APPLICATIONES MATHEMATICAE
24,4 (1997), pp. 445–455

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LINEARIZATION OF THE PRODUCT OF ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE

Abstract. Let $\{P_k\}$ be any sequence of classical orthogonal polynomials of a discrete variable. We give explicitly a recurrence relation (in k) for the coefficients in $P_iP_j = \sum_k c(i,j,k)P_k$, in terms of the coefficients σ and τ of the Pearson equation satisfied by the weight function ϱ, and the coefficients of the three-term recurrence relation and of two structure relations obeyed by $\{P_k\}$.

1. Introduction. Let $\{P_k(x)\}$ be any system of classical orthogonal polynomials of a discrete variable, i.e., Charlier polynomials $C_k(x;a)$, Meixner polynomials $M_k(x;\beta,c)$, Krawtchouk polynomials $K_k(x;p,N)$, or Hahn polynomials $Q_n(x;\alpha,\beta,N)$:

$$\sum_{x=0}^{B-1} \varrho(x)P_k(x)P_l(x) = \delta_{kl}h_k \qquad (k,l = 0,1,\ldots),$$

where $h_k > 0$ $(k = 0,1,\ldots)$; the set of orthogonality is $\{0,1,\ldots,B-1\}$, where B equals $+\infty$, $+\infty$, $N+1$ and N, respectively.

Askey and Gasper [2] have given explicit forms for the coefficients in

$$(1.1)\quad P_i(x)P_j(x) = \sum_{k=|i-j|}^{\min(i+j,B-1)} c_k^{ij} P_k(x) \qquad (i,j \ge 0; x \in \{0,1,\ldots,B-1\}),$$

called the *linearization coefficients* of the polynomials $\{P_k\}$ (see [1], Lecture 5), in terms of finite or infinite series.

1991 *Mathematics Subject Classification*: Primary 33C45, 33E30.
Key words and phrases: linearization coefficients; classical orthogonal polynomials of a discrete variable; recurrence relations.

The aim of this paper is to show that c_k^{ij} obey a linear recurrence relation

$$\mathcal{L}^* c_k^{ij} \equiv \sum_{h=0}^{r} A_h^*(k) c_{k+h}^{ij} = 0. \tag{1.2}$$

Recurrence (1.2) may serve as a basis for a very efficient backward recursion algorithm for evaluating these coefficients. The difference operator $\mathcal{L}^*$ is given explicitly in terms of the coefficients σ and τ of the Pearson equation (see (2.2) below) satisfied by the weight function ϱ, and the coefficients of the three-term recurrence relation (see (2.1)) and of structure relations obeyed by $\{P_k\}$ (see (2.5), (2.6)). This result is contained in Theorem 3.5; applications to some systems of polynomials are given.

The main tool used in the derivation of the recurrence relation is the fourth-order difference equation

$$\boldsymbol{Q}_4 w = 0, \tag{1.3}$$

obeyed by the product $w := P_i P_j$. We give a determinantal form (see Theorem 3.1), as well as two (equivalent) almost factorized forms of the fourth-order operator $\boldsymbol{Q}_4$ (see Corollary 3.2 and Theorem 3.4).

2. Properties of the classical orthogonal polynomials

2.1. *Basics of classical orthogonal polynomials of a discrete variable.* In the sequel, we make use of certain properties enjoyed by all classical families of orthogonal polynomials (see [4], Chapter VI; [5]; [6]; [9], Chapter II; or [10]). Besides the three-term recurrence relation

$$xP_k(x) = \xi_0(k)P_{k-1}(x) + \xi_1(k)P_k(x) + \xi_2(k)P_{k+1}(x) \qquad (k = 0, 1, \ldots;\ P_{-1}(x) \equiv 0,\ P_0(x) \equiv 1) \tag{2.1}$$

we need four other properties.

First, the weight function ϱ satisfies a difference equation of the Pearson type

$$\boldsymbol{\Delta}[\sigma(x)\varrho(x)] = \tau(x)\varrho(x), \tag{2.2}$$

where σ is a polynomial of degree at most 2, and τ is a first-degree polynomial.

Second, for arbitrary i, the polynomial P_i obeys the second order difference equation

$$\boldsymbol{P}_2^{(n)} P_i(x) \equiv \{\sigma(x)\boldsymbol{\Delta}\boldsymbol{\nabla} + \tau(x)\boldsymbol{\Delta} + \lambda_i \boldsymbol{I}\}P_i(x) = 0, \tag{2.3}$$

where $\boldsymbol{\Delta} := \boldsymbol{E} - \boldsymbol{I}$, $\boldsymbol{\nabla} := \boldsymbol{I} - \boldsymbol{E}^{-1}$, $\boldsymbol{E}^m$ ($m \in \mathbb{Z}$) is the mth shift operator, $\boldsymbol{E}^m f(x) = f(x+m)$, $\boldsymbol{I}$ is the identity operator, $\boldsymbol{I}f(x) = f(x)$, and λ_i is the constant given by

$$\lambda_i := -\tfrac{1}{2} i[(i-1)\sigma'' + 2\tau'] \qquad (i \in \mathbb{N}). \tag{2.4}$$

(By convention, all the bold letter operators act on the variable x.)

Third, we have a pair of the so-called *structure relations*,

$$[\sigma(x)+\tau(x)]\boldsymbol{\Delta} P_k(x) = d_0(k)P_{k-1}(x)+d_1(k)P_k(x)+d_2(k)P_{k+1}(x), \tag{2.5}$$

and

$$\sigma(x)\boldsymbol{\nabla} P_k(x) = d_0(k)P_{k-1}(x)+[d_1(k)+\lambda_k]P_k(x)+d_2(k)P_{k+1}(x). \tag{2.6}$$

Fourth,

$$\sigma(x)\varrho(x)x^k|_{x=0}^{x=B} = 0 \qquad (k=0,1,\ldots). \tag{2.7}$$

2.2. *Identities involving the discrete Fourier coefficients.* We shall need certain properties of the Fourier coefficients of an arbitrary polynomial f, $\deg f < B$, defined by

$$a_k[f] := \frac{1}{h_k} b_k[f] \qquad (k=0,1,\ldots,B-1), \tag{2.8}$$

where

$$b_k[f] := \sum_{x=0}^{B-1} \varrho(x)P_k(x)f(x) \tag{2.9}$$

i.e., the coefficients in the expansion

$$f = \sum_{k=0}^{\deg f} a_k[f]P_k.$$

Let $\mathcal{X}$, $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ be the difference operators (acting on k) defined by

$$\mathcal{X} := \xi_0(k)\mathcal{E}^{-1}+\xi_1(k)\mathcal{I}+\xi_2(k)\mathcal{E}, \tag{2.10}$$

$$\mathcal{D} := d_0(k)\mathcal{E}^{-1}+d_1(k)\mathcal{I}+d_2(k)\mathcal{E}, \tag{2.11}$$

$$\widetilde{\mathcal{D}} := \mathcal{D}+\lambda_k\mathcal{I} \tag{2.12}$$

(cf. (2.1), (2.5) and (2.6), respectively) where $\mathcal{I}$ is the identity operator, and $\mathcal{E}^m$ the mth shift operator: $\mathcal{I}b_k[f]=b_k[f]$, $\mathcal{E}^m b_k[f] = b_{k+m}[f]$ $(m\in\mathbb{Z})$. For the sake of simplicity, we write $\mathcal{E}$ in place of $\mathcal{E}^1$. (We adopt the convention that all the script letter operators act on the variable k.)

Further, define the difference operators $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{L}$ (acting on x) by

$$\boldsymbol{U} := \sigma(x)\boldsymbol{\nabla}+\tau(x)\boldsymbol{I}, \tag{2.13}$$

$$\boldsymbol{V} := [\sigma(x)+\tau(x)]\boldsymbol{\Delta}+\tau(x)\boldsymbol{I}, \tag{2.14}$$

$$\boldsymbol{L} := \boldsymbol{V}-\boldsymbol{U}, \tag{2.15}$$

respectively. Notice that since $\boldsymbol{\Delta}\boldsymbol{\nabla} = \boldsymbol{\Delta}-\boldsymbol{\nabla}$, we can write

$$\boldsymbol{P}_2^{(n)} = \boldsymbol{L}+\lambda_i\boldsymbol{I}. \tag{2.16}$$

Using (2.1)–(2.7), the following lemma can be proved.

LEMMA 2.1 ([8]). *The coefficients* (2.9) *obey the identities*:

$$b_k[qf] = q(\mathcal{X})b_k[f] \quad (q \text{ an arbitrary polynomial}),$$
$$b_k[\boldsymbol{U}f] = -\mathcal{D}b_k[f], \quad \widetilde{\mathcal{D}}b_k[\boldsymbol{\nabla}f] = \lambda_k b_k[f],$$
$$b_k[\boldsymbol{V}f] = -\widetilde{\mathcal{D}}b_k[f], \quad \mathcal{D}b_k[\boldsymbol{\Delta}f] = \lambda_k b_k[f],$$
$$b_k[\boldsymbol{L}f] = -\lambda_k b_k[f].$$

3. Main result

3.1. *Fourth-order difference equation for the product* P_iP_j. Using definitions (2.13) and (2.14), equation (2.3) can be written in the following equivalent form:

$$A(x)y(x+1) + B_n(x)y(x) + C(x)y(x-1) = 0, \tag{3.1}$$

with $y = P_n$, and

$$A := \sigma + \tau, \quad B_n := \lambda_n - 2\sigma - \tau, \quad C := \sigma. \tag{3.2}$$

In the sequel, we adopt the notation

$$f^{(m)}(x) := \boldsymbol{E}^m f(x) = f(x+m) \quad (m \in \mathbb{Z}). \tag{3.3}$$

The following theorem is a slightly improved version of a result of [7].

THEOREM 3.1. *The product* $w := P_iP_j$ $(i, j \geq 0,\ i \neq j)$ *satisfies the following difference equation of the fourth order*:

$$\boldsymbol{Q}_4 w \equiv \begin{vmatrix} C^{(1)}C^{(2)}\boldsymbol{R}_2 w & B_i & 1 \\ C^{(2)}\boldsymbol{R}_3 w & -B_j^{(1)} & 1 \\ \boldsymbol{R}_4 w & B_i^{(2)} & 1 \end{vmatrix} = 0, \tag{3.4}$$

where

$$\boldsymbol{R}_2 := A^2\boldsymbol{E} - B_iB_j\boldsymbol{I} - C^2\boldsymbol{E}^{-1}, \tag{3.5}$$
$$\boldsymbol{R}_3 := A\boldsymbol{E}\boldsymbol{R}_2 + F\boldsymbol{I}, \tag{3.6}$$
$$\boldsymbol{R}_4 := A\boldsymbol{E}\boldsymbol{R}_3 - G\boldsymbol{I}. \tag{3.7}$$

Here the notation used is in agreement with (3.3), *and*

$$F := C^{(1)}(B_iB_i^{(1)} + B_jB_j^{(1)}), \quad G := C^{(1)}C^{(2)}(B_iB_j^{(2)} + B_jB_i^{(2)}). \tag{3.8}$$

Proof. We have

$$AP_i^{(1)} + B_iP_i^{(0)} + CP_i^{(-1)} = 0, \tag{3.9}$$
$$AP_j^{(1)} + B_jP_j^{(0)} + CP_j^{(-1)} = 0. \tag{3.10}$$

Multiplying (3.9) by $AP_j^{(1)}$, and making use of (3.10), we obtain

$$\boldsymbol{R}_2 w = C[B_iP_i^{(0)}P_j^{(-1)} + B_jP_j^{(0)}P_i^{(-1)}] \tag{3.11}$$

with the operator $\boldsymbol{R}_2$ given by (3.5).

Applying the operator $\boldsymbol{A}\boldsymbol{E}$ to both sides of Eq. (3.11), and making use of (3.9) and (3.10), we get

$$\boldsymbol{R}_3 w = -CC^{(1)}[B_j^{(1)} P_i^{(0)} P_j^{(-1)} + B_i^{(1)} P_j^{(0)} P_i^{(-1)}] \tag{3.12}$$

with the operator $\boldsymbol{R}_3$ given by (3.6).

Repeating the above process for Eq. (3.12), we obtain

$$\boldsymbol{R}_4 w = CC^{(1)}C^{(2)}[B_i^{(2)} P_i^{(0)} P_j^{(-1)} + B_j^{(2)} P_j^{(0)} P_i^{(-1)}], \tag{3.13}$$

where the operator $\boldsymbol{R}_4$ is given by (3.7).

Eqs. (3.11), (3.12) and (3.13) imply

$$\begin{vmatrix} \boldsymbol{R}_2 w & B_i & B_j \\ \boldsymbol{R}_3 w & -C^{(1)} B_j^{(1)} & -C^{(1)} B_i^{(1)} \\ \boldsymbol{R}_4 w & C^{(1)}C^{(2)} B_i^{(2)} & C^{(1)}C^{(2)} B_j^{(2)} \end{vmatrix} = 0; \tag{3.14}$$

as $B_j^{(m)} = (\lambda_j - \lambda_i) + B_i^{(m)}$ (cf. (3.2)), this is equivalent to (3.4). ■

Corollary 3.2. *An equivalent form of the difference equation* (3.4) *is*

$$(\boldsymbol{S}_2 \boldsymbol{R}_2 + \boldsymbol{T}_1) w = 0, \tag{3.15}$$

where the difference operator $\boldsymbol{R}_2$ *is given in* (3.5), *and*

$$\boldsymbol{S}_2 := AA^{(1)} W_1 \boldsymbol{E}^2 + AC^{(2)} W_2 \boldsymbol{E} + C^{(1)}C^{(2)} W_3 \boldsymbol{I}, \tag{3.16}$$

$$\boldsymbol{T}_1 := AF^{(1)} W_1 \boldsymbol{E} + H\boldsymbol{I}. \tag{3.17}$$

Here we use the notation

$$W_1 := B_i + B_j^{(1)}, \quad W_2 := B_i^{(2)} - B_i, \quad W_3 := -B_i^{(2)} - B_j^{(1)},$$
$$H := C^{(2)} F W_2 - G W_1.$$

Proof. Expanding the determinant (3.4) with respect to the first column, we obtain

$$\boldsymbol{Q}_4 = C^{(1)}C^{(2)} W_3 \boldsymbol{R}_2 + C^{(2)} W_2 \boldsymbol{R}_3 + W_1 \boldsymbol{R}_4.$$

On using (3.6) and (3.7), and rearranging terms, the result follows. ■

If $i = j$, a slight modification of the argument given in the proof of Theorem 3.1 and Corollary 3.2 leads to the following result.

Theorem 3.3. *The square* $w := P_i^2$ $(i \in \mathbb{N})$ *obeys the third-order difference equation*

$$\boldsymbol{Q}_3 w \equiv \begin{vmatrix} C^{(1)} \boldsymbol{R}_2 w & B_i \\ \boldsymbol{R}_3 w & -B_i^{(1)} \end{vmatrix} = 0, \tag{3.18}$$

notation used being that of (3.5) *and* (3.6) (*with* $i = j$). *An equivalent form of this equation is*

$$(\boldsymbol{S}_1\boldsymbol{R}_2 + \boldsymbol{T}_0)w = 0, \tag{3.19}$$

where

$$\boldsymbol{R}_2 := A^2\boldsymbol{E} - B_i^2\boldsymbol{I} - C^2\boldsymbol{E}^{-1}, \tag{3.20}$$

$$\boldsymbol{S}_1 := AB_i\boldsymbol{E} + B_i^{(1)}C^{(1)}\boldsymbol{I}, \tag{3.21}$$

$$\boldsymbol{T}_0 := 2B_i^2B_i^{(1)}C^{(1)}\boldsymbol{I}. \tag{3.22}$$

In the next theorem, we give an alternative derivation of the fourth-order difference equation for P_iP_j. It should be stressed that this time the case of $i = j$ is not excluded.

THEOREM 3.4. *For any* i, $j \geq 0$, *the product* $w = P_iP_j$ *satisfies the fourth-order difference equation*

$$\widetilde{\boldsymbol{Q}}_4 w = 0 \tag{3.23}$$

with

$$\widetilde{\boldsymbol{Q}}_4 = \boldsymbol{N}_2\boldsymbol{M}_2 - \lambda_i\lambda_j\boldsymbol{K}_2, \tag{3.24}$$

where

$$\boldsymbol{N}_2 := \alpha(x)[\varphi_0(x)\boldsymbol{V} + \varphi_1(x)\boldsymbol{I}] - \beta(x)[\psi_0(x)\boldsymbol{U} + \psi_1(x)\boldsymbol{I}], \tag{3.25}$$

$$\boldsymbol{M}_2 := \boldsymbol{L} + (\lambda_i + \lambda_j)\boldsymbol{I}, \tag{3.26}$$

$$\boldsymbol{K}_2 := \alpha(x)[\boldsymbol{V} + \eta(x)\boldsymbol{I}] - \beta(x)[\boldsymbol{U} + \vartheta(x)\boldsymbol{I}], \tag{3.27}$$

and where

$$\begin{aligned} \alpha &:= A^{(-1)}[B_i + B_j + \boldsymbol{\nabla}(A + C)], & \psi_0 &:= C^{(-1)}, \\ \beta &:= C^{(1)}[B_i + B_j - \boldsymbol{\Delta}(A + C)], & \psi_1 &:= -A[A^{(-1)} + C^{(-1)}] - \tfrac{1}{2}\alpha, \\ \varphi_0 &:= A^{(1)} & \eta &:= C + C^{(1)}, \\ \varphi_1 &:= [A^{(1)} + C^{(1)}]C + \tfrac{1}{2}\beta, & \vartheta &:= -A - A^{(-1)}. \end{aligned} \tag{3.28}$$

Proof. Let $w := P_iP_j$. Using Leibniz' rules

$$\begin{cases} \boldsymbol{\Delta}(fg) = f\boldsymbol{\Delta}g + g^{(1)}\boldsymbol{\Delta}f, \\ \boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g^{(-1)}\boldsymbol{\nabla}f, \end{cases} \tag{3.29}$$

and the difference equations satisfied by P_i and P_j (cf. (2.3)), it can be checked that

$$\boldsymbol{M}_2 w = A\boldsymbol{\Delta}P_i\boldsymbol{\Delta}P_j + C\boldsymbol{\nabla}P_i\boldsymbol{\nabla}P_j, \tag{3.30}$$

where we use the notation (3.26) and (3.2). Using this result and the identity

$$C[\lambda_iP_i\boldsymbol{\nabla}P_j + \lambda_jP_j\boldsymbol{\nabla}P_i] - A[\lambda_iP_i\boldsymbol{\Delta}P_j + \lambda_jP_j\boldsymbol{\Delta}P_i] = 2\lambda_i\lambda_j w,$$

we obtain

$$(3.31)\quad A\boldsymbol{\Delta}(A\boldsymbol{M}_2w) = -(A^2 + CC^{(1)})\boldsymbol{M}_2w + \lambda_i\lambda_j[A\boldsymbol{\Delta} + (A + C^{(1)})\boldsymbol{I}]w - \beta A\boldsymbol{\Delta}P_i\boldsymbol{\Delta}P_j,$$

$$(3.32)\quad C\boldsymbol{\nabla}(C\boldsymbol{M}_2w) = (C^2 + AA^{(-1)})\boldsymbol{M}_2w + \lambda_i\lambda_j[C\boldsymbol{\nabla} - (C + A^{(-1)})\boldsymbol{I}]w + \alpha C\boldsymbol{\nabla}P_i\boldsymbol{\nabla}P_j.$$

On subtracting the equations (3.31) and (3.32), multiplied by α and β, respectively, and making use of (3.29) and (3.30), the result follows. ∎

3.2. *Recurrence relation for the linearization coefficients.* For some technical reasons, it is easier to construct a recurrence

$$(3.33)\qquad \mathcal{L}s_k^{ij} \equiv \sum_{h=0}^{r} A_h(k)s_{k+h}^{ij} = 0$$

for

$$(3.34)\qquad s_k^{ij} := \sum_{x=0}^{B-1} \varrho(x)P_i(x)P_j(x)P_k(x),$$

obviously equivalent to (1.2), in view of

$$(3.35)\qquad s_k^{ij} = h_k c_k^{ij}.$$

Now, we prove

THEOREM 3.5. *For arbitrary $i, j \ge 0$, the recurrence relation*

$$(3.36)\qquad \mathcal{L}s_k^{ij} = 0$$

holds, where

$$(3.37)\quad \mathcal{L} := \alpha(\mathcal{X})\{[\varphi_1(\mathcal{X}) - \varphi_0(\mathcal{X})\widetilde{\mathcal{D}}](\omega_k\mathcal{I}) - \lambda_i\lambda_j[\eta(\mathcal{X}) - \widetilde{\mathcal{D}}]\} - \beta(\mathcal{X})\{[\psi_1(\mathcal{X}) - \psi_0(\mathcal{X})\mathcal{D}](\omega_k\mathcal{I}) - \lambda_i\lambda_j[\vartheta(\mathcal{X}) - \mathcal{D}]\},$$

with $\omega_k := \lambda_i + \lambda_j - \lambda_k$, notation being that of (2.10)–(2.12), (3.28).

Proof. Let $w := P_iP_j$. Obviously,

$$s_k^{ij} = b_k[w], \qquad c_k^{ij} = a_k[w].$$

By virtue of Theorem 3.4,

$$b_k[\widetilde{\boldsymbol{Q}}_4w] = 0.$$

It suffices to show that the identity

$$b_k[\widetilde{\boldsymbol{Q}}_4w] = \mathcal{L}b_k[w]$$

holds. Now, observe that by Lemma 2.1, we have the following identities:

$$b_k[\boldsymbol{N}_2z] = \{\alpha(\mathcal{X})[\varphi_1(\mathcal{X}) - \varphi_0(\mathcal{X})\widetilde{\mathcal{D}}] - \beta(\mathcal{X})[\psi_1(\mathcal{X}) - \psi_0(\mathcal{X})\mathcal{D}]\}b_k[z],$$

$$b_k[\boldsymbol{M}_2w] = (\lambda_i + \lambda_j - \lambda_k)b_k[w],$$

$$b_k[\boldsymbol{K}_2 w] = \{\alpha(\mathcal{X})[\eta(\mathcal{X}) - \widetilde{\mathcal{D}}] - \beta(\mathcal{X})[\vartheta(\mathcal{X}) - \mathcal{D}]\} b_k[w].$$

From (3.24)–(3.27), applying again Lemma 2.1, we obtain (3.38). ∎

Obviously, we have the following.

COROLLARY 3.6. *The linearization coefficients* c_k^{ij} *in* (1.1) *obey the recurrence relation*

$$\mathcal{L}^* c_k^{ij} = 0 \tag{3.39}$$

with $\mathcal{L}^* := \mathcal{L}(h_k \mathcal{I})$, $\mathcal{L}$ *being the difference operator given in* (3.37).

EXAMPLE 3.7. The coefficients $\{c_k^{ij}\}$ in

$$C_i(x;a) C_j(x;a) = \sum_{k=|i-j|}^{i+j} c_k^{ij} C_k(x;a) \qquad (x \in \mathbb{N}_0),$$

where $C_m(x;a)$ is the mth monic Charlier polynomial (see Appendix, Table 1), satisfy the sixth-order recurrence relation

$$\sum_{h=-3}^{3} B_h(k) c_{k+h}^{ij} = 0 \qquad (|i-j| + 3 \le k \le i + j + 2),$$

with

$$\begin{aligned}
B_{-3}(k) &= 2(k - s - 3),\\
B_{-2}(k) &= (k - s - 2)(6k + 8a - s + 1) + 2ij,\\
B_{-1}(k) &= (k - s - 1)[6k^2 + 2(11a - s + 4)k - s + 1 + 2a(4a + 7)]\\
&\quad + ij(4k + 12a - s + 5),\\
B_0(k) &= (k - s)\{2k^3 + (7 - s + 20a)k^2\\
&\quad + 2(11a^2 + 23a + 3 - s)k + 2a^2(3s + 13) - a(s^2 - 25)\}\\
&\quad + ij[2k^2 + (7 - s + 22a)k + 6(a + 1)(4a + 1) - 2s(2a + 1)],\\
B_1(k) &= a(k - s + 1)\{6k^3 + 10(2a + 3)k^2 + [49 - s^2 + a(9s + 67) + 4a^2]k\\
&\quad + 4(s + 1)a^2 + (58 + 15s - s^2)a - 2s^2 + 26\}\\
&\quad + 2aij[5k^2 + 2(9 + 10a - s)k + 2(a + 2)(4a - s + 8) - 16],\\
B_2(k) &= a^2(k + 2)\{(k - s + 2)[3(k + 3)(2k + 3) + 3(s + 2a)(k + 1)\\
&\quad - (s - 6a - 4)(s + 1)] + 4ij(4k - s + 6a + 8)\},\\
B_3(k) &= 2a^3(k + 2)_2(k + i - j + 3)(k - i + j + 3),
\end{aligned}$$

where $s := i + j$. The initial conditions are $c_{i+j}^{ij} = 1$, and $c_m^{ij} = 0$ for $m > i + j$. Actual forms for B_h's were obtained using the computer algebra system MAPLE [3].

EXAMPLE 3.8. The coefficients $\{c_k^{ij}\}$ in

$$K_i(x;1/2,N)K_j(x;1/2,N) = \sum_{k=|i-j|}^{i+j} c_k^{ij} K_k(x;1/2,N) \quad (0 \le x \le N),$$

where $K_m(x;1/2,N)$ is a special case of the mth monic Krawtchouk polynomial (see Appendix, Table 2), satisfy the three-term recurrence relation

$$\begin{aligned} &16(k-s-2)(2N-s-k+2)c_{k-2}^{ij} \\ &\quad + 4[(k^2-d^2)(k-N-2)_2 - (k+1)_2(k-s)(2N-s-k)]c_k^{ij} \\ &\quad - (k+1)_2[(k+2)^2-d](k-N)_2 c_{k+2}^{ij} = 0 \quad (|i-j|+2 \le k \le i+j+1), \end{aligned}$$

where $s := i+j$, and $d := i-j$. The starting values are $c_{i+j}^{ij} = 1$, and $c_m^{ij} = 0$ for $m > i+j$. This result agrees with the explicit form given in [2].

Acknowledgments. A part of this work was done during a visit of one of the authors (S. L.) at the Université des Sciences et Technologies de Lille. He is very indebted to Professor Claude Brezinski, Directeur du Laboratoire d'Analyse Numérique et d'Optimisation, and Professor Jeannette Van Iseghem for the kind invitation and their warm hospitality.

Appendix

TABLE 1
Data for the monic Charlier and Hahn polynomials

	CHARLIER $C_k(x;a)$ $(a>0)$	HAHN $Q_k(x;\alpha,\beta,N)$ $(\alpha,\beta>-1, N\in\mathbb{Z}^+)$
σ	x	$x(N+\alpha-x)$
τ	$a-x$	$(\beta+1)(N-1)-(\gamma+1)x$
λ_k	k	$k(k+\gamma)$
$\mathcal{X}$	$ak\mathcal{E}^{-1}+(k+a)\mathcal{I}+\mathcal{E}$	$\frac{k(N-k)(k+\alpha)(k+\beta)(k+\gamma-1)(k+\gamma+N-1)}{(2k+\gamma-2)_2(2k+\gamma-1)_2}\mathcal{E}^{-1} + \left\{\frac{\alpha-\beta+2N-2}{4}+\frac{(\beta^2-\alpha^2)(\gamma+2N-1)}{4(2k+\gamma-1)(2k+\gamma+1)}\right\}\mathcal{I}+\mathcal{E}$
$\mathcal{D}$	$ak\mathcal{E}^{-1}$	$\frac{k(k+\alpha)(k+\beta)(k+\gamma-1)_2(N-k)(k+\gamma+N-1)}{(2k+\gamma-2)_2(2k+\gamma-1)_2}\mathcal{E}^{-1} - \frac{k(k+\gamma)[2k(k+\gamma)+(\gamma-\alpha)(\gamma-1)-N(\alpha-\beta)]}{(2k+\gamma-1)(2k+\gamma+1)}\mathcal{I}-k\mathcal{E}$
h_k	$k!a^k$	$\frac{k!\Gamma(k+\alpha+1)\Gamma(k+\beta+1)(2k+\gamma+1)_{N-k-1}}{(k+\gamma)_k(N-k-1)!}$

Note: $\gamma := \alpha+\beta+1$.

TABLE 2
Data for the monic Meixner and Krawtchouk polynomials

	Meixner $M_k(x;\beta,c)$ $(\beta>0, c\in(0,1))$	Krawtchouk $K_k(x;p,N)$ $(p\in(0,1),\ N\in\mathbb{Z}^+)$
σ	x	x
τ	$\beta c+(c-1)x$	$(1-p)^{-1}(Np-x)$
λ_k	$(1-c)k$	$(1-p)^{-1}k$
$\mathcal{X}$	$\frac{ck(k+\beta-1)}{(1-c)^2}\mathcal{E}^{-1}$	$p(1-p)k(N-k+1)\mathcal{E}^{-1}$
	$+\frac{[(c+1)k+\beta c]}{1-c}\mathcal{I}+\mathcal{E}$	$+[k+p(N-2k)]\mathcal{I}+\mathcal{E}$
$\mathcal{D}$	$\frac{ck(1-\beta-k)}{c-1}\mathcal{E}^{-1}+ck\mathcal{I}$	$pk(1+N-k)\mathcal{E}^{-1}-p(1-p)^{-1}k\mathcal{I}$
h_k	$\frac{k!(\beta)_k c^k}{(1-c)^{\beta+2k}}$	$\frac{N!k!}{(N-k)!}p^k(1-p)^k$

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Received on 7.11.1996