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ON MINIMAX SEQUENTIAL PROCEDURES FOR EXPONENTIAL FAMILIES OF STOCHASTIC PROCESSES

Abstract. The problem of finding minimax sequential estimation procedures for stochastic processes is considered. It is assumed that in addition to the loss associated with the error of estimation a cost of observing the process is incurred. A class of minimax sequential procedures is derived explicitly for a one-parameter exponential family of stochastic processes. The minimax sequential procedures are presented in some special models, in particular, for estimating a parameter of exponential families of diffusions, for estimating the mean or drift coefficients of the class of Ornstein–Uhlenbeck processes, for estimating the drift of a geometric Brownian motion and for estimating a parameter of a family of counting processes. A class of minimax sequential rules for a compound Poisson process with multinomial jumps is also found.

1. Introduction. The paper is devoted to the problem of determining minimax stopping rules and corresponding estimators in estimating unknown parameters of stochastic processes. The study of sequential estimation for continuous time stochastic processes was initiated by Dvoretzky, Kiefer and Wolfowitz (1953). They proved that for the Poisson, negative-binomial, gamma and Wiener processes the minimax sequential procedure for estimating the mean value parameter reduces to a fixed-time procedure if a weighted quadratic loss function is used. Their results were generalized by Magiera (1977) to a one-dimensional parameter exponential family of processes with stationary independent increments, and next by Trybuła (1985) and Franz (1985), using different loss functions, to the multidimensional exponential family. Trybuła (1985) also showed that for the Poisson process an inverse procedure is minimax. Wilczyński (1985) obtained an

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analogous result for the multinomial process. A minimax procedure for estimating the mean value of the Ornstein–Uhlenbeck process was found by Różański (1982). For a class of continuous Gaussian processes with a special form of covariance matrix minimax sequential procedures for estimating the vector parameter of the mean value function were obtained by Rhiel (1985). The optimal procedures he derived are fixed-time ones. The minimaxity of a sequential procedure for estimating the one-dimensional coefficient of the drift matrix of hypoelliptic homogeneous Gaussian diffusions was shown by Le Breton and Musiela (1985). Considering an exponential family of processes which includes the binomial process and some models for counting processes, it was proved by Magiera (1990) that an inverse procedure is minimax under a suitably weighted quadratic loss function.

In this paper a class of minimax sequential procedures is derived explicitly for a one-parameter exponential model for stochastic processes. The minimax procedures obtained are not in general fixed-time ones, in contrast to most of those in the special models mentioned above. The model considered covers several important classes of stochastic processes, such as an exponential family of processes with stationary independent increments, and many counting, diffusion-type etc. processes. Some special models, frequently occurring in theory and practice, are considered in Sections 6 and 7. Most of them have not been treated in the literature in the problem of finding minimax sequential procedures. The minimax sequential procedures are presented in estimating a parameter of a family of exponential-type diffusions, in estimating a drift parameter of hypoelliptic homogeneous Gaussian diffusions, in estimating the mean or drift coefficients of the class of Ornstein–Uhlenbeck processes, in estimating the drift of a geometric Brownian motion and in estimating a parameter of a family of counting processes.

In Section 8 a special model of a multiparameter exponential family of processes is considered. A class of sequential minimax rules is presented for a compound Poisson process with multinomial jumps.

2. Preliminaries. Let $X(t)$, $t \in T$, be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P_\vartheta)$ with values in $(\mathbb{R}^k, \mathcal{B}_{\mathbb{R}^k})$, where $T = [0, \infty)$ or $T = \{0, 1, 2, \ldots\}$ and ϑ is a parameter with values in an open set $\Theta \subseteq \mathbb{R}^n$. Let $P_{\vartheta,t}$ denote the restriction of P_ϑ to $\mathcal{F}_t = \sigma\{X(s) : s \le t\}$. Suppose that the family $P_{\vartheta,t}$, $\vartheta \in \Theta$, is dominated by a measure μ_t which is the restriction of a probability measure μ to $\mathcal{F}_t$ and that the density functions (likelihood functions) have the form

$$\frac{dP_{\vartheta,t}}{d\mu_t} = L(Y(t), t, \vartheta),$$

where $Y(t)$, $t \in T$, is a process with values in $(\mathbb{R}^l, \mathcal{B}_{\mathbb{R}^l})$ and adapted to the filtration $\mathcal{F}_t$, $t \in T$, and $L(\cdot, \cdot, \vartheta)$ is a continuous function. Clearly, $Y(t)$ is a sufficient statistic for ϑ relative to $\mathcal{F}_t$, $t \in T$. In the case of continuous time it is supposed that the process $Y(t), t \in T$, has *Skorokhod paths*, i.e., paths which are right-continuous and have limits from the left.

Let τ be a stopping time relative to $\mathcal{F}_t$, $t \in T$, such that $P_\vartheta(\tau < \infty) = 1$ for each $\vartheta \in \Theta$. Suppose that the process can be observed during the random time interval $[0, \tau]$. The σ-algebra $\mathcal{F}_\tau$ of events happening before τ is given by

$$A \in \mathcal{F}_\tau \Leftrightarrow A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \in T.$$

Define $U = \mathbb{R}^l \times T$, $\mathcal{B}_U = \mathcal{B}_{\mathbb{R}^l} \otimes \mathcal{B}_T$, and let m_ϑ be the measure induced by the mapping $(Y(\tau), \tau)$ on the measurable space $(U, \mathcal{B}_U)$ given P_ϑ, i.e.,

$$m_\vartheta(B) = P_\vartheta((Y(\tau), \tau)^{-1}(B)), \quad B \in \mathcal{B}_U,$$

and let m_τ be the measure induced by $(Y(\tau), \tau)$ given μ. The measure m is well defined, because for every finite stopping time τ the mapping $(Y(\tau), \tau) : \Omega \to \mathcal{B}_U$ is $\mathcal{F}_\tau$-measurable. The random variable $(Y(\tau), \tau)$ is a sufficient statistic for ϑ relative to $\mathcal{F}_\tau$ (Döhler (1981)).

By Döhler (1981), the following modification of a well-known lemma of Sudakov holds.

LEMMA. *Let τ be any finite stopping time relative to $\mathcal{F}_t$, $t \in T$. Then, for every $\vartheta \in \Theta$, $m_\vartheta \ll m_\tau$ and the Radon–Nikodym derivative of m_ϑ with respect to m_τ is*

$$\frac{dm_\vartheta}{dm_\tau}(y, t) = L(y, t, \vartheta), \quad y \in \mathbb{R}^l, \ t \in T.$$

Let $P_{\vartheta,\tau}$ and μ_τ denote the restrictions of P_ϑ and μ to $\mathcal{F}_\tau$. By the Lemma, $P_{\vartheta,\tau} \ll \mu_\tau$ and

$$\frac{dP_{\vartheta,\tau}}{d\mu_\tau} = L(Y(\tau), \tau, \vartheta).$$

This formula is the fundamental identity of sequential analysis. It means that the form of the likelihood function is independent of the sampling rule τ. By virtue of the fundamental identity the expectations (risks) are well defined for randomly stopped processes. In the sequel the function $L(Y(\tau), \tau, \vartheta)$ will be denoted simply by $L(\tau, \vartheta)$.

3. A minimax theorem. Suppose that, in addition to the loss associated with the error of estimation, the statistician incurs a cost of observation of the process. Let $\mathcal{L}(\vartheta, d(Y(t), t), Y(t), t)$ be the loss function determining the loss incurred by the statistician if ϑ is the true value of the parameter and d is the estimator chosen by him having observed $Y(t)$ at the moment of stopping t. Denote by $c(Y(t), t)$ the cost function which represents the cost of observing the process up to time t.

Sequential procedures of the form $\delta = (\tau, d)$ for estimating ϑ will be considered, where τ is a finite stopping time with respect to $\mathcal{F}_t$, $t \in T$, and $d = d(Y(\tau), \tau)$ is an $\mathcal{F}_\tau$-measurable random variable. The risk function of the sequential procedure $\delta = (\tau, d)$ is defined by

$$\mathcal{R}(\vartheta, \delta) = E_\vartheta[\mathcal{L}(\vartheta, d(Y(\tau), \tau), Y(\tau), \tau) + c(Y(\tau), \tau)].$$

In the sequel we only consider sequential procedures $\delta = (\tau, d)$ for which $\mathcal{R}(\vartheta, \delta) < \infty$ for each $\vartheta \in \Theta$. The class of all sequential procedures satisfying this condition will be denoted by $\mathcal{D}$. The problem is to find optimal stopping rules τ and the corresponding estimators $d(\tau) = d(Y(\tau), \tau)$ subject to the minimax criterion: a sequential procedure $\delta_0 = (\tau_0, d_0)$ is said to be *minimax* if

$$\sup_{\vartheta \in \Theta} \mathcal{R}(\vartheta, \delta_0) = \inf_{\delta \in \mathcal{D}} \sup_{\vartheta \in \Theta} \mathcal{R}(\vartheta, \delta).$$

Let π be the prior distribution of the parameter on the space $(\Theta, \mathcal{B}_\Theta)$ and suppose that $\mathcal{R}(\vartheta, \delta)$ is a $\mathcal{B}_\Theta$-measurable function of ϑ. Then the *Bayes risk* is defined by

$$r(\pi, \delta) = \int_\Theta \mathcal{R}(\vartheta, \delta)\, \pi(d\vartheta),$$

provided the integral exists. A sequential procedure $\widehat{\delta} = (\widehat{\tau}, \widehat{d})$ is called *Bayes* for π if

$$r(\pi, \widehat{\delta}) = \inf_{\delta \in \mathcal{D}} r(\pi, \delta).$$

Let $\pi(\cdot \mid Y(\tau) = y, \tau = t)$ denote the posterior distribution of the parameter ϑ given $Y(\tau) = y$, $\tau = t$. The *posterior risk* corresponding to π and an estimator d is defined by

$$\varrho(\pi(\cdot \mid Y(\tau) = y,\ \tau = t), d) = \int_\Theta \mathcal{L}(\vartheta, d(y,t), y, t)\, \pi(d\vartheta \mid Y(\tau) = y,\ \tau = t).$$

An estimator $d^* = d^*(y,t)$ is called (y,t)-*Bayes* for π if

$$\varrho(\pi(\cdot \mid Y(\tau) = y,\ \tau = t), d^*) = \inf_d \varrho(\pi(\cdot \mid Y(\tau) = y,\ \tau = t), d)$$

for all $(y,t) \in U$.

A minimax theorem will be presented which is a version of the theorem of Wilczyński (1985) formulated here for the general form $\mathcal{L}(\vartheta, d(Y(t), t), Y(t), t)$ of loss function.

Denote by $W(y,t)$ a measurable mapping from $(U, \mathcal{B}_U)$ into $(\mathbb{R}, \mathcal{B}_\mathbb{R})$.

THEOREM 1. *Assume that the cost function is of the form* $c(y,t) = c(W(y,t))$. *Suppose that there exists a sequence of priors* $\pi_n, n = 1, 2, \ldots,$ *of the parameter* ϑ *for which there are corresponding* (y,t)-*Bayes estimators*

d_n^* such that

$$\liminf_{n\to\infty} \varrho(\pi_n(\cdot \mid Y(\tau) = y,\ \tau = t), d_n^*) = K(W(y,t))$$

for each $(y,t) \in U$, where K is a real-valued measurable function defined on $\mathbb{R}$. Moreover, assume that $K(\cdot) + c(\cdot)$ attains its minimum over $\mathcal{Z}$ at a point z^*, where $\mathcal{Z}$ is the set of values of the process $W(Y(t),t)$, $t \in T$. If

$$\tau_{z^*} = \inf\{t \in T : W(Y(t),t) = z^*\}$$

is a finite stopping time for each $\vartheta \in \Theta$, and if there exists an estimator $d_{z^*}(\tau_{z^*}) = d_{z^*}(Y(\tau_{z^*}), \tau_{z^*})$ such that

$$\sup_{\vartheta\in\Theta} \mathcal{R}(\vartheta, \delta_{z^*}) \le K(z^*) + c(z^*),$$

where $\delta_{z^*} = (\tau_{z^*}, d_{z^*}(\tau_{z^*}))$, then the sequential procedure δ_{z^*} is minimax under the loss function $\mathcal{L}(\vartheta, d(Y(t),t), Y(t), t)$ in the class of all sequential procedures $\delta = (\tau, d(\tau)) \in \mathcal{D}$.

4. The exponential model of processes and conjugate priors. Suppose that the likelihood function of the process $X(t)$, $t \in T$, has the following exponential form:

$$L(t,\vartheta) = \frac{dP_{\vartheta,t}}{d\mu_t} = \exp[\vartheta Z(t) - \Phi(\vartheta)S(t)], \tag{1}$$

where $\vartheta \in \Theta \subseteq \mathbb{R}^n$; $\Phi(\vartheta)$ and $S(t)$ are one-dimensional, while $Z(t)$ is an n-dimensional vector process; $(Z(t), S(t))$, $t \in T$, is a stochastic process adapted to the filtration $\mathcal{F}_t$, $t \in T$. Of course, $(Z(t), S(t))$ is a sufficient statistic for ϑ relative to $\mathcal{F}_t$, $t \in T$.

The process $(Z(t), S(t))$, $t \in T$, is assumed to satisfy the following conditions: $Z(t)$ is right continuous as a function of t, P_ϑ-a.s., and $S(t)$, $t \in T$, are nonnegative random variables ($S(t)$ may be nonrandom as well) such that $S(t)$ is nondecreasing and continuous as a function of t and $S(t) \to \infty$ as $t \to \infty$, P_ϑ-a.s.

The family of (1) covers many counting, branching, diffusion-type etc. processes.

Some results concerning characterization of the family of prior distributions on Θ which should be conjugate to the family of (1) are needed in the method of finding optimal procedures.

Denote by $\mathcal{Y}$ the interior of the convex hull of the set of all possible values of the process $(Z(t), S(t))$, $t \in T$. Define a family $\pi(\vartheta; r, \alpha)$, $(r,\alpha) \in \mathbb{R}^{n+1}$, of measures on Θ with the density (with respect to the Lebesgue measure) given by

$$\frac{d\pi(\vartheta; r, \alpha)}{d\vartheta} = f(\vartheta; r, \alpha) = C(r,\alpha)\exp[r\vartheta - \alpha\Phi(\vartheta)]. \tag{2}$$

The following theorem due to Magiera and Wilczyński (1991) shows that if the hyperparameters r, α are chosen from the set $\mathcal{Y}$, then there exists a norming constant $C(r,\alpha)$ such that $\pi(\vartheta; r,\alpha)$ is a proper conjugate prior, and the expectation of the gradient $\nabla\Phi(\vartheta)$ equals r/α.

THEOREM 2. *If* $(r,\alpha) \in \mathcal{Y}$, *then*

(a) $\int_\Theta f(\vartheta; r,\alpha)\, d\vartheta < \infty$,
(b) $E\nabla\Phi(\vartheta) = \int_\Theta (\nabla\Phi(\vartheta)) f(\vartheta; r,\alpha)\, d\vartheta = r/\alpha$.

In the next section a class of minimax sequential estimation procedures will be determined explicitly for the one-parameter case of the statistical model considered. For this case define

$$A_0 = \Big\{(r,\alpha) \in \mathbb{R}^2 : \int_\Theta \Phi''(\vartheta) \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta < \infty\Big\}$$

and let $\widetilde{\pi}(\vartheta; r,\alpha), (r,\alpha) \in A_0$, be a family of prior distributions of the parameter ϑ according to the following form of densities (with respect to the Lebesgue measure $d\vartheta$):

$$d\widetilde{\pi}(\vartheta; r,\alpha) = \widetilde{C}(r,\alpha)\Phi''(\vartheta) \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta. \tag{3}$$

To ensure finite posterior expected loss under the weighted square error considered in the next section, certain restrictions on the hyperparameters r, α will be needed. Define

$$A_1 = \Big\{(r,\alpha) \in \mathbb{R}^2 : \int_\Theta \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta < \infty\Big\},$$

$$A_2 = \Big\{(r,\alpha) \in \mathbb{R}^2 : \int_\Theta [\Phi'(\vartheta)]^2 \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta < \infty\Big\}$$

and $A = A_0 \cap A_1 \cap A_2 \cap \mathcal{Y}$. Since, by Theorem 2, $\mathcal{Y} \subset A_1$, it follows that $A = A_0 \cap A_2 \cap \mathcal{Y}$. To derive the Bayes estimator of ϑ and the posterior expected loss, one has to consider the following conditions:

$$\int_\Theta \frac{d}{d\vartheta}\{\exp[r\vartheta - \alpha\Phi(\vartheta)]\}\, d\vartheta = 0, \tag{4}$$

$$\int_\Theta \frac{d}{d\vartheta}\{[r - \alpha\Phi'(\vartheta)] \exp[r\vartheta - \alpha\Phi(\vartheta)]\}\, d\vartheta = 0. \tag{5}$$

Suppose that the set A has the following representation:

$$A = \{(r,\alpha) \in \mathbb{R}^2 : (r,\alpha) \in \mathcal{Y},\ \alpha > \alpha_0\}, \tag{6}$$

where α_0 is nonnegative. It then follows from Theorem 2 that the condition (4) is always satisfied and the following formula holds:

$$\alpha \int_\Theta \Phi'(\vartheta) \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta = r \int_\Theta \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta. \tag{7}$$

If (5) is also satisfied almost everywhere for $(r,\alpha) \in A$, then the following relation is valid:

$$\int_{\Theta} [r - \alpha\Phi'(\vartheta)]^2 \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta = \alpha \int_{\Theta} \Phi''(\vartheta) \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta, \tag{8}$$

which will be used to derive posterior expected loss.

5. Minimax sequential procedures for the one-parameter exponential families of processes. It is assumed that the condition (5) is satisfied for all $(r,\alpha) \in A$, and, additionally, in the case when $\alpha_0 > 0$, that

$$\sup_{\vartheta\in\Theta} \frac{[\Phi'(\vartheta)]^2}{\Phi''(\vartheta)} = \frac{1}{\alpha_0}. \tag{9}$$

Conditions (5) and (9) are practically not restrictive. They are satisfied for all known processes of the exponential family.

Note that if for a finite stopping time τ the usual regularity conditions are satisfied which allow differentiating twice under the integral sign with respect to ϑ in the identity $\int \exp[\vartheta Z(\tau) - \Phi(\vartheta)S(\tau)]\, d\mu_\tau = 1$, then the Wald identities hold:

$$E_\vartheta Z(\tau) = \Phi'(\vartheta) E_\vartheta S(\tau), \tag{10}$$

$$E_\vartheta [Z(\tau) - \Phi'(\vartheta)S(\tau)]^2 = \Phi''(\vartheta) E_\vartheta S(\tau). \tag{11}$$

Let the loss function be defined by

$$\mathcal{L}(\vartheta, d(z,s)) = \frac{1}{\Phi''(\vartheta)} [d(z,s) - \Phi'(\vartheta)]^2,$$

and let the cost function be of the form $c(z,s) = c(s)$, where z, s are the values of $Z(\tau)$ and $S(\tau)$, respectively.

THEOREM 3. *If there exists $s^* > 0$ such that*

$$\frac{1}{\alpha_0 + s^*} + c(s^*) = \min_{s>0} \left[\frac{1}{\alpha_0 + s} + c(s) \right],$$

then the sequential procedure $\delta_{s^*} = (\tau_{s^*}, d^0(\tau_{s^*}))$ with

$$\tau_{s^*} = \inf\{t : S(t) = s^*\} \quad \textit{and} \quad d^0(\tau_{s^*}) = \frac{Z(\tau_{s^*})}{\alpha_0 + S(\tau_{s^*})} = \frac{Z(\tau_{s^*})}{\alpha_0 + s^*}$$

is minimax in $\mathcal{D}$.

Proof. Let $\widetilde{\pi}(\vartheta; r, \alpha)$, $(r,\alpha) \in A$, be the family of priors on Θ defined by (3). The posterior probability distribution of the parameter ϑ, given $Z(\tau) = z$, $S(\tau) = s$ is determined by $\widetilde{\pi}(\vartheta; \widetilde{r}, \widetilde{\alpha})$, where $\widetilde{r} = r + z$ and

$\widetilde{\alpha} = \alpha + s$. The posterior risk is

$$\varrho_\alpha(\widetilde{\pi}(\cdot \mid Z(\tau) = z,\ S(\tau) = s), d(z,s)) = \widetilde{C}(\widetilde{r},\widetilde{\alpha}) \int_\Theta [d(z,s) - \Phi'(\vartheta)]^2 \exp[\widetilde{r}\vartheta - \widetilde{\alpha}\Phi(\vartheta)]\, d\vartheta$$

and this risk attains its minimum for

$$d^*_\alpha(z,s) = \frac{\int_\Theta \Phi'(\vartheta)\exp[\widetilde{r}\vartheta - \widetilde{\alpha}\Phi(\vartheta)]\, d\vartheta}{\int_\Theta \exp[\widetilde{r}\vartheta - \widetilde{\alpha}\Phi(\vartheta)]\, d\vartheta}.$$

By (7) this estimator takes the form

$$d^*_\alpha(z,s) = \frac{\widetilde{r}}{\widetilde{\alpha}} = \frac{r+z}{\alpha+s}.$$

The posterior risk corresponding to this estimator is

$$\varrho_\alpha(\widetilde{\pi}(\cdot \mid Z(\tau) = z,\ S(\tau) = s), d^*_\alpha(z,s)) = \widetilde{C}(\widetilde{r},\widetilde{\alpha})\widetilde{\alpha}^{-2} \int_\Theta [\widetilde{r} - \widetilde{\alpha}\Phi'(\vartheta)]^2 \exp[\widetilde{r}\vartheta - \widetilde{\alpha}\Phi(\vartheta)]\, d\vartheta.$$

Taking into account the identity

$$\int_\Theta [\widetilde{r} - \widetilde{\alpha}\Phi'(\vartheta)]^2 \exp[\widetilde{r}\vartheta - \widetilde{\alpha}\Phi(\vartheta)]\, d\vartheta = \widetilde{\alpha} \int_\Theta \Phi''(\vartheta)\exp[\widetilde{r}\vartheta - \widetilde{\alpha}\Phi(\vartheta)]\, d\vartheta$$

(see formula (8)) gives

$$\varrho_\alpha(\widetilde{\pi}(\cdot \mid Z(\tau) = z,\ S(\tau) = s), d^*_\alpha(z,s)) = \frac{1}{\widetilde{\alpha}}\widetilde{C}(\widetilde{r},\widetilde{\alpha}) \int_\Theta \Phi''(\vartheta)\exp[\widetilde{r}\vartheta - \widetilde{\alpha}\Phi(\vartheta)]\, d\vartheta = \frac{1}{\alpha+s}.$$

Consider now the sequential procedure $\delta_s = (\tau_s, d^0(\tau_s))$ with

$$\tau_s = \inf\{t : S(t) = s\}, \quad s > 0,$$

and

$$d^0(\tau_s) = d^0(Z(\tau_s), S(\tau_s)) = \frac{Z(\tau_s)}{\alpha_0 + S(\tau_s)} = \frac{Z(\tau_s)}{\alpha_0 + s}.$$

By the assumptions on the process $S(t)$ the stopping time τ_s is finite for each $\vartheta \in \Theta$. Moreover, note that the likelihood function at τ_s belongs to a noncurved exponential family. It then follows from the well-known analytical properties of noncurved exponential families (see Barndorff-Nielsen (1978) or Brown (1986)) that the regularity conditions are satisfied. Thus, for the stopping time τ_s the Wald identities (10) and (11) take the form

$$E_\vartheta Z(\tau_s) = s\Phi'(\vartheta), \tag{12}$$

$$E_\vartheta [Z(\tau_s) - s\Phi'(\vartheta)]^2 = s\Phi''(\vartheta). \tag{13}$$

The risk corresponding to the estimator $d^0 = d^0(\tau_s)$ is

$$\mathcal{R}_0(\vartheta, d^0) = \frac{E_\vartheta[Z(\tau_s) - (\alpha_0 + s)\Phi'(\vartheta)]^2}{(\alpha_0 + s)^2 \Phi''(\vartheta)}.$$

Using the identities (12) and (13) yields

$$\mathcal{R}_0(\vartheta, d^0) = \frac{s\Phi''(\vartheta) + [\alpha_0 \Phi'(\vartheta)]^2}{(\alpha_0 + s)^2 \Phi''(\vartheta)}.$$

Thus, under condition (9),

$$\sup_{\vartheta \in \Theta} \mathcal{R}_0(\vartheta, d^0) = \frac{1}{\alpha_0 + s} = \lim_{\alpha \to \alpha_0} \varrho_\alpha(\widetilde{\pi}(\cdot \mid Z(\tau) = z, S(\tau) = s), d^*_\alpha(z, s)).$$

Referring to Theorem 1 with $W(z, s) = s$ for each $(z, s) \in \mathbb{R} \times \mathbb{R}_+$ and $K(s) = 1/(\alpha_0 + s)$ for each $s > 0$ yields the desired result.

6. Minimax sequential procedures for exponential families of diffusions. In this section we consider some families of diffusion-type processes $X(t)$, $t \geq 0$, for which the likelihood functions have the following special form of the exponential model defined by (1):

$$L(t, \vartheta) = \exp\left[\vartheta Z(t) - \frac{1}{2}\vartheta^2 S(t)\right], \tag{14}$$

where $\vartheta \in \Theta = (-\infty, \infty)$ and the process $Z(t)$, $t \geq 0$, takes its values in $\mathbb{R}$. In this model

$$\begin{aligned}\int_\Theta \Phi''(\vartheta) \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta &= \int_{-\infty}^{\infty} \exp\left[r\vartheta - \alpha\frac{\vartheta^2}{2}\right] d\vartheta \\ &= \left(\frac{2\pi}{\alpha}\right)^{1/2} \exp\left(\frac{r^2}{2\alpha}\right)\end{aligned}$$

and

$$\begin{aligned}\int_\Theta [\Phi'(\vartheta)]^2 \exp[r\vartheta - \alpha\Phi(\vartheta)]\, d\vartheta &= \int_{-\infty}^{\infty} \vartheta^2 \exp\left[r\vartheta - \alpha\frac{\vartheta^2}{2}\right] d\vartheta \\ &= (2\pi)^{1/2}\alpha^{-3/2}\left(1 + \frac{r^2}{\alpha}\right) \exp\left(\frac{r^2}{2\alpha}\right)\end{aligned}$$

for $-\infty < r < \infty$ and $\alpha > 0$. Thus, in this case, $A_0 = A_1 = A_2 = (-\infty, \infty) \times (0, \infty)$. The set A has the representation of (6) with $\alpha_0 = 0$, i.e., $A = (-\infty, \infty) \times (0, \infty) = \mathcal{Y}$. Moreover, it is easy to check that condition (5) is satisfied for all $(r, \alpha) \in A$.

The following corollary determines minimax sequential procedures $\delta = (\tau, d(\tau))$ for estimating ϑ under the loss function

$$\mathcal{L}(\vartheta, d(\tau)) = [d(\tau) - \vartheta]^2$$

and the cost function having the form $c(\tau) = c(S(\tau))$.

Corollary 1. *If there exists $s^* > 0$ such that*

$$\frac{1}{s^*} + c(s^*) = \min_{s>0}\left[\frac{1}{s} + c(s)\right],$$

then the procedure $\delta_{s^*} = (\tau_{s^*}, d^0(\tau_{s^*}))$ with

$$\tau_{s^*} = \inf\{t : S(t) = s^*\} \quad \textit{and} \quad d^0(\tau_{s^*}) = \frac{Z(\tau_{s^*})}{s^*}$$

is minimax in $\mathcal{D}$.

6.1. *A family of exponential-type diffusions.* Let $X(t)$, $t \geq 0$, be a stochastic process satisfying the following stochastic differential equation:

$$\begin{aligned} dX(t) &= [a_1(t, X(t)) + \vartheta a_2(t, X(t))]dt + b(t, X(t))dW(t), \\ X(0) &= x_0, \end{aligned} \tag{15}$$

with $b(t, X(t)) > 0$, where $W(t)$, $t \geq 0$, denotes the standard Wiener process and all processes and functions take their values in $\mathbb{R}$. The class of solutions of (15) induces a family of probability measures P_ϑ, $\vartheta \in \Theta$, on the space of continuous functions from $[0,\infty)$ into $\mathbb{R}$ with the filtration generated by the cylinder sets. If the process

$$S(t) = \int_0^t [a_2(s, X(s))]^2 [b(s, X(s))]^{-2}\, ds$$

satisfies $P_\vartheta(S(t) < \infty) = 1$ for all $\vartheta \in \Theta$ and all $t > 0$, then for all $t > 0$ the measures $P_{\vartheta,t}$, $\vartheta \in \Theta$, are equivalent, and the likelihood function $L(t,\vartheta) = dP_{\vartheta,t}/dP_{0,t}$ is of the form (14), where

$$\begin{aligned} Z(t) = &\int_0^t a_2(s, X(s))[b(s, X(s))]^{-2}\, dX(s) \\ &- \int_0^t a_1(s, X(s))[b(s, X(s))]^{-2} a_2(s, X(s))\, ds, \end{aligned}$$

provided the last integral exists (see Küchler and Sørensen (1994)). Corollary 1 determines a class of minimax sequential procedures in this model.

6.2. *Hypoelliptic homogeneous Gaussian diffusions.* Consider a multidimensional process defined by an autonomous linear stochastic differential equation of the following form:

$$dX(t) = \vartheta AX(t)dt + GdW(t), \quad X(0) = x_0,$$

where $W(t)$, $t \geq 0$, is the standard k-dimensional Wiener process and A and G are constant $n \times n$ and $n \times k$ matrices, respectively. Recall that (see Liptser and Shiryaev (1978)) the process $X(t)$, $t \geq 0$, is a Gaussian Markov process with mean function $E_\vartheta X(t) = \exp(\vartheta At)x_0$, $t \geq 0$, and the covariance function

$$B_\vartheta(t,s) = E_\vartheta[(X(t) - E_\vartheta X(t))(X(s) - E_\vartheta X(s))^*] = \begin{cases} \exp[\vartheta A(t-s)]B_\vartheta(s) & \text{if } t \geq s, \\ B_\vartheta(t)\exp[\vartheta A^*(s-t)] & \text{if } t \leq s, \end{cases}$$

where the variance function $B_\vartheta(t)$, $t \geq 0$, is given by

$$\dot{B}_\vartheta(t) = \vartheta A B_\vartheta(t) + \vartheta B_\vartheta(t) A^* + GG^*, \qquad t \geq 0, \qquad B_\vartheta(0) = 0,$$

or equivalently

$$B_\vartheta(t) = \exp(\vartheta At)\int_0^t \exp(-\vartheta As)GG^*\exp(-\vartheta A^* s)\,ds\,\exp(\vartheta A^* t), \quad t \geq 0$$

(here the star denotes transposition). It is assumed that the differential generator corresponding to the homogeneous Gaussian diffusion $X(t)$, $t \geq 0$, is hypoelliptic, or equivalently, that $\operatorname{rank}[G, AG, \ldots, A^{n-1}G] = n$. Then, for every $0 \leq s < t$, the integral $\int_s^t \exp(-\vartheta Au)GG^*\exp(-\vartheta A^* u)\,du$ is a positive definite matrix and in particular, for every $t > 0$, the variance matrix $B_\vartheta(t)$ is regular.

Define $H = GG^*$ and let H^+ be its pseudoinverse. Assume that $HH^+A \neq 0$. The log likelihood function is equal to

$$\vartheta\int_0^t X^*(s)A^*H^+\,dX(s) - \frac{1}{2}\vartheta^2\int_0^t X^*(s)A^*H^+AX(s)\,ds$$

(see Liptser and Shiryaev (1978) or Le Breton and Musiela (1985)). By Corollary 1, if there exists $s^* > 0$ such that

$$\frac{1}{s^*} + c(s^*) = \min_{s>0}\left[\frac{1}{s} + c(s)\right],$$

then the procedure $\delta_{s^*} = (\tau_{s^*}, d^0(\tau_{s^*}))$ with

$$\tau_{s^*} = \inf\left\{t : \int_0^t X^*(s)A^*H^+AX(s)\,ds = s^*\right\}$$

and

$$d^0(\tau_{s^*}) = \frac{1}{s^*}\int_0^{\tau_{s^*}} X^*(s)A^*H^+\,dX(s)$$

is minimax in $\mathcal{D}$.

In the problem of sequential estimation of ϑ when the cost of observation is not taken into account the minimaxity of the sequential procedure $\delta_{s^*} = (\tau_{s^*}, d^0(\tau_{s^*}))$ was shown by Le Breton and Musiela (1985) by using another method.

6.3. *Minimax sequential procedures for the class of Ornstein–Uhlenbeck processes*

6.3.1. Let $X(t)$, $t \geq 0$, be the Ornstein–Uhlenbeck process with the mean coefficient ν, the drift coefficient β and the scale coefficient σ, defined as a solution of the stochastic differential equation

$$dX(t) = \beta[\nu - X(t)]dt + \sigma dW(t), \tag{16}$$

$-\infty < \nu < \infty$, $\beta > 0$, $\sigma > 0$.

1. Assume that $X(0) = x_0$, that ν is the parameter of interest, and that β, σ are known. Then the model is a special case of (15) for $a_1(t, X(t)) = -\beta X(t)$, $a_2(t, X(t)) = \beta$, $b(t, X(t)) = \sigma$, $\vartheta = \nu$. The solution of equation (16) has the mean $E_\nu X(t) = \nu + (x_0 - \nu)\exp(-\beta t)$ and the covariance function $B(s,t) = (\sigma^2/(2\beta))\{\exp(-\beta|t-s|) - \exp[-\beta(t+s)]\}$. The likelihood function is

$$L(t,\nu) = \frac{dP_{\nu,t}}{dP_{0,t}} = \exp\left\{\frac{\beta}{\sigma^2}\left[\nu\Big(X(t) - x_0 + \beta\int_0^t X(s)\,ds\Big) - \frac{1}{2}\nu^2\beta t\right]\right\}.$$

It follows from Corollary 1 that if there exists $t^* > 0$ such that

$$\frac{\sigma^2}{\beta^2 t^*} + c(t^*) = \min_{t>0}\left[\frac{\sigma^2}{\beta^2 t} + c(t)\right],$$

then the fixed-time procedure $\delta_{t^*} = (t^*, d^0(t^*))$ with

$$d^0(t^*) = \frac{X(t^*) - x_0 + \beta\int_0^{t^*} X(s)\,ds}{\beta t^*}$$

is minimax for estimating the mean value parameter ν.

2. Suppose that $X(0) =_{\mathcal{D}} \mathcal{N}(\nu, \sigma^2/(2\beta))$ ($X(0)$ being independent of $W(t)$, $t \geq 0$). Assume that ν is the parameter of interest and β, σ are known. The process $X(t)$, $t \geq 0$, is a stationary Gaussian Markov process with $E_\nu X(t) = \nu$ and the covariance function $B(s,t) = (\sigma^2/(2\beta))\exp(-\beta|t-s|)$. Here the likelihood function is

$$L(t,\nu) = \exp\left\{\frac{\beta}{\sigma^2}\left[\nu\Big(X(t) + X(0) + \beta\int_0^t X(s)\,ds\Big) - \frac{1}{2}\nu^2(2 + \beta t)\right]\right\}.$$

Assuming $c(t) = c(\beta\sigma^{-2}(2 + \beta t))$, Corollary 1 gives the following result: If there exists $t^* > 0$ such that

$$\frac{\sigma^2}{\beta(2+\beta t^*)} + c(t^*) = \min_{t>0}\left[\frac{\sigma^2}{\beta(2+\beta t)} + c(t)\right],$$

then the fixed-time procedure $\delta_{t^*} = (t^*, d^0(t^*))$ with

$$d^0(t^*) = \frac{X(t^*) + X(0) + \beta\int_0^{t^*} X(s)\,ds}{2+\beta t^*}$$

is minimax for estimating the mean value ν. This result was obtained by Różański (1982).

6.3.2. Suppose the observed process $X(t)$, $t \geq 0$, has the stochastic differential

$$dX(t) = \vartheta X(t)dt + \sigma dW(t), \quad X(0) = x_0,$$

where $\sigma > 0$ is known. The solution of this equation has the mean $E_\vartheta X(t) = x_0 \exp(\vartheta t)$ and the variance $\mathrm{Var}_\vartheta X(t) = (\sigma^2/(2\vartheta))[\exp(2\vartheta t)-1]$. This model is obtained from (15) for $a_1(t,X(t)) = 0$, $a_2(t,X(t)) = X(t)$, $b(t,X(t)) = \sigma$, and it is a one-dimensional case of the model of Section 6.2. For this model the statistics $Z(t)$ and $S(t)$ appearing in (14) are given by

$$Z(t) = \frac{1}{\sigma^2}\int_0^t X(s)\,dX(s) = \frac{X^2(t) - x_0^2 - \sigma^2 t}{2\sigma^2}$$

(by Ito's formula) and

$$S(t) = \frac{1}{\sigma^2}\int_0^t X^2(s)\,ds.$$

If there exists $s^* > 0$ such that

$$\frac{1}{s^*} + c(s^*) = \min_{s>0}\left[\frac{1}{s} + c(s)\right],$$

then the sequential procedure $\delta_{s^*} = (\tau_{s^*}, d^0(\tau_{s^*}))$ with

$$\tau_{s^*} = \inf\left\{t : \int_0^t X^2(s)\,ds = s^*\right\} \quad \text{and} \quad d^0(\tau_{s^*}) = \frac{X^2(\tau_{s^*}) - x_0^2 - \sigma^2\tau_{s^*}}{2s^*}$$

is minimax for estimating the drift coefficient ϑ.

6.4. *Estimating the drift of a geometric Brownian motion.* Let $X(t)$, $t \geq 0$, be a diffusion process satisfying the stochastic differential equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t).$$

The state space of $X(t)$ is $(0,\infty)$ and its initial state is x_0. By a straightforward application of Ito's formula $X(t)$ can be represented in the form

$$X(t) = x_0 \exp\left[\sigma W(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right], \quad t \geq 0.$$

The process $X(t)$ is called a *geometric Brownian motion* with drift parameter μ and volatility σ. The geometric Brownian motion serves as a model of the stock price dynamics and forms the basis for the famous option pricing theory of Black and Scholes (1973).

For every $t > 0$ the density of the random variable $X(t)$ is equal to

$$\frac{1}{X(t)\sigma(2\pi t)^{1/2}} \exp\left\{-\frac{[\log(X(t)/x_0) - (\mu - \sigma^2/2)t]^2}{2\sigma^2 t}\right\},$$

i.e., $X(t)$ is log normally distributed with parameters $((\mu - \sigma^2/2)t + \log x_0, \sigma^2 t)$. Suppose we start observing the process $X(t)$ at a fixed moment $t_0 \geq 0$. The likelihood function based on the observation up to time $t_0 + t$ is

$$L(t, \mu) = \exp\left\{\frac{1}{\sigma^2}\left[\mu\left(\frac{\sigma^2}{2}(t + t_0) + \log\frac{X(t + t_0)}{x_0}\right) - \frac{1}{2}\mu^2(t + t_0)\right]\right\}.$$

Observe that, unlike $X(t)$, the process $\log(X(t)/x_0) + \frac{1}{2}\sigma^2 t = \sigma W(t) + \mu t$ has independent increments.

COROLLARY 2. *If there exists $t^* > 0$ such that*

$$\frac{\sigma^2}{t^* + t_0} + c(t^*) = \min_{t>0}\left[\frac{\sigma^2}{t + t_0} + c(t)\right],$$

then the fixed-time procedure $\delta_{t^*} = (t^*, d^0(t^*))$ with

$$d^0(t^*) = \frac{\sigma^2}{2} + \frac{1}{t^* + t_0}\log\frac{X(t^* + t_0)}{x_0}$$

is minimax for estimating the drift parameter μ.

7. An exponential family of counting processes. Let $X(t)$, $t \geq 0$, be a counting process and let $X(t) = M(t) + \Lambda(t)$ denote its Doob–Meyer decomposition, where $M(t)$ is the martingale part and $\Lambda(t)$ is the compensator. Assume that $\Lambda(t) = \mu S(t)$, where $\mu > 0$ and $S(t)$ is continuous. It is well known (see Liptser and Shiryaev (1978)) that under certain conditions the likelihood function is given by

$$L(t, \mu) = \exp[(X(t) - x_0)\log\mu - \mu S(t)],$$

where $X(0) = x_0$.

An example is obtained by taking $S(t) = \int_0^t H(s)\,ds$, where $H(t)$ is a positive, predictable stochastic process. In particular, $H(t) \equiv 1$ for the Poisson process, $H(t) = bt^{b-1}$ for the Weibull process (b being a known value), $H(t) = X(t-)$ for the pure birth process, and $H(t) = X(t-)[M - X(t-)]^+$ for the logistic birth process, where M is a known constant.

In this model the set A of prior parameters is $A = (0, \infty) \times (0, \infty)$, i.e., $\alpha_0 = 0$. Moreover, it is easy to check that condition (5) is satisfied for all

$(r,\alpha) \in A$. Suppose that the loss incurred at the moment of stopping τ in estimating the parameter μ using an estimator $d(\tau)$ is defined by

$$\mathcal{L}(\mu, d(\tau)) = \frac{1}{\mu}[d(\tau) - \mu]^2,$$

and that the cost function has the form $c(\tau) = c(S(\tau))$. It then follows from Theorem 3 that if there exists $s^* > 0$ such that

$$\frac{1}{s^*} + c(s^*) = \min_{s>0} \left[\frac{1}{s} + c(s)\right],$$

then the procedure $\delta_{s^*} = (\tau_{s^*}, d^0(\tau_{s^*}))$ with

$$\tau_{s^*} = \inf\Big\{t : \int_0^t H(s)\,ds = s^*\Big\} \quad \text{and} \quad d^0(\tau_{s^*}) = \frac{X(\tau_{s^*}) - x_0}{s^*}$$

is minimax in $\mathcal{D}$.

8. Minimax sequential procedures for a compound Poisson process with multinomial jumps. In this section an example of a multiparameter exponential model for stochastic processes will be considered. Take into account the following process. Jumps occur according to a Poisson process with intensity, say, $\lambda(t)$. Denoting by $X(i)$ the magnitude of the ith jump (at time t_i) we assume that $X(i)$, $i = 1, 2, \ldots$, are independent identically distributed random variables which are also independent of the Poisson process $k(t)$, $t \geq 0$, which produces these jumps.

The likelihood function based on the realization $(k(t); t_1, \ldots, t_{k(t)}; X(1), \ldots, X(k(t)))$ is given by

$$\prod_{j=1}^{k(t)} \lambda(t_j)\Big\{\exp\Big[-\int_0^t \lambda(s)\,ds\Big]\Big\}\prod_{j=1}^{k(t)} p(X(j)),$$

$k(t) \geq 1$, where $p(X(j))$ is the density of $X(j)$, the jump size.

Suppose the jumps $X(1), X(2), \ldots$ have the multinomial distribution with parameter $p \in \mathcal{P} = \{(p_1, \ldots, p_m) : p_i > 0,\ i = 1, \ldots, m;\ \sum_{i=1}^m p_i = 1\}$, i.e., $X(i) = (X_1(i), \ldots, X_m(i))$ with $P(X(i) = e_j) = p_j$, $j = 1, \ldots, m$, $i = 1, 2, \ldots$, where $e_1 = (1, 0, \ldots, 0), \ldots, e_m = (0, \ldots, 0, 1)$. Define $Y_i(k(t)) = \sum_{j=1}^{k(t)} X_i(j), i = 1, \ldots, m$. If the intensity $\lambda(t)$ is known then the likelihood function is defined by

$$p_1^{Y_1(k(t))} \ldots p_m^{Y_m(k(t))},$$

where $\sum_{i=1}^m Y_i(k(t)) = k(t)$. This formula can be rewritten in the following exponential form:

$$\exp\Big[\sum_{i=1}^{m-1} \vartheta_i Y_i(k(t)) - \varPhi(\vartheta)k(t)\Big],$$

where

$$\vartheta_i = \log\left[\frac{p_i}{1-\sum_{i=1}^{m-1} p_i}\right], \qquad i = 1,\ldots,m-1,$$

$$\Phi(\vartheta) = \log\left[1 + \sum_{i=1}^{m-1} \exp\vartheta_i\right],$$

which is a special form of the general model defined by (1).

Now a class of minimax sequential procedures for estimating the parameter $p = (p_1,\ldots,p_m)$ will be determined. Let the loss function $\mathcal{L}(p,d)$ be defined by

$$\mathcal{L}(p,d) = \sum_{i=1}^{m} \frac{(d_i - p_i)^2}{p_i} \tag{17}$$

and let the cost function be of the form $c(Y_1(k(t)),\ldots,Y_{m-1}(k(t)),\ k(t)) = c(k(t))$. Let

$$\tau_k = \inf\{t \ge 0 : k(t) = k\}, \qquad k \ge 1. \tag{18}$$

Exploiting the method of Theorem 1, in an analogous way to Wilczyński (1985) for the multinomial process one obtains the following result.

THEOREM 4. *If there exists $k_0 \in \{1,2,\ldots\}$ for which*

$$\frac{m-1}{k_0+m-1} + c(k_0) = \min_k\left\{\frac{m-1}{k+m-1} + c(k)\right\},$$

then the sequential procedure $(\tau_{k_0},\ d_{k_0})$, $d_{k_0} = (d_{1,k_0},\ldots,d_{m,k_0})$ with

$$d_{i,k_0} = \frac{Y_i(k_0)}{k_0+m-1},$$

$i = 1,\ldots,m$, *is minimax in the class $\mathcal{D}$ under the loss function* (17).

Sketch of proof. Assuming conjugate priors $\overline{\pi}$ for $\mathbf{p}$ with densities according to

$$d\overline{\pi}(p;\alpha) = \frac{\Gamma(m\alpha)}{[\Gamma(\alpha)]^m}(p_1\ldots p_m)^{\alpha-1}dp, \qquad \alpha > 0,$$

we note that for any stopping time τ the posterior risk is

$$\varrho(\overline{\pi}(\cdot \mid Y(k(\tau)) = y,\ k(\tau) = k), d)$$
$$= \frac{\Gamma(k+m\alpha)}{\prod_{i=1}^m \Gamma(y_i+\alpha)} \int\!\!\ldots\!\!\int_{\mathcal{P}} \sum_{i=1}^{m} \frac{(d_i-p_i)^2}{p_i} p_1^{y_1+\alpha-1}\ldots p_m^{y_m+\alpha-1}dp_1\ldots dp_{m-1}.$$

For each $\alpha > 1$ this risk is minimized by

$$d^*_{\alpha,i}(y) = \frac{y_i+\alpha-1}{k+m\alpha-1}$$

$(i = 1, \ldots, m;\ k = \sum_{i=1}^m y_i)$, and

$$\varrho(\overline{\pi}(\cdot \mid Y(k(\tau)) = y,\ k(\tau) = k), d_\alpha^*) = \frac{m-1}{k+m\alpha-1} \to \frac{m-1}{k+m-1}$$

as $\alpha \to 1$. For the sequential procedure $\delta_k = (\tau_k, d_k)$, where τ_k is defined by (18) and $d_k = (d_{1,k}, \ldots, d_{m,k})$ with

$$d_{i,k} = \frac{Y_i(k(\tau_k))}{k(\tau_k)+m-1} = \frac{Y_i(k)}{k+m-1}, \qquad i = 1, \ldots, m,$$

the risk function is

$$\mathcal{R}(p, \delta_k) = E_p\bigg[\sum_{i=1}^m \frac{(d_{i,k}-p_i)^2}{p_i} + c(k(\tau_k))\bigg] = \frac{m-1}{k+m-1} + c(k).$$

Thus, upon putting $W(Y_1(k(t)), \ldots, Y_{m-1}(k(t)), k(t)) = k(t)$ and $K(k) = (m-1)/(k+m-1)$, the result follows from Theorem 1.

An analogous result holds for the loss function

$$\mathcal{L}(p, d) = \sum_{i=1}^m \frac{(d_i - p_i)^2}{1-p_i}. \tag{19}$$

Theorem 5. *If there exists* $k_0 \in \{1, 2, \ldots\}$ *for which*

$$\frac{1}{k_0 + (m-1)^{-1}} + c(k_0) = \min_k \bigg\{\frac{1}{k+(m-1)^{-1}} + c(k)\bigg\},$$

then the sequential procedure (τ_{k_0}, d_{k_0}), $d_{k_0} = (d_{1,k_0}, \ldots, d_{m,k_0})$ *with*

$$d_{i,k_0} = \frac{Y_i(k_0) + (m-1)^{-1}}{k_0 + (m-1)^{-1}}, \qquad i = 1, \ldots, m,$$

is minimax in the class $\mathcal{D}$ *under the loss function* (19).

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