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POINT DERIVATIONS FOR LIPSCHITZ FUNCTIONS AND CLARKE'S GENERALIZED DERIVATIVE

Abstract. Clarke's generalized derivative $f^0(x,v)$ is studied as a function on the Banach algebra $\mathrm{Lip}(X,d)$ of bounded Lipschitz functions f defined on an open subset X of a normed vector space E. For fixed $x\in X$ and fixed $v\in E$ the function $f^0(x,v)$ is continuous and sublinear in $f\in \mathrm{Lip}(X,d)$. It is shown that all linear functionals in the support set of this continuous sublinear function satisfy Leibniz's product rule and are thus point derivations. A characterization of the support set in terms of point derivations is given.

1. Introduction. A derivative concept for a class of functions $f: X\to\mathbb{R}$ defined on a subset X of a real vector space E may be viewed as an operator $\mathcal{D}$ which assigns reals $\mathcal{D}(f,x,v)$ to triples (f,x,v), where f is an element of a function class, $x\in X$, and $v\in E$. In applications derivative concepts are mostly used as approximation tools for a function f in a neighborhood of a point x. Therefore studies of derivative concepts usually focus on the properties of the function $\mathcal{D}(f,\cdot,\cdot)$ with f thought of as being fixed. However, if one is interested in characterizing a derivative concept as an operator on a function space it seems more natural to focus attention on the function $\mathcal{D}(\cdot,x,v)$, where x and v are fixed. This is the viewpoint of the theory of point derivations as introduced in [5]. In the sequel we shall study Clarke's generalized derivative [2]

$$\mathcal{D}(f,x_0,v):=f^0(x_0,v)=\limsup_{\substack{x\to x_0\\ t\downarrow 0}}\frac{f(x+tv)-f(x)}{t}$$

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from the latter point of view. Clarke's generalized derivative is continuous and sublinear, not only as a function of the direction v, but also as a function on the Banach space of Lipschitz functions f. Continuous sublinear functions on Banach spaces are completely characterized by their support sets which are subsets of the dual space. We show in the sequel that the support set of the function $\mathcal{D}_{\mathrm{cl}}|_{x_0,v} = \mathcal{D}(\cdot, x_0, v)$ consists of point derivations in the sense of [5] and characterizes those point derivations which are contained in the support set.

In the next section we review some properties of the algebra of Lipschitz functions and explain the concept of its point derivations. Section 3 explores the relation between point derivations and Clarke's generalized derivative. For the reader's convenience we include the short proofs of [5].

2. Lipschitz functions and point derivations

2.1. *The Lipschitz algebra.* The following brief description of the Banach algebra of Lipschitz functions follows the presentation given in [5]. Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called a *Lipschitz function* if there exists a constant $K \geq 0$ such that for all $x, y \in X$,

$$|f(x) - f(y)| \leq K d(x, y).$$

The set of all bounded Lipschitz functions defined on (X, d) is a real algebra and will be denoted by $\mathrm{Lip}(X, d)$. For $f \in \mathrm{Lip}(X, d)$ the following two constants are finite:

$$\|f\|_d := \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X,\ x \neq y\right\},$$
$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

A norm on $\mathrm{Lip}(X, d)$ is defined by

$$\|f\| := \|f\|_d + \|f\|_\infty.$$

Arens and Eells show in [1] that $(\mathrm{Lip}(X, d), \|\cdot\|)$ is always the dual space of some normed linear space and hence complete. Moreover, $(\mathrm{Lip}(X, d), \|\cdot\|)$ is a Banach algebra, i.e. for all $f, g \in \mathrm{Lip}(X, d)$ the inequality $\|fg\| \leq \|f\| \cdot \|g\|$ holds. This follows from the inequality

$$\frac{|(fg)(x) - (fg)(y)|}{d(x, y)} \leq |f(x)| \frac{|(g)(x) - (g)(y)|}{d(x, y)} + |g(y)| \frac{|(f)(x) - (f)(y)|}{d(x, y)},$$

which implies

$$\|fg\|_d \leq \|f\|_\infty \cdot \|g\|_d + \|g\|_\infty \cdot \|f\|_d$$

and thus yields

$$\begin{aligned}\|fg\| &= \|fg\|_\infty + \|fg\|_d \\ &\le \|f\|_\infty \cdot \|g\|_\infty + \|f\|_\infty \cdot \|g\|_d + \|g\|_\infty \cdot \|f\|_d \\ &= \|f\|_\infty \cdot (\|g\|_\infty + \|g\|_d) + \|g\|_\infty \cdot \|f\|_d \\ &\le \|f\| \cdot \|g\|.\end{aligned}$$

The Banach algebra $(\mathrm{Lip}(X,d), \|\cdot\|)$ will be called the *Lipschitz algebra* on (X,d). The unit element is the characteristic function on X, which is denoted by $\mathbf{1}$.

2.2. *Point derivations.* Point derivations are linear functionals on $\mathrm{Lip}(X,d)$ which satisfy Leibniz's product rule.

DEFINITION 2.1. Let (X,d) be a metric space and $x_0 \in X$. A continuous linear functional $l \in \mathrm{Lip}(X,d)^*$ is said to be a *point derivation* at $x_0 \in X$ if for all $f, g \in \mathrm{Lip}(X,d)$ Leibniz's rule

$$l(fg) = f(x_0) \cdot l(g) + g(x_0) \cdot l(f)$$

holds.

The linear space of all point derivations at $x_0 \in X$ will be denoted by $\mathrm{Der}_{x_0}(\mathrm{Lip}(X,d))$. It is a weak-$*$-closed subspace of $\mathrm{Lip}(X,d)^*$ (cf. [5], Proposition 8.2).

It is interesting that point derivations which reflect merely the product rule can in fact be completely characterized by another property of the classical derivative, namely the fact that the derivative of f vanishes provided f is the unit function or f has a root of second order. To formalize this, let us denote by

$$\boldsymbol{m}(x_0) := \{f \in \mathrm{Lip}(X,d) : f(x_0) = 0\}$$

the closed ideal in $\mathrm{Lip}(X,d)$ of functions which vanish at $x_0 \in X$. Then the ideal of functions in $\mathrm{Lip}(X,d)$ with a root of second order is given by

$$\boldsymbol{m}^2(x_0) := \mathrm{cl}\left(\left\{f := \sum_{j=1}^{k} f_j g_j : f_j, g_j \in \boldsymbol{m}(x_0),\ j = 1, \ldots, k,\ k \in \mathbb{N}\right\}\right),$$

where cl denotes the closure in $\mathrm{Lip}(X,d)$. I. Singer and J. Wermer have shown in [6] that point derivations can indeed be characterized by their values for the unit function and their values on the latter ideal.

PROPOSITION 2.2. *Let (X,d) be a metric space and $x_0 \in X$. Then a continuous linear functional $l \in \mathrm{Lip}(X,d)^*$ satisfies*

(i) $l(\mathbf{1}) = 0$,
(ii) $l|\boldsymbol{m}^2(x_0) = 0$,

if and only if for all $f, g \in \mathrm{Lip}(X, d)$ *Leibniz's rule*

$$l(fg) = f(x_0) \cdot l(g) + g(x_0) \cdot l(f)$$

is satisfied at x_0.

P r o o f. "$\Leftarrow$" Assume that $l \in \mathrm{Lip}(X, d)^*$ satisfies Leibniz's rule. Since $\mathbf{1}^2 = \mathbf{1}$, Leibniz's rule for $f = g = \mathbf{1}$ implies that $l(\mathbf{1}) = 2l(\mathbf{1})$, which means that $l(\mathbf{1}) = 0$.

Now assume that $f, g \in \boldsymbol{m}(x_0)$. Then $l(fg) = f(x_0) \cdot l(g) + g(x_0) \cdot l(f) = 0$, and since l is continuous, it follows that $l|\boldsymbol{m}^2(x_0) = 0$.

"$\Rightarrow$" Assume that the continuous linear functional l satisfies (i) and (ii). Then for every $f, g \in \mathrm{Lip}(X, d)$ we have

$$\begin{aligned} l(fg) &= l(fg - f(x_0)g(x_0)\mathbf{1}) \\ &= l((f - f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1}) + f(x_0)(g - g(x_0)\mathbf{1}) \\ &\quad + g(x_0)(f - f(x_0)\mathbf{1})) \\ &= l((f - f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1})) + f(x_0) \cdot l(g - g(x_0)\mathbf{1}) \\ &\quad + g(x_0) \cdot l(f - f(x_0)\mathbf{1}) \\ &= f(x_0) \cdot l(g - g(x_0)\mathbf{1}) + g(x_0) \cdot l(f - f(x_0)\mathbf{1}) \\ &= f(x_0) \cdot l(g) + g(x_0) \cdot l(f), \end{aligned}$$

since $(f - f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1}) \in \boldsymbol{m}^2(x_0)$. ∎

In [5] D. R. Sherbert determines all point derivations in $\mathrm{Lip}(X, d)$. We briefly outline his construction.

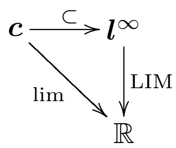
Consider the real Banach space

$$\boldsymbol{l}^\infty := \{x := (x_n)_{n \in \mathbb{N}} : (x_n)_{n \in \mathbb{N}} \text{ bounded sequence}\}$$

endowed with the supremum norm

$$\|x\|_\infty := \sup_{n \in \mathbb{N}} |x_n|.$$

Let $\boldsymbol{c} \subset \boldsymbol{l}^\infty$ denote the closed subspace of all convergent sequences and let $\lim : \boldsymbol{c} \to \mathbb{R}$ be the continuous linear functional which assigns to every convergent sequence its limit. Consider a norm-preserving Hahn–Banach extension LIM of the functional lim to $\boldsymbol{l}^\infty$:



with the following additional properties:

(i) $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_{n+1}$,
(ii) $\liminf_{n\to\infty} x_n \le \mathrm{LIM}_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$.

We shall use the notation $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}(x)$ for $x := (x_n)_{n\in\mathbb{N}} \in \boldsymbol{l}^\infty$. Such functionals LIM are called *translation invariant Banach limits.* For their construction we refer to [3], Chapter II.4, Exercise 22.

Now let $x_0 \in X$ be a nonisolated point and let $w := (x_n, y_n)_{n\in\mathbb{N}} \subset \{(s,t) \in X \times X : \ s \neq t\}$ converge to the point (x_0, x_0). Then

$$T_w : \mathrm{Lip}(X,d) \to \boldsymbol{l}^\infty \quad \text{with} \quad T_w(f) := \left(\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right)_{n\in\mathbb{N}}$$

is a continuous linear operator, since $\|T_w(f)\|_\infty \le \|f\|_d \le \|f\|$.

It is easy to see that for any translation invariant Banach limit LIM the continuous linear fuctional $D_w : \mathrm{Lip}(X,d) \to \mathbb{R}$ with $D_w(f) = \mathrm{LIM}(T_w(f))$ is a point derivation at $x_0 \in X$. For abbreviation put $\Delta := \{(s,t) \in X \times X : s = t\}$.

Proposition 2.3 ([5], Lemma 9.4). *Let $x_0 \in X$ be a nonisolated point of a metric space (X,d) and $w := (x_n, y_n)_{n\in\mathbb{N}} \subset (X \times X) \setminus \Delta$ be a sequence which converges to (x_0, x_0). Then for every translation invariant Banach limit* $\mathrm{LIM} : \boldsymbol{l}^\infty \to \mathbb{R}$ *the continuous linear fuctional*

$$D_w : \mathrm{Lip}(X,d) \to \mathbb{R} \quad \textit{with} \quad D_w(f) = \mathrm{LIM}(T_w(f))$$

is a point derivation at $x_0 \in X$.

Proof. First observe that for every convergent sequence $(a_n)_{n\in\mathbb{N}} \in \boldsymbol{c}$ and every bounded sequence $(b_n)_{n\in\mathbb{N}} \in \boldsymbol{l}^\infty$ the formula

$$\mathop{\mathrm{LIM}}_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \mathop{\mathrm{LIM}}_{n\to\infty} b_n$$

holds. Indeed, put $\alpha := \lim_{n\to\infty} a_n$. Then $\mathrm{LIM}_{n\to\infty}(a_n \cdot b_n - \alpha b_n) = 0$, since $(a_n \cdot b_n - \alpha b_n)_{n\in\mathbb{N}}$ converges to zero and hence $\mathrm{LIM}_{n\to\infty}(a_n \cdot b_n) = \alpha \,\mathrm{LIM}_{n\to\infty} b_n$.

Now let $f, g \in \mathrm{Lip}(X,d)$ be given. From the above observation it follows that

$$\begin{aligned}
D_w(fg) &= \mathrm{LIM}(T_w(fg)) \\
&= \mathop{\mathrm{LIM}}_{n\to\infty}\left(\frac{(fg)(y_n) - (fg)(x_n)}{d(y_n, x_n)}\right) \\
&= \mathop{\mathrm{LIM}}_{n\to\infty}\left(f(y_n)\frac{g(y_n) - g(x_n)}{d(y_n, x_n)} + g(x_n)\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0) \mathop{\mathrm{LIM}}_{n\to\infty}\left(\frac{g(y_n) - g(x_n)}{d(y_n, x_n)}\right) + g(x_0) \mathop{\mathrm{LIM}}_{n\to\infty}\left(\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0)\,\mathrm{LIM}(T_w(g)) + g(x_0)\,\mathrm{LIM}(T_w(f)) \\
&= f(x_0) D_w(g) + g(x_0) D_w(f).
\end{aligned}$$

Since D_w is continuous, it is a point derivation at $x_0 \in X$. ■

In Theorem 9.5 of [5] Sherbert shows that the space of all point derivations at a particular point x_0 is spanned by point derivations which are constructed in the above way, where the Banach limit is arbitrary but fixed.

THEOREM 2.4. *Let* $x_0 \in X$ *be a nonisolated point of a metric space* (X,d) *and* $\mathcal{W}_{x_0} := \{w := (x_n, y_n)_{n\in\mathbb{N}} \subset X \times X \setminus \Delta : \lim x_n = \lim y_n = x_0\}$. *Moreover*, *let* $\mathrm{LIM} : \boldsymbol{l}^\infty \to \mathbb{R}$ *be a fixed translation invariant Banach limit. Then*

$$\mathrm{Der}_{x_0}(\mathrm{Lip}(X,d)) = \mathrm{cl\,span}\{D_w = \mathrm{LIM}(T_w) : w \in \mathcal{W}_{x_0}\},$$

where cl span *denotes the weak-∗-closure of the linear hull in* $\mathrm{Lip}(X,d)^*$.

3. Clarke's generalized derivative and point derivations. In this section we assume that $X \subset E$ is an open subset of a real normed vector space $(E, \|\cdot\|_E)$. For every point $x_0 \in X$ and every direction $v \in E$ Clarke's generalized directional derivative as defined in [2] is given by

$$\mathcal{D}_{\mathrm{cl}}|_{x_0,v} : \mathrm{Lip}(X,d) \to \mathbb{R},$$
$$\mathcal{D}_{\mathrm{cl}}|_{x_0,v}(f) := f^0(x_0,v) = \limsup_{\substack{x\to x_0\\ t\downarrow 0}} \frac{f(x+tv)-f(x)}{t}.$$

It is well known that Clarke's generalized derivative is a continuous sublinear function of the direction v. This led to the invention of Clarke's subdifferential as an extension of the subdifferential of a convex function [2]. It is, however, less known, although straightforward, that Clarke's concept also leads to a continuous sublinear function when considered as a function of $f \in \mathrm{Lip}(X,d)$.

PROPOSITION 3.1. *Let* $(E, \|\cdot\|_E)$ *be a real normed vector space*, $X \subset E$ *an open subset*, $x_0 \in X$ *and* $v \in E$. *Then the generalized directional derivative*

$$\mathcal{D}_{\mathrm{cl}}|_{x_0,v} : \mathrm{Lip}(X,d) \to \mathbb{R}$$

is a continuous sublinear function.

Proof. The function $\mathcal{D}_{\mathrm{cl}}|_{x_0,v} : \mathrm{Lip}(X,d) \to \mathbb{R}$ is obviously sublinear. To see that it is continuous, it suffices to show continuity at $0 \in \mathrm{Lip}(X,d)$. This, however, is again obvious since

$$|f^0(x_0,v)| = \Big|\limsup_{\substack{x\to x_0\\ t\downarrow 0}} \frac{f(x+tv)-f(x)}{t}\Big| \le \|f\|_d \cdot \|v\|_E \le \|v\|_E \cdot \|f\|. \ \blacksquare$$

Due to the Hahn–Banach theorem the continuous sublinear function $\mathcal{D}_{\mathrm{cl}}|_{x_0,v}$ is completely determined by its support set

$$\partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0 = \{l \in \mathrm{Lip}(X,d)^* : l(f) \le \mathcal{D}_{\mathrm{cl}}|_{x_0,v}(f)\}$$

via the relation

$$\mathcal{D}_{\mathrm{cl}}|_{x_0,v}(f) = \sup_{l \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0} l(f).$$

The support set of a continuous sublinear function on a locally convex space has been studied in [4]. It is a nonempty, convex, weak-$*$-compact subset of E^*. The aim of this section is the study of the set $\partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$.

It is well known that Clarke's generalized derivative does not obey Leibniz's product rule (cf. [2], Proposition 2.3.13). However, our first result shows that all elements of the support set $\partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$ are point derivations and thus obey Leibniz's product rule.

PROPOSITION 3.2. *Let $(E, \|\cdot\|_E)$ be a real normed vector space, $X \subset E$ an open subset, $x_0 \in X$ and $v \in E$. Then every element $l \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$ is a point derivation at x_0.*

Proof. To use Proposition 2.2 we show that for every $l \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$:

(i) $l(\mathbf{1}) = 0$,

(ii) $l|\boldsymbol{m}^2(x_0) = 0$.

To prove (i), let $l \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$ and $c \in \mathbb{R}$. Then $c \cdot l(\mathbf{1}) \le (c\mathbf{1})^0(x_0, v) = 0$. For $c = 1$ this implies $l(\mathbf{1}) \le 0$ and for $c = -1$ it implies $l(\mathbf{1}) \ge 0$; hence $l(\mathbf{1}) = 0$.

To prove (ii), let again $l \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$, i.e. $l(f) \le f^0(x_0, v)$ for every $f \in \mathrm{Lip}(X, d)$. Suppose $f \in \boldsymbol{m}(x_0)$. Then Proposition 2.1.1 of [2] implies that, on the one hand,

$$l(f^2) \le (f^2)^0(x_0, v) \le 2|f(x_0)| \cdot f^0(x_0, v) = 0$$

and, on the other hand,

$$l(-f^2) \le (-f^2)^0(x_0, v) = (f^2)^0(x_0, -v) \le 2|f(x_0)| \cdot f^0(x_0, -v) = 0.$$

Therefore $l(f^2) = 0$ for every $f \in \boldsymbol{m}(x_0)$. Since $gh = \frac{1}{2}((g+h)^2 - g^2 - h^2)$, it follows from the continuity of l that $l|\boldsymbol{m}^2(x_0) = 0$. ∎

The following result characterizes the set $\partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$ and is complementary to Theorem 2.4.

THEOREM 3.3. *Let $(E, \|\cdot\|_E)$ be a real normed vector space, $X \subset E$ an open subset $x_0 \in X$ and $v \in E$ with $\|v\|_E = 1$. Moreover, let*

$$\mathcal{D}_{\mathrm{cl}}|_{x_0,v} : \mathrm{Lip}(X, d) \to \mathbb{R}$$

with

$$\mathcal{D}_{\mathrm{cl}}|_{x_0,v}(f) = f^0(x_0, v) = \limsup_{\substack{x \to x_0 \\ t \downarrow 0}} \frac{f(x + tv) - f(x)}{t}$$

be the generalized directional derivative. Then

$$\partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0 = \operatorname{cl}\operatorname{conv}\{D_w : w \in \mathcal{W}_{x_0}(v)\},$$

where cl conv *denotes the weak-$*$-closure of the convex hull in* $\mathrm{Lip}(X,d)^*$, *and*

$$\mathcal{W}_{x_0}(v) := \{w := (x_n, y_n)_{n\in\mathbb{N}} \in \mathcal{W}_{x_0} : y_n - x_n \in \mathbb{R}_+ \cdot v\}.$$

Proof. Let a translation invariant Banach limit $\mathrm{LIM} : \boldsymbol{l}^\infty \to \mathbb{R}$ and a sequence $w := (x_n, y_n)_{n\in\mathbb{N}} \in \mathcal{W}_{x_0}(v)$ be given. Then there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of positive numbers which converges to zero such that $y_n = x_n + \tau_n v$ for all $n \in \mathbb{N}$. Hence for every $f \in \mathrm{Lip}(X,d)$,

$$\begin{aligned} D_w(f) &= \mathrm{LIM}(T_w(f)) = \mathop{\mathrm{LIM}}_{n\to\infty}\left(\frac{f(y_n) - f(x_n)}{\|y_n - x_n\|_E}\right) \\ &= \mathop{\mathrm{LIM}}_{n\to\infty}\left(\frac{f(x_n + \tau_n v) - f(x_n)}{\tau_n}\right) \\ &\le \limsup_{n\to\infty} \frac{f(x_n + \tau_n v) - f(x_n)}{\tau_n} \le f^0(x_0, v) \end{aligned}$$

since $\mathrm{LIM}_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n$ for every bounded sequence $b := (b_n)_{n\in\mathbb{N}}$. Hence

$$\operatorname{cl}\operatorname{conv}\{D_w : w \in \mathcal{W}_{x_0}(v)\} \subseteq \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0.$$

To prove equality, we show that for every $f \in \mathrm{Lip}(X,d)$ there exists a $D_{\widehat{w}} \in \operatorname{cl}\operatorname{conv}\{D_w : w \in \mathcal{W}_{x_0}(v)\}$ such that $D_{\widehat{w}}(f) = f^0(x_0, v)$. Note that for every $f \in \mathrm{Lip}(X,d)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x_0 and a sequence of positive real numbers $(\tau_n)_{n\in\mathbb{N}}$ converging to zero such that

$$f^0(x_0, v) = \limsup_{\substack{x\to x_0\\ t\downarrow 0}} \frac{f(x+tv) - f(x)}{t} = \lim_{n\to\infty} \frac{f(x_n + \tau_n v) - f(x_n)}{\tau_n}.$$

Hence for the sequence $\widehat{w} := (x_n, x_n + \tau_n v)_{n\in\mathbb{N}} \in \mathcal{W}_{x_0}(v)$ we have

$$\begin{aligned} D_{\widehat{w}}(f) &= \mathrm{LIM}(T_{\widehat{w}}(f)) = \mathop{\mathrm{LIM}}_{n\to\infty}\left(\frac{f(x_n + \tau_n v) - f(x_n)}{\tau_n}\right) \\ &= \lim_{n\to\infty} \frac{f(x_n + \tau_n v) - f(x_n)}{\tau_n} = f^0(x_0, v). \ \blacksquare \end{aligned}$$

In [5], p. 266, Sherbert points out that there is no strict analogy between the classical derivative of a function and point derivations and mentions the example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2 \sin\frac{1}{x}$ for $x \ne 0$ and $f(0) = 0$. This function is differentiable at $x_0 = 0$ but has at $x_0 = 0$ point derivations with values in the interval $[-1, 1]$ (see also Example 2.2.3 of [2]).

However, as pointed out in Proposition 2.2.4 of [2], there is a close analogy between Clarke's generalized derivative and the concept of strict differentiability in the sense of Bourbaki. More precisely, let $(E, \|\cdot\|_E)$ be a real normed vector space, $X \subset E$ an open subset, and $x_0 \in X$. Then a locally Lipschitz function $f : X \to \mathbb{R}$ is said to be *strictly differentiable* at $x_0 \in X$ in the sense of Bourbaki if there exists a continuous linear functional $l \in E^*$ such that

$$\lim_{\substack{x\to x_0\\ t\downarrow 0}} \frac{f(x+tv)-f(x)}{t} = l(v)$$

for all $v \in E$. The functional $l \in E^*$ is called the *strict derivative* of f at $x_0 \in X$.

In view of the relation between Clarke's generalized derivative and the concept of strict differentiability we obtain the following result.

PROPOSITION 3.4. *Let* $(E, \|\cdot\|_E)$ *be a real normed vector space*, $X \subset E$ *an open subset*, $x_0 \in X$ *and* $f \in \mathrm{Lip}(X, d)$. *Then the following statements are equivalent*:

(a) *There exists a linear functional* $l \in E^*$ *such that* $D(f) = l(v)$ *for every* $v \in E$ *and every* $D \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$.

(b) *The function* f *is strictly differentiable at* x_0 *in the sense of Bourbaki.*

Moreover, *if any of these statements holds*, *then* l *is the strict derivative of* f.

Proof. Suppose statement (a) holds. Then there exists a linear functional $l \in E^*$ such that $l(v) = D(f)$ for all $D \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$. Hence

$$\mathcal{D}_{\mathrm{cl}}|_{x_0,v}(f) = \sup_{D\in\partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0} D(f) = l(v),$$

which shows that Clarke's generalized derivative is linear in $v \in E$. Proposition 2.2.4 of [2] shows that statement (b) holds.

Conversely, suppose (b) holds. Using again Proposition 2.2.4 of [2] we deduce that $\mathcal{D}_{\mathrm{cl}}|_{x_0,v}(f)$ is a continuous linear function of v. Moreover, from Theorem 3.3 it follows that for all $v \in E$ and all $D, D' \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$ we have $D'(f) = D(f)$, since a Banach limit coincides on convergent sequences with the ordinary limit. Hence for all $D \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$ we have $D(f) = \mathcal{D}_{\mathrm{cl}}|_{x_0,v}(f)$, which means that $D(f)$ is constant for every $D \in \partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$. ■

The last proposition shows that the orthogonal complement of the linear span of the union of all sets $\partial(\mathcal{D}_{\mathrm{cl}}|_{x_0,v})|_0$ over $v \in E$ is the set of all Lipschitz functions which are strictly differentiable at x_0 in the sense of Bourbaki and have a critical point at x_0.

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References

[1] R. Arens and J. Eells, Jr., *On embedding uniform and topological spaces*, Pacific J. Math. 6 (1956), 397–403.
[2] F. H. Clarke, *Optimization and Nonsmooth Analysis*, CRM, Université de Montréal, 1989.
[3] N. Dunford and J. T. Schwartz, *Linear Operators: Part I*, Interscience Publ. New York, 1957.
[4] L. Hörmander, *Sur la fonction d'appui des ensembles convexes dans un espace localement convexe*, Ark. Mat. 3 (1954), 181–186.
[5] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964), 240–272.
[6] I. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. 129 (1955), 260–264.

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