APPLICATIONES MATHEMATICAE
24,4 (1997), pp. 429–444

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DISPERSIVE FUNCTIONS AND STOCHASTIC ORDERS

Abstract. Generalizations of the hazard functions are proposed and general hazard rate orders are introduced. Some stochastic orders are defined as general ones. A unified derivation of relations between the dispersive order and some other orders of distributions is presented.

1. Introduction. Denote by ϕ, ψ, χ real functions, A, B, C, D intervals on the real line $\mathbb{R}$, X, Y, Z random variables, F, G, H their respective probability distribution functions and by f, g, h their respective density functions, if they exist. Define $\overline{F} = 1 - F$ and $F^{-1}(u) = \inf\{x : F(x) \ge u\}$, $u \in (0,1)$ (and analogously for G). We identify the distribution functions F, G, H with the respective probability distributions and denote their supports, which are intervals, by S_F, S_G, S_H respectively. We denote by $\phi\psi$ the superposition of functions ϕ and ψ: in particular, $G^{-1}F$ denotes the superposition of G^{-1} and F. We use *increasing* in place of *nondecreasing* and *decreasing* in place of *nonincreasing*.

Alzaid and Proschan (1992) have introduced the following definition.

Definition 1. A function $\phi : A \to \mathbb{R}$ is *dispersive* on the interval $A \subset \mathbb{R}$ if $\phi(x) - x$ is increasing on A.

It is easy to see that the superposition of two dispersive functions is also dispersive.

We give two theorems on dispersive functions which are modifications of two known results.

1991 *Mathematics Subject Classification*: 60E05, 62N05.

Key words and phrases: dispersive function, preservation theorem, hazard function, mean residual life, partial orders, stationary renewal distribution.

Research supported by KBN (State Committee for Scientific Research, Poland), under Grant 2 PO3A 003310.

Bartoszewicz (1987) has proved a theorem on the preservation of the dispersiveness under transformations. Its modified version is the following (see also Shaked and Shanthikumar (1994)).

THEOREM 1. *Let A, B, C, D be intervals on the real line $\mathbb{R}$ and $B \subset C \subset A$. Let $\phi : A \to B$ be a dispersive function on A such that there exists x_0 for which*

$$\phi(x) \le x \quad \text{if } x < x_0,$$

and

$$\phi(x) \ge x \quad \text{if } x > x_0.$$

If a function $\psi : C \to D$ is increasing (decreasing) concave (convex) on the set $\{x : x = \psi^{-1}(u) < x_0,\ u \in D\}$ and convex (concave) on the set $\{x : x = \psi^{-1}(u) > x_0,\ u \in D\}$, then the function $\psi\phi\psi^{-1}$ is dispersive on D.

We recall that $\phi : \mathbb{R} \to \mathbb{R}$ is *superadditive* (*subadditive*) on $A \subset \mathbb{R}$ if $\phi(x+y) \ge (\le)\ \phi(x) + \phi(y)$ for $x, y, x+y \in A$. It is well known that if ϕ is convex (concave), then it is superadditive (subadditive) and if ϕ is increasing superadditive (subadditive), then ϕ^{-1} is subadditive (superadditive). It is also well known that the superposition of two increasing superadditive (subadditive) functions is also superadditive (subadditive).

Theorem 2.3 of Ahmed *et al.* (1986) can be modified as follows (see also Bartoszewicz (1987)).

THEOREM 2. *Let $A \subset \mathbb{R}$ be an interval and $\phi : A \to \mathbb{R}$ be an increasing function such that there exists x_0 for which*

$$\lim_{x \to x_0 - 0} \frac{\phi(x)}{x} \le 1 \quad \text{and} \quad \lim_{x \to x_0 + 0} \frac{\phi(x)}{x} \ge 1.$$

If ϕ is subadditive for $x < x_0$ and superadditive for $x > x_0$, then ϕ is dispersive on A.

Recall the definitions of two stochastic orders. *F is stochastically smaller than G* ($F \le_{\mathrm{st}} G$ or $X \le_{\mathrm{st}} Y$) if $F(x) \ge G(x)$ for every x. *F is less dispersive than G* ($F \le_{\mathrm{disp}} G$ or $X \le_{\mathrm{disp}} Y$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \le G^{-1}(\beta) - G^{-1}(\alpha) \text{ whenever } 0 < \alpha < \beta < 1. \tag{1}$$

Shaked (1982) has proved that $F \le_{\mathrm{disp}} G$ if and only if $G^{-1}F(x) - x$ is increasing on S_F, i.e. $\phi = G^{-1}F$ is dispersive on S_F. Thus Theorem 1 may be treated as a preservation theorem for the dispersive order. In particular, we have the following statement (Bartoszewicz (1987), Shaked and Shanthikumar (1994)).

COROLLARY 1. *Let $F(0) = G(0) = 0$, S_F and S_G be intervals and ψ be an increasing convex function. If $X \le_{\mathrm{st}} Y$ and $X \le_{\mathrm{disp}} Y$, then $\psi(X) \le_{\mathrm{disp}} \psi(Y)$.*

Bartoszewicz (1987) has used Theorems 1 and 2 to prove some relations between the dispersive and the hazard rate orders of distributions. In this paper we remark that many other stochastic orders may be defined by dispersive functions. In Section 2 a general approach is proposed by introducing operators and general hazard functions. Particular cases are considered. Many orders of distributions are defined by dispersiveness of these hazard functions. Properties of the hazard operators are given in Section 3. Some relations between the dispersive and some other orders are established in Section 4.

2. Generalizations of the hazard function. Let $\mathcal{F}$ be the class of distributions with a common support S which is an interval. It is well known that the function $R_F(x) = -\log \overline{F}(x)$, $x \in S$, is the hazard function of $F \in \mathcal{F}$. If the density f exists, we have

$$R_F(x) = \int_{-\infty}^{x} r_F(t)\,dt,$$

where

$$r_F(t) = f(t)/\overline{F}(t) \tag{2}$$

is the *hazard rate function* of the distribution F. Thus F may be represented in the form

$$F(x) = 1 - e^{-R_F(x)}, \qquad x \in S. \tag{3}$$

We generalize (3) in two ways.

2.1. *Hazard functions defined on S.* Let $\mathcal{H}$ be a class of real functions defined on S. Consider a one-to-one mapping (operator) $\Lambda : \mathcal{F} \to \mathcal{H}$ with $\mathcal{H} = \Lambda(\mathcal{F})$. Denote by $\Lambda^{-1} : \mathcal{H} \to \mathcal{F}$ the inverse operator. Thus for every $F \in \mathcal{F}$ we have

$$F(x) = (\Lambda^{-1} \circ \Lambda(F))(x), \qquad x \in S,$$

where $\circ$ denotes the superposition of operators. By analogy we introduce the following definition.

Definition 2. The function $R_F(x) = \Lambda(F)(x)$, $x \in S$, is called the *Λ-hazard function* of the distribution $F \in \mathcal{F}$.

Also by analogy with the hazard rate order (see Example 1 below) we propose the following definition.

Definition 3. Let $F, G \in \mathcal{F}$. We say that F *is smaller than G in the Λ-order* ($F \le_\Lambda G$ or $X \le_\Lambda Y$) if $R_F(x) - R_G(x)$ is increasing on S.

It is easy to see that under general assumptions about Λ the Λ-order is a preorder. If R_G^{-1}, the inverse of R_G, exists and is increasing, then we may define the Λ-order in the following way:

$$F \le_{\Lambda} G \;\Leftrightarrow\; R_F R_G^{-1} \text{ is dispersive on } R_G(S).$$

Now we give examples of stochastic orders which may be defined as Λ-orders. Properties of these may be found in Shaked and Shanthikumar (1994). The orders defined in Examples 1, 3 and 4 are commonly called *uniform stochastic orders* and were studied in detail by Keilson and Sumita (1982) and Lynch *et al.* (1987).

EXAMPLE 1 (The hazard rate order). Let $\mathcal{F}$ be the class of continuous distributions with common support $S = [0, \infty)$. Let $F, G \in \mathcal{F}$. We say that F is *smaller than G in the hazard rate order* ($F \le_{\mathrm{hr}} G$ or $X \le_{\mathrm{hr}} Y$) if

$$\overline{F}/\overline{G} \text{ decreases in } x \in S. \tag{4}$$

If the densities f and g exist, then (4) holds if, and only if,

$$r_F(x) \ge r_G(x), \qquad x \in S,$$

where r_F and r_G are the hazard rate functions of F and G respectively, defined by (2). We define $\Lambda(F) = -\log \overline{F}$ for $F \in \mathcal{F}$, and $\Lambda^{-1}(\phi) = 1 - e^{-\phi}$ for $\phi \in \mathcal{H}$. Hence

$$F \le_{\Lambda} G \equiv F \le_{\mathrm{hr}} G.$$

EXAMPLE 2 (The generalized hazard rate order). Let $\mathcal{F}$ be the class of continuous distributions with a common support S which is an interval. Let H be a distinguished distribution from $\mathcal{F}$ and

$$\Lambda(F) = H^{-1}F.$$

We may call this Λ-order a *generalized hazard order* and denote it by $\le_{\mathrm{ghr}}$. The function $R_F(x) = H^{-1}F(x)$, $x \in S$, is called a *generalized hazard function*. If the density h exists, the function

$$\frac{d}{dx} R_F(x) = \frac{f(x)}{hH^{-1}F(x)}$$

is called a *generalized hazard rate function* of F (Barlow *et al.* (1972)). If $S = [0, \infty)$ and $H(x) = 1 - e^{-x}$, $x \ge 0$, we obtain the usual hazard rate function and the usual hazard rate order.

EXAMPLE 3 (The reversed hazard rate order). Let now $\mathcal{F}$ be the class of continuous distributions with common support $S = (-\infty, a]$, $a < \infty$. Let $F, G \in \mathcal{F}$. We say that F *is smaller than G in the reversed hazard rate order* ($F \le_{\mathrm{rh}} G$ or $X \le_{\mathrm{rh}} Y$) if

$$F(x)/G(x) \text{ decreases in } x \in S. \tag{5}$$

If the densities f and g exist, we may define the corresponding reversed hazard rate functions

$$\breve{r}_F(x) = \frac{f(x)}{F(x)} \quad \text{and} \quad \breve{r}_G(x) = \frac{g(x)}{G(x)}, \quad x \in S,$$

and then (5) holds if, and only if,

$$\breve{r}_F(x) \le \breve{r}_G(x), \quad x \in S.$$

We define $\Lambda(F) = -\log F$ for $F \in \mathcal{F}$, and $\Lambda^{-1}(\phi) = e^{-\phi}$ for $\phi \in \mathcal{H}$. Thus from (5) we have

$$F \le_\Lambda G \equiv F \le_{\rm rh} G.$$

Remark 1. The reversed hazard rate order may also be treated as the generalized hazard rate order defined in Example 2 with $H(x) = e^{x-a}$, $x \le a$.

Example 4 (The likelihood ratio order). Let $\mathcal{F}$ be the class of absolutely continuous distributions with respect to the Lebesgue measure with a common support S. Let $F, G \in \mathcal{F}$. We say that F *is smaller than* G *in the likelihood ratio order* ($F \le_{\rm lr} G$ or $X \le_{\rm lr} Y$) if

$$f(x)/g(x) \text{ decreases in } x \in S.$$

We define the operator $\Lambda : \mathcal{F} \to \mathcal{H}$ by

$$\Lambda(F) = -\log \frac{d}{dx} F, \quad F \in \mathcal{F}. \tag{6}$$

The inverse operator has the form

$$\Lambda^{-1}(\phi) = \int e^{-\phi}, \quad \phi \in \mathcal{H}.$$

Thus we have

$$F \le_\Lambda G \equiv F \le_{\rm lr} G.$$

Example 5 (The mean residual life order). Let $\mathcal{F}$ be the class of continuous distributions on $S = [0, \infty)$ with finite means. If X is a random variable with distribution $F \in \mathcal{F}$, we define the *mean residual life* of X at $x > 0$ as

$$m_F(x) = E[X - x \mid X > x] = \frac{\int_x^\infty \overline{F}(u)\,du}{\overline{F}(x)}. \tag{7}$$

Since $\mu_F = E(X) = \int_0^\infty \overline{F}(x)\,dx < \infty$ we have $m_F(x) < \infty$ for $x < \infty$. For a random variable Y with distribution function $G \in \mathcal{F}$, m_G and μ_G are similarly defined.

We say that F *is smaller than* G *in the mean residual life order* ($F \le_{\rm mrl} G$ or $X \le_{\rm mrl} Y$) if

$$m_F(x) \le m_G(x), \quad x > 0.$$

One can easily prove (Shaked and Shanthikumar (1994)) that

$$F \leq_{\mathrm{mrl}} G \;\Leftrightarrow\; \frac{\int_x^\infty \overline{F}(u)\,du}{\int_x^\infty \overline{G}(u)\,du} \text{ decreases in } x > 0.$$

We define the operator $\Lambda : \mathcal{F} \to \mathcal{H}$ by

$$\Lambda(F)(x) = R_F(x) = -\log \int_x^\infty \overline{F}(u)\,du, \qquad x > 0. \tag{8}$$

The inverse operator Λ^{-1} is of the form

$$\Lambda^{-1}(\phi) = 1 + \frac{d}{du} e^{-\phi}, \qquad \phi \in \mathcal{H}.$$

Thus we have

$$F \leq_\Lambda G \equiv F \leq_{\mathrm{mrl}} G.$$

Since R_G is increasing, $F \leq_{\mathrm{mrl}} G \Leftrightarrow R_F R_G^{-1}$ is dispersive on $(-\log \mu_G, \infty)$.

Example 6 (The memory order). Let $\mathcal{F}$ be the class of continuous distributions on $S = [0,\infty)$ with finite means and $F(0) = 0$ for all $F \in \mathcal{F}$. Let m_F and m_G be the respective mean residual lifes of distributions F and G from $\mathcal{F}$, defined by (7), and μ_F and μ_G their respective means. Following Ebrahimi and Zahedi (1992) we say that F *is smaller than* G *in the memory order* ($F \leq_{\mathrm{m}} G$ or $X \leq_{\mathrm{m}} Y$) if

$$\mu_F \geq \mu_G \quad \text{and} \quad m_F(x) - m_G(x) \text{ increases in } x > 0.$$

Ebrahimi and Zahedi (1992) have proved that $F \leq_{\mathrm{m}} G$ implies $G \leq_{\mathrm{mrl}} F$. We define the operator $\Lambda : \mathcal{F} \to \mathcal{H}$ by

$$\Lambda(F)(x) = R_F(x) = m_F(x) = \frac{\int_x^\infty \overline{F}(u)\,du}{\overline{F}(x)}.$$

Thus for $F, G \in \mathcal{F}$ with $\mu_F \geq \mu_G$, we have

$$F \leq_\Lambda G \equiv F \leq_{\mathrm{m}} G.$$

Example 7 (Monotone convex and concave orders). Let X and Y be two random variables and F and G be their respective distribution functions. We say that F *is smaller than* G *in the increasing convex* [*concave*] *order* ($F \leq_{\mathrm{icx}} G$ or $X \leq_{\mathrm{icx}} Y$ [$F \leq_{\mathrm{icv}} G$ or $X \leq_{\mathrm{icv}} Y$]) if

$$E[\phi(X)] \leq E[\phi(Y)] \text{ for all increasing convex [concave] functions } \phi : \mathbb{R} \to \mathbb{R}.$$

One can prove (Shaked and Shanthikumar (1994)) that

$$F \leq_{\mathrm{icx}} G \;\Leftrightarrow\; \int_x^\infty \overline{F}(u)\,du \leq \int_x^\infty \overline{G}(u)\,du \text{ for all } x$$

and

$$F \leq_{\mathrm{icv}} G \Leftrightarrow \int_{-\infty}^{x} F(u)\,du \geq \int_{-\infty}^{x} G(u)\,du \text{ for all } x.$$

It is easy to notice that we may define these orders as Λ-orders. For the increasing convex order we put

$$\Lambda(F)(x) = -\int_{-\infty}^{x}\int_{t}^{\infty} \overline{F}(u)\,du\,dt$$

and for the increasing concave order we put

$$\Lambda(F)(x) = \int_{-\infty}^{x}\int_{-\infty}^{t} F(u)\,du\,dt.$$

2.2. *Hazard functions defined on* $(0,1)$. Let $\widetilde{\Lambda} : \mathcal{F} \to \widetilde{\mathcal{H}}$ be a one-to-one operator, where $\widetilde{\mathcal{H}} = \widetilde{\Lambda}(\mathcal{F})$ is the class of real functions defined on (0,1). Denote by $\widetilde{\Lambda}^{-1} : \widetilde{\mathcal{H}} \to \mathcal{F}$ the inverse operator. We introduce the following definition.

Definition 5. The function $\widetilde{R}_F(u) = \widetilde{\Lambda}(F)(u)$, $u \in (0,1)$, is called the $\widetilde{\Lambda}$-*hazard function* of the distribution $F \in \mathcal{F}$.

We also propose the following definition.

Definition 6. Let $F, G \in \mathcal{F}$. We say that F *is smaller than* G *in the* $\widetilde{\Lambda}$-*order* ($F \leq_{\tilde{\Lambda}} G$ or $X \leq_{\tilde{\Lambda}} Y$) if $\widetilde{R}_G(u) - \widetilde{R}_F(u)$ is increasing on $(0,1)$.

Similarly to the Λ-order, it is easy to see that under general assumptions about $\widetilde{\Lambda}$ the $\widetilde{\Lambda}$-order is a preorder. If $\widetilde{R}_F^{-1}$, the inverse of $\widetilde{R}_F$, exists and is increasing, then we may define the $\widetilde{\Lambda}$-order as follows:

$$F \leq_{\tilde{\Lambda}} G \;\Leftrightarrow\; \widetilde{R}_G \widetilde{R}_F^{-1} \text{ is dispersive on } \widetilde{R}_F((0,1)). \tag{9}$$

Example 8 (The dispersive order). This order defined by (1) may also be defined as the $\widetilde{\Lambda}$-order for $\widetilde{\Lambda}(F) = F^{-1}$, i.e. $\widetilde{R}_F(u) = F^{-1}(u)$, $u \in (0,1)$. The inverse operator is $\widetilde{\Lambda}^{-1}(\phi) = \phi^{-1}$, $\phi \in \widetilde{\mathcal{H}}$. Thus (1) holds iff $\widetilde{R}_G(u) - \widetilde{R}_F(u)$ is increasing on $(0,1)$, i.e.

$$F \leq_{\tilde{\Lambda}} G \equiv F \leq_{\mathrm{disp}} G.$$

Example 9 (The star order). Let $\mathcal{F}$ be the class of continuous distributions on $(0,\infty)$ and F, $G \in \mathcal{F}$. We say that F *is smaller than* G *in the star order* ($F \leq_{*} G$ or $X \leq_{*} Y$) if $G^{-1}F$ is starshaped, i.e. $G^{-1}F(x)/x$ is increasing in $x > 0$, i.e. $G^{-1}(u)/F^{-1}(u)$ is increasing in $u \in (0,1)$. Putting $\widetilde{\Lambda}(F) = \log F^{-1}$ and $\widetilde{\Lambda}^{-1}(\phi) = \psi^{-1}$, where $\psi = e^{\phi}$, $\phi \in \widetilde{\mathcal{H}}$, we have

$$F \leq_{\tilde{\Lambda}} G \equiv F \leq_{*} G.$$

Example 10 (The convex transform order). We say that F *is smaller than* G *in the convex transform order* ($F \le_{\mathrm{c}} G$ or $X \le_{\mathrm{c}} Y$) if

$$G^{-1}F \text{ is convex on } S_F. \tag{10}$$

If F, $G \in \mathcal{F}$, the class of absolutely continuous distributions on an interval S, then (10) is equivalent to

$$\frac{fF^{-1}(u)}{gG^{-1}(u)} \quad \text{is increasing on } (0,1).$$

Thus putting $\widetilde{\Lambda}(F) = \log \frac{d}{du} F^{-1}$ and $\widetilde{\Lambda}^{-1}(\phi) = \psi^{-1}$, where $\psi = \int e^{\phi}$, $\phi \in \widetilde{\mathcal{H}}$, we have

$$F \le_{\tilde{\Lambda}} G \equiv F \le_{\mathrm{c}} G.$$

Example 11 (The s-order). Let F and G be continuous symmetric distributions on $\mathbb{R}$, i.e. $F(x) = 1 - F(-x)$, $x \in \mathbb{R}$ (and analogously for G). We say that F *is smaller than* G *in the* s-*order* ($F \le_{\mathrm{s}} G$ or $X \le_{\mathrm{s}} Y$) (van Zwet (1964)) if

$$G^{-1}F \text{ is concave on } (-\infty, 0) \text{ and convex on } (0, \infty). \tag{11}$$

If $F, G \in \mathcal{F}$, the class of absolutely continuous symmetric distributions on $\mathbb{R}$, then (11) is equivalent to

$$\frac{fF^{-1}(u)}{gG^{-1}(u)} \text{ is decreasing on } (0, 1/2) \text{ and increasing on } (1/2, 1).$$

Thus putting

$$\widetilde{\Lambda}(F)(u) = \widetilde{R}_F(u) = \begin{cases} -\log \frac{d}{du} F^{-1}(u), & u \in (0, 1/2), \\ \log \frac{d}{du} F^{-1}(u), & u \in (1/2, 1), \end{cases}$$

and $\widetilde{\Lambda}^{-1}(\phi) = \psi^{-1}$, where $\psi = \operatorname{sign}(\psi) \int e^{\phi}$, $\phi \in \widetilde{\mathcal{H}}$, we have for $F, G \in \mathcal{F}$,

$$F \le_{\tilde{\Lambda}} G \equiv F \le_{\mathrm{s}} G.$$

Example 12 (The Lorenz order). Let $\mathcal{F}$ be the class of distributions on $[0, \infty)$ with a common mean μ. Let $F, G \in \mathcal{F}$. We say that F *is smaller than* G *in the Lorenz order* ($F \le_{\mathrm{L}} G$ or $X \le_{\mathrm{L}} Y$) if

$$\int_0^x F^{-1}(t)\,dt \ge \int_0^x G^{-1}(t)\,dt, \qquad x \in [0,1].$$

Putting

$$\widetilde{\Lambda}(F)(u) = -\int_0^u \int_0^x F^{-1}(t)\,dt\,dx, \qquad u \in (0,1),$$

and $\widetilde{\Lambda}^{-1}(\phi) = \psi^{-1}$, where $\psi = -\frac{d^2}{du^2}\phi$, $\phi \in \widetilde{\mathcal{H}}$, we have

$$F \le_{\tilde{\Lambda}} G \equiv F \le_{\mathrm{L}} G.$$

EXAMPLE 13 (The usual stochastic order). The order $\le_{\mathrm{st}}$ may be defined as a Λ-order and a $\widetilde{\Lambda}$-order as well. Let $\mathcal{F}$ be the class of distribution functions with the common support $S = [a, \infty)$, $a > -\infty$. We define the operator $\Lambda : \mathcal{F} \to \mathcal{H}$ by

$$\Lambda(F)(x) = R_F(x) = \int_a^x F(u)\, du, \qquad x \in S.$$

The inverse operator is $\Lambda^{-1}(\phi) = \frac{d}{du}\phi$, $\phi \in \mathcal{H}$. Thus for $F, G \in \mathcal{F}$,

$$F(x) \ge G(x) \text{ for all } x \in S \;\Leftrightarrow\; R_F(x) - R_G(x) \text{ increases in } x \in S,$$

i.e.

$$F \le_\Lambda G \equiv F \le_{\mathrm{st}} G.$$

It is clear that $F \le_{\mathrm{st}} G$ if and only if $F^{-1}(u) \le G^{-1}(u)$ for all $u \in (0,1)$. Define the operator $\widetilde{\Lambda} : \mathcal{F} \to \widetilde{\mathcal{H}}$ by

$$\widetilde{\Lambda}(F)(u) = \widetilde{R}_F(u) = \int_0^u F^{-1}(t)\, dt, \qquad u \in (0,1).$$

The inverse operator is $\widetilde{\Lambda}^{-1}(F) = \psi^{-1}$, where $\psi = \frac{d}{du}\phi$, $\phi \in \widetilde{\mathcal{H}}$. Thus for $F, G \in \mathcal{F}$,

$$F(x) \ge G(x) \text{ for all } x \in S \Leftrightarrow \widetilde{R}_G(u) - \widetilde{R}_F(u) \text{ increases in } u \in (0,1),$$

i.e.

$$F \le_{\widetilde{\Lambda}} G \equiv F \le_{\mathrm{st}} G.$$

3. Preservation theorems. Let $\mathcal{F}$ be a class of distributions on which the operators Λ and $\widetilde{\Lambda}$ are defined. It is easy to see that if

$$R_F = \Lambda(F) = \psi F, \qquad F \in \mathcal{F},$$

where ψ is a function for which ψ^{-1} exists, then for $F, G \in \mathcal{F}$ we have

$$R_G^{-1} R_F = G^{-1} F. \tag{12}$$

Similarly, if

$$\widetilde{R}_F = \widetilde{\Lambda}(F) = \psi F^{-1}, \qquad F \in \mathcal{F},$$

and ψ^{-1} exists, then

$$\widetilde{R}_G^{-1} \widetilde{R}_F = G F^{-1}.$$

The following lemma gives conditions under which (12) holds.

LEMMA 1. *Let $F, G \in \mathcal{F}$, Λ and $\widetilde{\Lambda}$ be defined on $\mathcal{F}$ such that R_F^{-1}, R_G^{-1}, $\widetilde{R}_F^{-1}$, $\widetilde{R}_G^{-1}$ exist. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function for which $F\phi^{-1} \in \mathcal{F}$. Then*:

(a) $R_G^{-1} R_F = G^{-1} F$ *if*, *and only if*,

$$\Lambda(F\phi^{-1}) = \Lambda(F)\phi^{-1}; \tag{13}$$

(b) $\widetilde{R}_G\widetilde{R}_F^{-1} = G^{-1}F$ *if*, *and only if*,

$$\widetilde{\Lambda}(F\phi^{-1}) = \phi\widetilde{\Lambda}(F). \tag{14}$$

Proof. We prove the statement (a) only. The proof of (b) is similar.

Sufficiency. Assume (13) holds. Then for $\phi = R_G^{-1}R_F$ we have

$$\Lambda(FR_F^{-1}R_G) = \Lambda(F)R_F^{-1}R_G = R_FR_F^{-1}R_G = R_G = \Lambda(G).$$

Therefore we have $FR_F^{-1}R_G = G$, i.e. $R_G^{-1}R_F = G^{-1}F$.

Necessity. Assume (12) holds for every F, $G \in \mathcal{F}$. From (12) we have

$$\Lambda(G) = \Lambda(F)F^{-1}G. \tag{15}$$

Putting $G := F\phi^{-1}$ in (15) we have

$$\Lambda(F\phi^{-1}) = \Lambda(F)F^{-1}F\phi^{-1} = \Lambda(F)\phi^{-1}.$$

From Lemma 1 and Definition 1 we obtain the following corollaries.

COROLLARY 2. *Let $F, G \in \mathcal{F}$, Λ satisfy* (13) *and $\mathcal{H} = \Lambda(\mathcal{F})$ be a class of strictly increasing functions. Then*

$$F \le_{\text{disp}} G \;\Leftrightarrow\; R_G^{-1}R_F \textit{ is dispersive on } S.$$

COROLLARY 3. *Let $F, G \in \mathcal{F}$ and Λ satisfy* (13). *If R_F^{-1} and R_G^{-1} exist and are increasing*, *then*

$$X \le_\Lambda Y \;\Leftrightarrow\; R_F(X) \le_{\text{disp}} R_F(Y) \;\Leftrightarrow\; R_G(X) \le_{\text{disp}} R_G(Y).$$

Proof. Since $F_0 = FR_F^{-1}$ and $G_0 = GR_F^{-1}$ are the distribution functions of $R_F(X)$ and $R_F(Y)$ respectively, we have $R_FR_G^{-1} = R_FR_G^{-1}R_FR_F^{-1} = R_FG^{-1}FR_F^{-1} = G_0^{-1}F_0$.

Similarly we prove the second equivalence.

Immediately from Lemma 1 and Theorems 1 and 2 we obtain the following results.

COROLLARY 4. *Let $F, G \in \mathcal{F}$, Λ satisfy* (13) *and $\mathcal{H} = \Lambda(\mathcal{F})$ be a class of strictly increasing functions.*

(a) *If $\phi = R_FR_G^{-1}$ and $\psi = R_G^{-1}$ (or $\psi = R_F^{-1}$) satisfy the assumptions of Theorem* 1 *with $A = C = R_G(S)$, $B = R_F(S)$ and $D = S$, then*

$$F \le_\Lambda G \;\Rightarrow\; F \le_{\text{disp}} G.$$

(b) *If $\phi = R_G^{-1}R_F$ and $\psi = R_F$ (or $\psi = R_G$) satisfy the assumptions of Theorem* 1 *with $A = B = C = S$ and $D = R_F(S)$ (or $D = R_G(S)$), then*

$$F \le_{\text{disp}} G \;\Rightarrow\; F \le_\Lambda G.$$

COROLLARY 5. *Let $F, G \in \mathcal{F}$, Λ satisfy* (13) *and $\mathcal{H} = \Lambda(\mathcal{F})$ be a class of strictly increasing functions. Let $x_0 \in \overline{S}$ be a point such that*

$$\lim_{x\to x_0-0} \frac{G^{-1}F(x)}{x} \le 1 \quad \text{and} \quad \lim_{x\to x_0+0} \frac{G^{-1}F(x)}{x} \ge 1.$$

If R_F is subadditive for $x < x_0$ and superadditive for $x > x_0$ and R_G is superadditive for $x < x_0$ and subadditive for $x > x_0$, then

$$F \le_{\mathrm{disp}} G \ \Leftrightarrow\ F \le_{\Lambda} G.$$

Similar corollaries may be formulated for the $\widetilde{\Lambda}$-order, putting in Corollaries 4 and 5: $R_F := \widetilde{R}_F^{-1}$ and $R_G := \widetilde{R}_G^{-1}$ and assuming that $\widetilde{\Lambda}$ satisfies (14).

4. The relation between the dispersive and some other stochastic orders. Some Λ- and $\widetilde{\Lambda}$-hazard functions satisfy the condition (13) or (14) and hence, applying Lemma 1 and Corollaries 4 and 5, we may obtain particular relations between the dispersive order and some other stochastic orders.

4.1. *The generalized hazard rate order*. Consider the class $\mathcal{F}$ of continuous distributions with the common support $S = [0, \infty)$ and let $H \in \mathcal{F}$ be a distinguished distribution. It is easy to see that $\Lambda(F) = H^{-1}F$ satisfies (13). Expressing the convexity of R_F in terms of the convex transform order, i.e. $F \le_{\mathrm{c}} H$ (see Example 10) we may reformulate Corollary 4 as follows.

COROLLARY 4.1. *Let $F, G \in \mathcal{F}$.*

(a) *If $F \le_{\mathrm{ghr}} G$ and $H \le_{\mathrm{c}} F$ or $H \le_{\mathrm{c}} G$, then $F \le_{\mathrm{disp}} G$.*
(b) *If $F \le_{\mathrm{disp}} G$ and $F \le_{\mathrm{c}} H$ or $G \le_{\mathrm{c}} H$, then $F \le_{\mathrm{ghr}} G$.*

Recall now the following definition (Barlow and Proschan (1975)). We say that F *is smaller than H in the superadditive order* ($F \le_{\mathrm{su}} G$ or $X \le_{\mathrm{su}} Y$) if $H^{-1}F$ is superadditive on S_F. Thus Corollary 5 may be stated in the following form.

COROLLARY 5.1. *Let $F, G \in \mathcal{F}$. If $F \le_{\mathrm{su}} H$ and $H \le_{\mathrm{su}} G$, then*

$$F \le_{\mathrm{disp}} G \Leftrightarrow F \le_{\mathrm{ghr}} G.$$

Let $\mathcal{F}_0$ be the class of continuous distributions on $\mathbb{R}$ symmetric with respect to the origin and let $H \in \mathcal{F}_0$ be a distinguished distribution. It is easy to see that for $F, G \in \mathcal{F}_0$, $F \le_{\mathrm{ghr}} G$ implies $F(x) \le G(x)$ for $x \le 0$ and $F(x) \ge G(x)$ for $x \ge 0$. Expressing the concavity and convexity of R_F in terms of the s-order (see Example 11) we may now formulate Corollary 4 in the following form.

Corollary 4.2. *Let* $F, G \in \mathcal{F}_0$.

(a) *If* $F \le_{\mathrm{ghr}} G$ *and* $H \le_{\mathrm{s}} F$ *or* $H \le_{\mathrm{s}} G$, *then* $F \le_{\mathrm{disp}} G$.
(b) *If* $F \le_{\mathrm{disp}} G$ *and* $F \le_{\mathrm{s}} H$ *or* $G \le_{\mathrm{s}} H$, *then* $F \le_{\mathrm{ghr}} G$.

4.2. *The hazard rate order*. Let $\mathcal{F}$ be the class of continuous distributions on $[0,\infty)$. Putting $H(x) = 1-e^{-x}$, $x \ge 0$, in Section 4.1 we obtain results for the usual hazard rate order. It is well known (Barlow and Proschan (1975)) that if $\log \overline{F}$ is convex ($H \le_{\mathrm{c}} F$), then F is a DFR (*decreasing failure rate*) distribution and if $\log \overline{F}$ is concave ($F \le_{\mathrm{c}} H$), then F is an IFR (*increasing failure rate*) distribution. If $\log \overline{F}$ is superadditive ($H \le_{\mathrm{su}} F$), then F is an NWU (*new worse than used*) distribution and if $\log \overline{F}$ is subadditive ($F \le_{\mathrm{su}} H$), then F is an NBU (*new better than used*) distribution. Thus Corollary 4 may be formulated for this particular case as follows.

Corollary 4.3 (Bartoszewicz (1987)). *Let* $F, G \in \mathcal{F}$.

(a) *If* $F \le_{\mathrm{hr}} G$ *and* F *or* G *is DFR*, *then* $F \le_{\mathrm{disp}} G$.
(b) *If* $F \le_{\mathrm{disp}} G$ *and* F *or* G *is IFR*, *then* $F \le_{\mathrm{hr}} G$.

Similarly Corollary 5 may be stated in the following form.

Corollary 5.2 (Bartoszewicz (1987)). *Let* $F, G \in \mathcal{F}$. *If* F *is NBU and* G *is NWU*, *then*

$$F \le_{\mathrm{disp}} G \iff F \le_{\mathrm{hr}} G.$$

Remark 2. It seems interesting to compare Corollary 4.3(a) with the following result.

Theorem 3 (Bartoszewicz (1985)). *If* $S_F = [0, a_1]$, $S_G = [0, a_2]$, $0 < a_1 \le a_2 \le \infty$ *and* $F \le_{\mathrm{c}} G$ *or* $S_F = [0,\infty)$, $S_G = [b,\infty)$, $G(b) = 0$, $b \ge 0$ *and* $G \le_{\mathrm{c}} F$, *then*

$$F \le_{\mathrm{st}} G \Rightarrow F \le_{\mathrm{disp}} G.$$

4.3. *The reversed hazard rate order*. Consider the class $\mathcal{F}$ of distributions and the reversed hazard rate order defined in Example 3. We say that $F \in \mathcal{F}$ has *increasing reversed failure rate* (F is IRFR) if $\log F$ is convex, and $F \in \mathcal{F}$ has *decreasing reversed failure rate* (F is DRFR) if $\log F$ is concave. Since $\Lambda(F)$ satisfies (13), after some modifications ($\Lambda(F) = -\log F$ is decreasing), we obtain from Corollary 4 the following result.

Corollary 4.4. *Let* $F, G \in \mathcal{F}$.

(a) *If* $F \le_{\mathrm{rh}} G$ *and* F *or* G *is IRFR*, *then* $G \le_{\mathrm{disp}} F$.
(b) *If* $F \le_{\mathrm{disp}} G$ *and* F *or* G *is DRFR*, *then* $G \le_{\mathrm{rh}} F$.

4.4. *The mean residual life order*. Consider now the class $\mathcal{F}$ of distributions defined in Example 5 and the mean residual life order defined as the Λ-order with $\Lambda(F)$ given by (8). We say that F is a DMRL (*decreasing mean residual life*) distribution if m_F is decreasing, i.e. R_F is convex. We say that F is an IMRL (*increasing mean residual life*) distribution if m_F is increasing, i.e. R_F is concave. It is well known (see e.g. Hollander and Proschan (1984)) that if F is IFR, then F is DMRL, and if F is DFR, then F is IMRL. Unfortunately, $\Lambda(F)$ does not satisfy (13) and we cannot obtain relations between the mean residual life and the dispersive orders directly from Corollary 4. Notice, however, that

$$R_F(x) = -\log \int_x^\infty \overline{F}(u)du = -\log\Big[\mu_F - \int_0^x \overline{F}(u)\,du\Big]$$
$$= -\log[1-F^*(x)] - \log\mu_F = -\log\overline{F^*}(x) - \log\mu_F,$$

where

$$F^*(x) = \frac{1}{\mu_F}\int_0^x \overline{F}(u)\,du$$

is the *stationary renewal distribution* corresponding to F (Barlow and Proschan (1975)). G^* is defined similarly. It is obvious that

$$m_F(x) = \frac{\mu_F\overline{F^*}}{\overline{F}(x)} = \frac{1}{r_{F^*}(x)}, \qquad x \in S,$$

and thus

$$F \le_{\mathrm{mrl}} G \equiv F^* \le_{\mathrm{hr}} G^*, \tag{16}$$

i.e.

$$F \le_\Lambda G \equiv F^* \le_{\mathrm{hr}} G^*.$$

From (16) and Corollary 4.3 we have the following implications.

Lemma 2.

(a) *F is IMRL $\Leftrightarrow F^*$ is DFR $\Rightarrow F^*$ is IMRL*;
(b) *F is DMRL $\Leftrightarrow F^*$ is IFR $\Rightarrow F^*$ is DMRL*;
(c) *F is IMRL $\Leftrightarrow F \le_{\mathrm{hr}} F^*$*;
(d) *F is DMRL $\Leftrightarrow F^* \le_{\mathrm{hr}} F$*;
(e) *F is IMRL $\Rightarrow F \le_{\mathrm{disp}} F^*$.*

We also have the following lemma.

Lemma 3. *Let $F, G \in \mathcal{F}$. If F is DMRL and G is IMRL, then*

$$F \le_{\mathrm{st}} G \;\Rightarrow\; F^* \le_{\mathrm{disp}} G^*$$

and also

$$F \le_{\mathrm{disp}} G \;\Rightarrow\; F^* \le_{\mathrm{disp}} G^*. \tag{17}$$

Proof. If F is DMRL and G is IMRL, then from Lemma 2(d) we have $F^* \le_{\mathrm{st}} F$ and $G \le_{\mathrm{st}} G^*$. Thus $F^* \le_{\mathrm{st}} G^*$. From Lemma 2(b) we have $F^* \le_{\mathrm{c}} K$, where K is the exponential distribution and also $K \le_{\mathrm{c}} G^*$.

Thus $F^* \le_{\mathrm{c}} G^*$. Therefore by Theorem 3 we have $F^* \le_{\mathrm{disp}} G^*$. Since $F \le_{\mathrm{disp}} G \Rightarrow F \le_{\mathrm{st}} G$, (17) also holds.

Directly from (16), Corollary 4.3 and Lemma 3 we obtain the relation between the mean residual life and the dispersive orders.

Corollary 4.5. *Let $F, G \in \mathcal{F}$.*

(a) *If $F \le_{\mathrm{mrl}} G$ and F or G is IMRL, then $F^* \le_{\mathrm{disp}} G^*$.*
(b) *If $F^* \le_{\mathrm{disp}} G^*$ and F or G is DMRL, then $F \le_{\mathrm{mrl}} G$.*
(c) *If $F \le_{\mathrm{disp}} G$ and F is DMRL and G is IMRL, then $F \le_{\mathrm{mrl}} G$.*

4.5. *The likelihood ratio order*. We consider the particular case presented in Example 4. Let $\mathcal{F}_1$ be the class of absolutely continuous distributions with respect to the Lebesgue measure with common support $S = [0, \infty)$ and with decreasing densities. Let $\Lambda : \mathcal{F}_1 \to \mathcal{H}$ be defined by (6). Thus for $F, G \in \mathcal{F}_1$ the functions R_F^{-1} and R_G^{-1} exist. As in Section 3.3, $\Lambda(F)$ does not satisfy (13) and also we cannot obtain relations between the likelihood ratio and the dispersive orders in the class $\mathcal{F}_1$ directly from Corollary 4. However, notice that

$$\begin{aligned} R_F(x) &= -\log f(x) = -\log \frac{f(x)}{f(0)} - \log f(0) \\ &= -\log \overline{F_*}(x) - \log f(0), \end{aligned}$$

where

$$F_*(x) = 1 - \frac{f(x)}{f(0)}, \qquad x \in S,$$

is a distribution function. It is easy to notice that $\mu_{F_*} = 1/f(0)$ and

$$F(x) = \frac{1}{\mu_{F_*}} \int_0^x \overline{F_*}(u)\, du,$$

so F is the stationary renewal distribution corresponding to F_* (G_* is similarly defined). Since

$$\frac{f(x)}{g(x)} = \frac{f(0)}{g(0)} \cdot \frac{\overline{F_*}(x)}{\overline{G_*}(x)},$$

we have

$$F \le_{\mathrm{lr}} G \equiv F_* \le_{\mathrm{hr}} G_*, \tag{18}$$

i.e.

$$F \le_{\Lambda} G \equiv F_* \le_{\mathrm{hr}} G_*.$$

One can easily verify the following implications.

Lemma 4. (a) *$\log f$ is convex $\Leftrightarrow$ F_* is DFR $\Rightarrow$ F is DFR*;
(b) *$\log f$ is concave $\Leftrightarrow$ F_* is IFR $\Rightarrow$ F is IFR*;
(c) *F_* is IMRL $\Leftrightarrow$ F is DFR*;

(d) F_* *is DMRL* $\Leftrightarrow F$ *is IFR*;
(e) F *is DFR* $\Leftrightarrow F_* \le_{\mathrm{hr}} F$;
(f) F *is IFR* $\Leftrightarrow F \le_{\mathrm{hr}} F_*$;
(g) $\log f$ *is convex* $\Rightarrow F_* \le_{\mathrm{disp}} F$.

Putting in Lemma 3: $F := F_*$, $G := G_*$ and $F^* := F$, $G^* := G$ we obtain the following statement.

LEMMA 5. *Let* $F, G \in \mathcal{F}_1$. *If* F *is IFR and* G *is DFR, then*

$$F_* \le_{\mathrm{st}} G_* \Rightarrow F \le_{\mathrm{disp}} G$$

and also

$$F_* \le_{\mathrm{disp}} G_* \Rightarrow F \le_{\mathrm{disp}} G.$$

The relation between the likelihood ratio and the dispersive orders in the class $\mathcal{F}_1$ follows from (18), Corollary 4.3 and Lemma 5.

COROLLARY 4.6. *Let* $F, G \in \mathcal{F}_1$.

(a) *If* $F \le_{\mathrm{lr}} G$ *and* $\log f$ *or* $\log g$ *is convex, then* $F_* \le_{\mathrm{disp}} G_*$.
(b) *If* $F_* \le_{\mathrm{disp}} G_*$ *and* $\log f$ *or* $\log g$ *is concave, then* $F \le_{\mathrm{lr}} G$.
(c) *If* $F \le_{\mathrm{lr}} G$, $\log f$ *is concave and* $\log g$ *is convex, then* $F \le_{\mathrm{disp}} G$.

Remark 3. It is well known that if f is logarithmically concave (convex), then F is IFR (DFR) (Barlow and Proschan (1975)). If F is IFR (DFR), then F is DMRL (IMRL) and also NBU (NWU) (Hollander and Proschan (1984)). On the other hand, for distributions F, G with the common support $S = [0, \infty)$, we have the implications (Shaked and Shanthikumar (1994))

$$\begin{array}{ccccc} F \le_{\mathrm{lr}} G & \Rightarrow & F \le_{\mathrm{hr}} G & \Rightarrow & F \le_{\mathrm{mrl}} G \\ & & \Downarrow & & \\ F \le_{\mathrm{disp}} G & \Rightarrow & F \le_{\mathrm{st}} G & & \end{array}$$

Comparing Corollaries 4.3(b), 4.5(b)–(c) and 4.6(b) we see that, as usual, stronger assumptions imply stronger results.

4.6. *The star order*. Consider the star order defined in Example 9 and $\mathcal{F}$ the class of continuous distributions on $(0, \infty)$. For $F, G \in \mathcal{F}$ we have

$$F \le_* G \ \Leftrightarrow\ \widetilde{R}_G(u) - \widetilde{R}_F(u) \text{ increases in } u \in (0,1). \tag{19}$$

Since $\widetilde{R}_F(u) = \log F^{-1}$ is increasing, (19) is equivalent to

$$F \le_* G \ \Leftrightarrow\ \widetilde{R}_G \widetilde{R}_F^{-1}(x) \text{ is dispersive.}$$

Assume that $F \le_{\mathrm{st}} G$ and $F \le_* G$. Applying Theorem 1 to the functions $\phi(x) = \widetilde{R}_G \widetilde{R}_F^{-1}(x) = \log G^{-1}F(e^x)$ and $\psi(x) = e^x$ we find that $\psi\phi\psi^{-1} = G^{-1}F$ is dispersive, i.e. $F \le_{\mathrm{disp}} G$. Thus we obtain the following known result (Sathe (1984), Shaked and Shanthikumar (1994)).

COROLLARY 6. *Let* $F, G \in \mathcal{F}$. *If* $F \le_{\mathrm{st}} G$ *and* $F \le_* G$, *then* $F \le_{\mathrm{disp}} G$.

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Received on 4.10.1996