to formulate the following result, which is a counterpart of Lemma 3.2.

LEMMA 3.3. *Consider the points* (g_i, h_i), $i = j-1, j, j+1$, *which correspond to the three consecutive levels* $s_{j-1} > s_j > s_{j+1}$ *of the stock price at time* T. *Suppose that*, *for* $i = j-1, j$, *the pair of points* (g_i, h_i) *and* (g_{i+1}, h_{i+1}) *satisfies* (22). *Then the intersection points* $(\widehat{G}_{j-1}, \widehat{H}_{j-1})$ *and* $(\widehat{G}_j, \widehat{H}_j)$ *also satisfy* (22).

The proof of the next result relies on arguments similar to those used in the proof of Proposition 3.1. Therefore we leave the details to the reader.

PROPOSITION 3.2. *Suppose that condition* (21) *is satisfied. If a claim* $X = (g(S_T), h(S_T))$ *satisfies condition* (TC.4) *then the optimal hedging strategy for the seller of* X *coincides with the unique trading strategy that replicates* X.

The following corollary provides sufficient conditions for the optimality of replication for short call and put options, that is, from the perspective of an option's buyer.

COROLLARY 3.2. *Suppose that for the unique generic value* s *of the stock price at time* $T-1$ *which satisfies* $su > K > sd$, *we have* $su(1-\mu) > K > sd(1+\lambda)$. *If*, *in addition*, *condition* (21) *is satisfied*, *then the optimal hedging strategy for the buyer of a call and a put option coincides with the option's replicating strategy. The initial portfolio of the replicating strategy for the short call* (*put*, *resp.*) *option involves shorting* (*purchasing*, *resp.*) *shares. In both cases a downward* (*upward*, *resp.*) *movement of the stock price corresponds to a purchase* (*a sale*, *resp.*) *of additional shares of stock.*

REMARK. If the inequality $d(1+\lambda) < u(1-\mu)$ fails to hold, the optimal hedging strategy for a buyer does not necessarily coincide with the replicating strategy.

3.4.1. *Martingale approach.* We shall now focus on the martingale approach to the perfect hedging against long positions in European options, that is, from the buyer's perspective. As before, we take $T = 1$ and we assume that $\widetilde{x}_0 = 0$. First consider the buyer's price of a call option. The replicating strategy of the short call option involves selling of shares at time 0 so that

$$\alpha_0 = -m_0, \qquad \beta_0 = c + (1-\mu_0)m_0 S_0,$$

where $c = \widetilde{y}_0$ and m_0 is a non-negative real number. If the stock price rises, at time 1 we need to sell more shares so that we get the following relationships:

$$\alpha_1^u = \alpha_0 - m_1^u = g(S_1^u), \qquad \beta_1^u = \beta_0(1+r) + (1-\mu_1)m_1^u S_1^u \geq h(S_1^u),$$

for some $m_1^u \geq 0$. On the other hand, if the stock price falls, we buy shares and thus

$$\alpha_1^d = \alpha_0 + l_1^d = g(S_1^d), \qquad \beta_1^d = \beta_0(1+r) - (1+\lambda_1)l_1^d S_1^d \geq h(S_1^d),$$

for some $l_1^d \geq 0$ in the down-state. For any claim X whose replicating strategy has such features, we get

$$\pi_0^{\mathrm{b}}(X|0) = \frac{\widehat{p}_{\mathrm{b}}(h(S_1^u) + (1-\mu_1)S_1^u g(S_1^u)) + (1-\widehat{p}_{\mathrm{b}})(h(S_1^d) + (1+\lambda_1)S_1^d g(S_1^d))}{1+r},$$

where

$$\widehat{p}_{\mathrm{b}} := \frac{(1-\mu_0)(1+r)S_0 - (1+\lambda_1)S_1^d}{(1-\mu_1)S_1^u - (1+\lambda_1)S_1^d} = \frac{(1-\mu_0)(1+r) - (1+\lambda_1)d}{(1-\mu_1)u - (1+\lambda_1)d}.$$

The pair $\{\widehat{p}_{\mathrm{b}}, 1-\widehat{p}_{\mathrm{b}}\}$ corresponds to the martingale measure $\widehat{\mathbf{P}}_{\mathrm{b}}$ for the process $\widehat{S}^{\mathrm{b}}$ which equals $\widehat{S}_0^{\mathrm{b}} = (1-\mu_0)S_0$ at time 0 and satisfies

$$\widehat{S}_1^{\mathrm{b}} = \begin{cases} (1-\mu_1)(1+r)^{-1}S_1^u & \text{in the up-state,} \\ (1+\lambda_1)(1+r)^{-1}S_1^d & \text{in the down-state,} \end{cases}$$

at time 1. In particular, the buyer's price of a call option satisfies

$$\pi_0^{\mathrm{b}}(C_T|0) = (1+r)^{-1}\widehat{p}_{\mathrm{b}}((1-\mu_1)uS_0 - K) = \mathbf{E}_{\widehat{\mathbf{P}}_{\mathrm{b}}}((1+r)^{-1}(\widehat{S}_T^{\mathrm{b}} - K)^+).$$

It remains to examine the buyer's price of a put option. We have

$$\alpha_0 = l_0, \qquad \beta_0 = c - (1+\lambda_0)l_0 S_0,$$

for some $l_0 \geq 0$, and the equations associated with the portfolio's revision at time 1 remain the same as above. Hence, the buyer's price of any claim X, which has similar features as a put option, is

$$\pi_0^{\mathrm{b}}(X|0) = \frac{\widetilde{p}_{\mathrm{b}}(h(S_1^u) + (1-\mu_1)S_1^u g(S_1^u)) + (1-\widetilde{p}_{\mathrm{b}})(h(S_1^d) + (1+\lambda_1)S_1^d g(S_1^d))}{1+r},$$

where

$$\widetilde{p}_{\mathrm{b}} := \frac{(1+\lambda_0)(1+r)S_0 - (1+\lambda_1)S_1^d}{(1-\mu_1)S_1^u - (1+\lambda_1)S_1^d} = \frac{(1+\lambda_0)(1+r) - (1+\lambda_1)d}{(1-\mu_1)u - (1+\lambda_1)d}.$$

The pair $\{\widetilde{p}_{\mathrm{b}}, 1-\widetilde{p}_{\mathrm{b}}\}$ represents the martingale measure, $\widetilde{\mathbf{P}}_{\mathrm{b}}$ say, for the process $\widetilde{S}$ which equals $\widetilde{S}_0^{\mathrm{b}} = (1+\lambda_0)S_0$ and

$$\widetilde{S}_1^{\mathrm{b}} = \begin{cases} (1-\mu_1)(1+r)^{-1}S_1^u & \text{in the up-state,} \\ (1+\lambda_1)(1+r)^{-1}S_1^d & \text{in the down-state.} \end{cases}$$

Finally, we have

$$\pi_0^{\mathrm{b}}(P_T|0) = (1+r)^{-1}(1-\widetilde{p}_{\mathrm{b}})(K-(1+\lambda_1)dS_0) = \mathbf{E}_{\widetilde{\mathbf{P}}_{\mathrm{b}}}((1+r)^{-1}(K-\widetilde{S}_T^{\mathrm{b}})^+).$$

This completes the study of short positions in the one-period case. The multi-period case is left to the reader.

3.5. *Case of small costs.* Let us return to the small costs assumption. Our purpose is to show that the exact replication is an optimal way of hedging for any contingent claim (long or short), provided that transaction costs are small enough. For the sake of generality, we no longer assume that the cost coefficients are constant over time. In the case of time-dependent cost coefficients, condition (20) is modified as follows.

Assumption (TC.5). The cost coefficients satisfy

$$\text{(23)} \qquad \frac{u(1-\mu_t)}{1+r} > 1+\lambda_{t-1}, \quad \frac{d(1+\lambda_t)}{1+r} < 1-\mu_{t-1}, \qquad \forall t = 1,\ldots,T.$$

We define

$$A_d = \frac{d(1-\mu_t)}{1+r}, \quad B_d = \frac{d(1+\lambda_t)}{1+r}, \quad A_u = \frac{u(1-\mu_t)}{1+r}, \quad B_u = \frac{u(1+\lambda_t)}{1+r},$$

so that obviously $A_d < B_d$, $A_u < B_u$, $A_d < A_u$ and $B_d < B_u$. By combining these inequalities with (23), we obtain $0 < A_d < B_d < 1-\mu_{t-1}$ and $B_u > A_u > 1+\lambda_{t-1}$. Consequently, the absolute value of the slope of the piecewise linear boundary of the convex set of feasible post-trading portfolios at time t may, in principle, take the following values: (A_d, B_u), (A_d, A_u, B_u), (A_d, B_d, B_u), (A_d, B_d, A_u, B_u) and (A_d, A_u, B_d, B_u) (in ascending order). Observe that the last case assumes implicitly that $A_u \le B_d$. However, using (23) we get

$$A_u = \frac{u(1-\mu_t)}{1+r} > 1+\lambda_{t-1} \ge 1, \qquad B_d = \frac{d(1+\lambda_t)}{1+r} < 1-\mu_{t-1} \le 1;$$

it is thus clear that we cannot have $A_u \le B_d$. This shows that we need to analyse only the remaining four cases. It is easily seen that the pre-trading portfolio is, at any time and at any node of the binomial lattice, a convex cone with the slopes $-(1-\mu_t)$ and $-(1+\lambda_t)$. We conclude that replication is an optimal hedging strategy for any contingent claim. We are in a position to formulate the last result of this paper (let us mention that Proposition 3.3 generalizes Theorem 3.2 in [2]).

PROPOSITION 3.3. *Under* (TC.5), *for any* ([5]) *European contingent claim* X *which settles at time* T *the replicating strategy is the optimal perfect hedging strategy for both the seller and the buyer of* X. *In particular*, $\pi_0^{\mathrm{s}}(X) = p_0^{\mathrm{s}}(X)$ *and* $\pi_0^{\mathrm{b}}(X) = p_0^{\mathrm{b}}(X)$.

REMARK. We could have considered also the case of time-dependent coefficients u, d and r. Assume, for instance, that (cf. (2)) $\xi_t = S_t/S_{t-1} \in \{u_t, d_t\}$ for every $t = 1, \ldots, T$, where u_t and d_t satisfy $0 < d_t < 1 + r_t < u_t$. Assumption (TC.5) would have the following form:

$$\frac{u_t(1-\mu_t)}{1+r_t} > 1 + \lambda_{t-1}, \quad \frac{d_t(1+\lambda_t)}{1+r_t} < 1 - \mu_{t-1}, \quad \forall t = 1, \ldots, T. \tag{24}$$

Under assumption (24), the optimality of exact replication of a contingent claim is still valid.

References

[1] M. Avallaneda and A. Parás, *Dynamic hedging portfolios for derivative securities in the presence of large transaction costs*, Appl. Math. Finance 1 (1994), 165–194.

[2] B. Bensaid, J.-P. Lesne, H. Pagès and J. Scheinkman, *Derivative asset pricing with transaction costs*, Math. Finance 2 (1992), 63–68.

[3] P. Boyle and T. Vorst, *Option replication in discrete time with transaction costs*, J. Finance 47 (1992), 271–293.

[4] M. Broadie, J. Cvitanić and M. Soner, *Optimal replication of contingent claims under portfolio constraints*, preprint, Columbia Univ., 1996.

[5] J. C. Cox, S. A. Ross and M. Rubinstein, *Option pricing*: *A simplified approach*, J. Financial Economics 7 (1979), 229–263.

[6] J. Cvitanić and I. Karatzas, *Hedging and portfolio optimization under transaction costs*: *a martingale approach*, Math. Finance 6 (1996), 133–165.

[7] M. H. A. Davis, *Local time on the stock exchange*, in: *Stochastic Calculus in Application*, J. R. Norris (ed.), Pitman Res. Notes Math. Ser. 197, Longman, London, 1988, 4–28.

[8] M. H. A. Davis and J. M. C. Clark, *A note on super-replicating strategies*, Philos. Trans. Roy. Soc. London Ser. A 347 (1994), 485–494.

[9] M. H. A. Davis and A. R. Norman, *Portfolio selection with transaction costs*, Math. Oper. Res. 15 (1990), 676–713.

[10] M. H. A. Davis and V. P. Panas, *The writing price of a European contingent claim under proportional transaction costs*, Comput. Appl. Math. 13 (1994), 115–157.

([5]) Indeed, this result holds not only for path-independent claims of the form $X = (g(S_T), h(S_T))$, but for any claim $X = (g_T, h_T)$, where g_T and h_T are random variables measurable with respect to the σ-field $\mathcal{F}_T = \sigma(S_0, \ldots, S_T)$.

[11] M. H. A. Davis, V. P. Panas and T. Zariphopoulou, *European option pricing with transaction costs*, SIAM J. Control Optim. 31 (1993), 470–493.

[12] J. C. Dermody and R. T. Rockafellar, *Cash stream valuation in the face of transaction costs and taxes*, Math. Finance 1 (1991), 31–54.

[13] C. Edirisinghe, V. Naik and R. Uppal, *Optimal replication of options with transaction costs and trading restrictions*, J. Finance Quant. Anal. 28 (1993), 117–138.

[14] S. Figlewski, *Options arbitrage in imperfect markets*, J. Finance 44 (1989), 1289–1311.

[15] J. E. Gilster and W. Lee, *The effects of transaction costs and different borrowing and lending rates on the option pricing model: a note*, ibid. 39 (1984), 1215–1221.

[16] J. P. Gould and D. Galai, *Transaction costs and the relationship between put and call prices*, J. Financial Economics 1 (1974), 105–130.

[17] E. R. Grannan and G. H. Swindle, *Minimizing transaction costs of option hedging strategies*, Math. Finance 6 (1996), 341–364.

[18] S. Hodges and A. Neuberger, *Optimal replication of contingent claims under transaction costs*, Rev. Futures Markets 8 (1989), 222–239.

[19] E. Jouini and H. Kallal, *Martingales and arbitrage in securities markets with transaction costs*, J. Econom. Theory 66 (1995), 178–197.

[20] Yu. M. Kabanov and M. Safarian, *On Leland's strategy of option pricing with transaction costs*, Finance Stochast. 1 (1997), 239–250.

[21] S. Kusuoka, *Limit theorem on option replication cost with transaction costs*, Ann. Appl. Probab. 5 (1995), 198–221.

[22] H. E. Leland, *Option pricing and replication with transaction costs*, J. Finance 40 (1985), 1283–1301.

[23] M. J. P. Magill and G. M. Constantinides, *Portfolio selection with transactions costs*, J. Econom. Theory 13 (1976), 245–263.

[24] F. Mercurio and T. Vorst, *Option pricing and hedging in discrete time with transaction costs and incomplete markets*, in: *Mathematics of Derivative Securities*, M. Dempster and S. Pliska (eds.), Cambridge Univ. Press, Cambridge, 1996, to appear.

[25] Q. Shen, *Bid-ask prices for call options with transaction costs*, preprint, Univ. of Pennsylvania, 1990.

[26] H. Shirakawa and H. Konno, *Pricing of options under the proportional transaction costs*, preprint, Tokyo Inst. of Technology, 1995.

[27] S. E. Shreve and H. M. Soner, *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab. 4 (1994), 609–692.

[28] S. E. Shreve, H. M. Soner and G.-L. Xu, *Optimal investment and consumption with two bonds and transaction costs*, Math. Finance 1 (1991), 53–84.

[29] H. M. Soner, S. E. Shreve and J. Cvitanić, *There is no nontrivial hedging portfolio for option pricing with transaction costs*, Ann. Appl. Probab. 5 (1995), 327–355.

[30] Ł. Stettner, *Option pricing in the CRR model with proportional transaction costs: a cone transformation approach*, Appl. Math. (Warsaw) 24 (1997), 475–514.

[31] S. Swidler and J. D. Diltz, *Implied volatilities and transaction costs*, J. Finance Quant. Anal. 27 (1992), 437–447.

[32] M. Taksar, M. J. Klass and D. Assaf, *A diffusion model for optimal portfolio selection in the presence of brokerage fees*, Math. Oper. Res. 13 (1988), 277–294.

[33] A. Tourin and T. Zariphopoulou, *Portfolio selection with transaction costs*, in: *Seminar on Stochastic Analysis*, *Random Fields and Applications*, Progr. Probab. 36, E. Bolthausen *et al.* (eds.), Birkhäuser, 1995, 385–391.
[34] E. Whalley and P. Wilmott, *A hedging strategy and option valuation model with transaction costs*, preprint, Oxford Univ., 1992.
[35] —, —, *Derivative pricing and transaction costs*, preprint, Oxford Univ., 1992.

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