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A NOTE ON THE CHARACTERIZATION OF SOME MINIFICATION PROCESSES

Abstract. We present a stochastic model which yields a stationary Markov process whose invariant distribution is maximum stable with respect to the geometrically distributed sample size. In particular, we obtain the autoregressive Pareto processes and the autoregressive logistic processes introduced earlier by Yeh *et al.* [9] and Arnold and Robertson [2].

1. Introduction. Let $\{\varepsilon_n,\, n \ge 0\}$ be a sequence of independent identically distributed (i.i.d.) random variables with common distribution function F. Define a Markov process $\{X_n,\, n \ge 0\}$ by $X_0 = \varepsilon_0$ and for $n \ge 1$,

$$(1)\qquad X_n = \begin{cases} \dfrac{X_{n-1} - b(p)}{a(p)} & \text{with probability } p, \\ \min\left\{\dfrac{X_{n-1} - b(p)}{a(p)}, \varepsilon_n\right\} & \text{with probability } q = 1 - p, \end{cases}$$

for some $a(p) > 0$ and $b(p)$, $0 < p < 1$. The sequence $\{\varepsilon_n\}$ is often referred to as an *innovation process*. Because of the structure of (1) the process $\{X_n\}$ is called a *minification process with the zero-defect* (Lewis and McKenzie [6], Kalamkar [5], Arnold and Hallett [1], Gaver and Lewis [3]).

The process $\{X_n\}$ defined by (1) is a stationary Markov process if and only if the distribution function

$$F(x) = P(\varepsilon_0 \le x)$$

satisfies

$$(2)\quad F(x) = pF(a(p)x + b(p)) + q[1 - \{1 - F(a(p)x + b(p))\} \cdot \{1 - F(x)\}], \quad -\infty < x < \infty.$$

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The equation (2) can be written as

$$F(a(p)x+b(p)) = \frac{pF(x)}{1-qF(x)}, \qquad -\infty < x < \infty, \tag{3}$$

where $a(p) > 0$ and $b(p)$, $0 < p < 1$, $q = 1-p$, are given in (1).

In this paper we shall characterize the minification processes $\{X_n, n \ge 0\}$ generated by (1) which are stationary for every $0 < p < 1$. Note that in this situation the stationary marginal distribution F for $\{X_n\}$ must satisfy (3) for every $0 < p < 1$. The case when the equation (3) is satisfied for some $0 < p < 1$ was studied in Pillai [7].

2. Maximum stability. Let F be a non-degenerate distribution function and $\{p_n, n \ge 1\}$ a probability distribution on the positive integers with $p_1 < 1$. Then F is called *maximum stable with respect to* $\{p_n\}$ if there exist real numbers $a > 0$ and b such that

$$p_1F(x) + p_2F^2(x) + \ldots = F(ax+b) \qquad \text{for all } x \tag{4}$$

(see e.g. Voorn [8]).

If the distribution $\{p_n\}$ is geometric:

$$p_n = pq^{n-1}, \qquad 0 < p < 1,\ q = 1-p,\ n = 1, 2, \ldots, \tag{5}$$

then the equation (4) may be written as

$$F(ax+b) = \sum_{n=1}^{\infty} pq^{n-1}F^n(x) = \frac{pF(x)}{1-qF(x)}. \tag{6}$$

Janjić [4] has found the class of distribution functions which satisfy, for every $0 < p < 1$, the equation (6) for some $a = a(p) > 0$ and $b = b(p)$. He has shown that the triple $(F(x), a(p), b(p))$ is the solution of the equation (6) for every $0 < p < 1$ if and only if either

$$\begin{aligned} &F(x) = 1/(1+ce^{-\alpha x}), \qquad -\infty < x < \infty,\ \alpha > 0,\ c > 0,\\ &a(p) = 1, \qquad b(p) = (\ln p)/\alpha, \qquad 0 < p < 1, \end{aligned} \tag{7}$$

or

$$\begin{aligned} &F(x) = \begin{cases} 0, & x \le \beta,\\ 1/(1+\delta(x-\beta)^{-\alpha}), & x > \beta,\ \alpha > 0,\ \delta > 0, \end{cases}\\ &a(p) = p^{1/\alpha}, \qquad b(p) = \beta(1-p^{1/\alpha}), \qquad 0 < p < 1,\ \alpha > 0, \end{aligned} \tag{8}$$

or

$$\begin{aligned} &F(x) = \begin{cases} 1/(1+\delta(-x+\beta)^{\alpha}), & x < \beta,\ \alpha > 0,\ \delta > 0,\\ 1, & x \ge \beta, \end{cases}\\ &a(p) = p^{-1/\alpha}, \qquad b(p) = \beta(1-p^{-1/\alpha}), \qquad 0 < p < 1,\ \alpha > 0. \end{aligned} \tag{9}$$

3. Autoregressive processes. We may now give the main result. It summarizes our considerations of Sections 1 and 2.

THEOREM 1. *Let* $\{X_n,\ n \geq 0\}$ *be a minification process given by* (1). *Then the process* $\{X_n\}$ *is strictly stationary for every* $0 < p < 1$ *if and only if its marginal distribution function* F *is maximum stable with respect to the geometric distribution* (5) *for every* $0 < p < 1$. *Thus* F *has one of the forms* (7)–(9).

In particular, if F is given by (7) with

$$c = e^{\mu/\sigma}, \quad \alpha = 1/\sigma, \quad \sigma > 0,\ -\infty < \mu < \infty, \tag{10}$$

we obtain the autoregressive logistic process

$$X_n = \begin{cases} X_{n-1} - \sigma \ln p & \text{with probability } p, \\ \min\{X_{n-1} - \sigma \ln p, \varepsilon_n\} & \text{with probability } 1-p, \end{cases} \tag{11}$$

which was studied by Arnold and Robertson [2].

Now let F be of the form (8). By taking

$$\beta = 0, \quad \delta = \sigma^{1/\gamma}, \quad \alpha = 1/\gamma, \quad \gamma > 0,\ \sigma > 0, \tag{12}$$

we have the autoregressive Pareto process

$$X_n = \begin{cases} p^{-\gamma} X_{n-1} & \text{with probability } p, \\ \min\{p^{-\gamma} X_{n-1}, \varepsilon_n\} & \text{with probability } 1-p, \end{cases} \tag{13}$$

which was introduced by Yeh *et al.* [9].

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