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CHARACTERISTIC PROPERTIES OF GENERALIZED ORDER STATISTICS FROM EXPONENTIAL DISTRIBUTIONS

Abstract. Exponential distributions are characterized by distributional properties of generalized order statistics. These characterizations include known results for ordinary order statistics and record values as particular cases.

1. Introduction. Various characterizations of exponential distributions based on distributional properties of order statistics are found in the literature.

Let $X_{1,n} \leq \ldots \leq X_{n,n}$ denote the order statistics of i.i.d. random variables $X_1, \ldots, X_n$, $n \geq 2$, each with distribution function F.

The starting point for many characterizations of exponential distributions via identically distributed functions of order statistics is the well-known result of Sukhatme (1937) stating that the normalized spacings

$$D_{1,n} = nX_{1,n} \quad \text{and} \quad D_{r,n} = (n-r+1)(X_{r,n} - X_{r-1,n}), \quad 2 \leq r \leq n,$$

from an exponential distribution with parameter λ, i.e. $F(x) = 1 - e^{-\lambda x}$, $x > 0$, $\lambda > 0$ ($F \equiv \mathrm{Exp}(\lambda)$ for short), are again independent and identically distributed as $\mathrm{Exp}(\lambda)$.

Ahsanullah (1978, 1981b) and Gajek & Gather (1989) consider identical distributions of $D_{r,n}$ and $D_{s,n}$ as well as weaker conditions for some integers r and s with $1 \leq r < s \leq n$.

Moreover, in the case of an exponential distribution we have identical distributions of $X_{s,n} - X_{r,n}$ and $X_{s-r,n-r}$ for all $1 \leq r < s \leq n$. Characterization results based on this property are discussed in Iwińska (1986), Gather (1988) and Gajek & Gather (1989).

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A detailed survey of characterizations of distributions via identically distributed functions of order statistics is given in Gather *et al*. (1997). Related results for record values are shown in Ahsanullah (1981a), Iwińska (1986) and Gajek & Gather (1989).

In Kamps (1995), a concept of generalized order statistics is introduced as a unified approach to order statistics and record values. Furthermore, a variety of other models of ordered random variables is contained in this concept. For a detailed discussion of several of these models, such as sequential order statistics, kth record values and Pfeifer's record model, we refer to Kamps (1995, Ch. I).

In this paper we present characterizations of exponential distributions via distributional properties of generalized order statistics including the known results for ordinary order statistics and record values as particular cases.

For the proofs of our results we need the aging properties IFR/DFR and NBU/NWU as weaker conditions, respectively. An absolutely continuous distribution function F with density function f is said to be IFR (DFR) if its failure rate $f/(1-F)$ increases (decreases). A distribution function F is said to be NBU (NWU) if $1-F(x+y) \le (\ge)\ (1-F(x))(1-F(y))$ for all x, y, $x+y$ in the support of F.

2. Generalized order statistics. Let $X(1,n,m,k),\dots,X(n,n,m,k)$ be generalized order statistics based on the absolutely continuous distribution function F with density function f, which means that the joint density function of the above quantities is given by

$$f^{X(1,n,m,k),\dots,X(n,n,m,k)}(x_1,\dots,x_n)$$
$$= k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}(1-F(x_i))^m f(x_i)\Big)(1-F(x_n))^{k-1}f(x_n),$$
$$F^{-1}(0+) < x_1 \le \ldots \le x_n < F^{-1}(1),$$

with $n\in\mathbb{N}$, $k>0$, $m\in\mathbb{R}$ such that $\gamma_r = k+(n-r)(m+1) > 0$ for all $1\le r\le n$.

In the case $m=0$ and $k=1$ this model reduces to the joint density of ordinary order statistics, and in the case $m=-1$ and $k\in\mathbb{N}$ we obtain the joint density of the first n kth record values based on a sequence $X_1, X_2, \ldots$ of i.i.d. random variables with distribution function F.

The marginal density function of the rth generalized order statistic is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1-F(x))^{\gamma_r-1}f(x)g_m^{r-1}(F(x))$$

(see Kamps 1995, p. 64), and the density function of the spacings

$$W_{r-1,r,n} = X(r,n,m,k) - X(r-1,n,m,k), \qquad 2 \le r \le n,$$

has the following representation:

$$f^{X(r,n,m,k)-X(r-1,n,m,k)}(y)$$
$$= \frac{c_{r-1}}{(r-2)!} \int_{-\infty}^{\infty} (1-F(x))^m f(x) g_m^{r-2}(F(x))(1-F(x+y))^{\gamma_r - 1} f(x+y)\,dx$$

with

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \qquad 1 \le r \le n,$$

$$h_m(x) = \int (1-x)^m\,dx = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1, \\ \log \dfrac{1}{1-x}, & m = -1, \end{cases}$$

$$g_m(x) = h_m(x) - h_m(0), \qquad x \in [0,1) \qquad \text{(see Kamps 1995, p. 69).}$$

Generalizing Sukhatme's (1937) result it is shown in Kamps (1995, p.81) that the normalized spacings

$$D(1,n,m,k) = \gamma_1 X(1,n,m,k),$$
$$D(r,n,m,k) = \gamma_r(X(r,n,m,k) - X(r-1,n,m,k)), \qquad 2 \le r \le n,$$

based on an exponential distribution with parameter λ are independent and identically distributed according to $\mathrm{Exp}(\lambda)$.

As a consequence, $W_{r-1,r,n}$ and $X(1, n-r+1, m, k)$ are identically distributed, since

$$W_{r-1,r,n} \sim \frac{\gamma_1}{\gamma_r} X(1,n,m,k) = Z$$

and

$$f^Z(z) = \frac{\gamma_r}{\gamma_1} f^{X(1,n,m,k)}\left(\frac{\gamma_r}{\gamma_1} z\right) = \gamma_r \left(1 - F\left(\frac{\gamma_r}{\gamma_1} z\right)\right)^{\gamma_1 - 1} f\left(\frac{\gamma_r}{\gamma_1} z\right)$$
$$= \gamma_r \exp\{-\lambda \gamma_r z\} = f^{X(1,n-r+1,m,k)}(z).$$

Hence, in the following characterization results it remains to show that the respective properties determine exponential distributions uniquely.

3. Characterization results. In Theorem 1 it is shown that, under certain regularity conditions, a weaker assumption than identical distributions of $D(r,n,m,k)$ and $D(s,n,m,k)$ is sufficient to characterize exponential distributions within the class of distributions with the IFR or DFR property. This result includes the characterizations by ordinary order statistics and record values established in Gajek & Gather (1989).

Let $r_Y(x) = g(x)/(1-G(x))$ denote the failure rate of a random variable Y with distribution function G and density function g. The appearing failure rates as well as the density f in Theorem 1 and in Remark 1 are supposed to be continuous from the right. If r_Y is monotone, then the limit $r_Y(0) = \lim_{x\to 0} r_Y(x)$ is assumed to be finite (cf. Gajek & Gather 1989).

THEOREM 1. *Let F be absolutely continuous with density function f, $F(0) = 0$, and suppose that F is strictly increasing on $(0,\infty)$, and either IFR or DFR. Then $F \equiv \mathrm{Exp}(\lambda)$ for some $\lambda > 0$ iff there exist integers r, s and n, $1 \le r < s \le n$, such that $r_{D(r,n,m,k)}(0) = r_{D(s,n,m,k)}(0)$.*

Proof. Let $r \ge 2$. Since

$$f^{D(r,n,m,k)}(x) = \frac{1}{\gamma_r} f^{W_{r-1,r,n}}\left(\frac{x}{\gamma_r}\right)$$
$$= \frac{1}{\gamma_r}\frac{c_{r-1}}{(r-2)!}\int_0^\infty (1-F(y))^m f(y) g_m^{r-2}(F(y))$$
$$\times \left(1 - F\left(\frac{x}{\gamma_r} + y\right)\right)^{\gamma_r - 1} f\left(\frac{x}{\gamma_r} + y\right) dy$$

and $\frac{1}{\gamma_r} c_{r-1} = c_{r-2}$, we have $(r(y) = f(y)/(1-F(y)))$

$$r_{D(r,n,m,k)}(0) = \frac{c_{r-2}}{(r-2)!}\int_0^\infty \psi_r(y) r(y) f(y)\,dy$$

with

$$\psi_r(y) = (1-F(y))^{\gamma_r + m} g_m^{r-2}(F(y)).$$

On the other hand, we obtain

$$f^{X(r-1,n,m,k)}(x) = \frac{c_{r-2}}{(r-2)!}\psi_r(x) f(x),$$

which implies

$$\int_0^\infty \psi_r(x) f(x) dx = \left(\frac{c_{r-2}}{(r-2)!}\right)^{-1}.$$

Thus we find

$$r_{D(r,n,m,k)}(0) = r_{D(s,n,m,k)}(0)$$
$$\Leftrightarrow \left(\int_0^\infty \psi_r(x) r(x) f(x)\,dx\right)\left(\int_0^\infty \psi_r(x) f(x)\,dx\right)^{-1}$$
$$= \left(\int_0^\infty \psi_s(x) r(x) f(x)\,dx\right)\left(\int_0^\infty \psi_s(x) f(x)\,dx\right)^{-1}$$
$$\Leftrightarrow (I =) \int_0^\infty\int_0^\infty (\psi_r(x)\psi_s(y) - \psi_s(x)\psi_r(y)) r(x) f(x) f(y)\,dx\,dy = 0.$$

We rewrite the integral I as follows:

$$\begin{aligned}
I &= \iint\limits_{\{(x,y)\in(0,\infty)^2:x\le y\}} \ldots + \iint\limits_{\{(x,y)\in(0,\infty)^2:x>y\}} \ldots \\
&= \iint\limits_{x\le y} ((\psi_r(x)\psi_s(y)-\psi_s(x)\psi_r(y))r(x) \\
&\qquad\qquad + (\psi_r(y)\psi_s(x)-\psi_s(y)\psi_r(x))r(y))f(x)f(y)\,dx\,dy \\
&= \iint\limits_{x\le y} (\psi_r(x)\psi_s(y)-\psi_s(x)\psi_r(y))(r(x)-r(y))f(x)f(y)\,dx\,dy.
\end{aligned}$$

We now have

$$\begin{aligned}
&\psi_r(x)\psi_s(y) > \psi_s(x)\psi_r(y) \\
&\qquad \Leftrightarrow g_m(F(x))/(1-F(x))^{m+1} < g_m(F(y))/(1-F(y))^{m+1},
\end{aligned}$$

and the latter inequality can be seen to hold true for $x<y$. Since r is increasing (decreasing) this yields

$$r(x) = r(y) \quad \text{for all } x<y.$$

Thus we have got a constant failure rate and hence the assertion.

Let $r=1$. Then we find

$$f^{D(1,n,m,k)}(x) = (1-F(x/\gamma_1))^{\gamma_1-1}f(x/\gamma_1) \quad \text{and} \quad r_{D(1,n,m,k)}(0) = f(0).$$

Thus,

$$\begin{aligned}
&r_{D(1,n,m,k)}(0) = r_{D(s,n,m,k)}(0) \\
&\qquad \Leftrightarrow f(0) = \Big(\int_0^\infty \psi_s(x)r(x)f(x)\,dx\Big)\Big(\int_0^\infty \psi_s(x)f(x)\,dx\Big)^{-1},
\end{aligned}$$

which implies

$$\int_0^\infty \psi_s(x)f(x)(f(0)-r(x))\,dx = 0.$$

The assertion follows since $r(x)\ge(\le)\ r(0)=f(0), x>0$.

Remark 1. It is easily seen that the property $r_{D(r,n,m,k)}(0)=r(0)$ for some $2\le r\le n$ is also a characteristic property of exponential distributions. This assertion corresponds to Remark 2.1 in Gajek & Gather (1989) and generalizes Theorem 2.2 in Ahsanullah (1981b) for ordinary order statistics and Theorem 2.3 in Ahsanullah (1981a) for record values. As in the case $r=1$ in Theorem 1 it is obvious that the IFR or DFR assumption can be replaced by the condition that zero is an extremal point of the failure rate of F.

The following theorem generalizes Theorem 2.1 in Ahsanullah (1981b) as well as Theorem 2.4 in Ahsanullah (1981a), which, in the case of ordinary order statistics and record values, characterize exponential distributions by the equality of expectations of successive (normalized) spacings.

THEOREM 2. *Let F be absolutely continuous with density function f, $F(0) = 0$, $F(x) < 1$ for all $x > 0$, and suppose that F is IFR or DFR. Moreover, let $m \geq -1$. Then $F \equiv \mathrm{Exp}(\lambda)$ for some $\lambda > 0$ iff there exist integers r and n, $1 \leq r \leq n-1$, such that $ED(r,n,m,k) = ED(r+1,n,m,k)$.*

Proof. Let $r \geq 2$, and let F be IFR. By interchanging the order of integration, we obtain

$$\begin{aligned}
&1 - F^{D(r,n,m,k)}(x) \\
&\quad = \frac{c_{r-2}}{(r-2)!} \int_x^\infty \int_0^\infty (1-F(y))^m f(y) g_m^{r-2}(F(y)) \\
&\qquad \times \left(1 - F\left(\frac{z}{\gamma_r} + y\right)\right)^{\gamma_r - 1} f\left(\frac{z}{\gamma_r} + y\right) dy\, dz \\
&\quad = \frac{c_{r-2}}{(r-2)!} \int_0^\infty (1-F(y))^m f(y) g_m^{r-2}(F(y)) \\
&\qquad \times \int_x^\infty \left(1 - F\left(\frac{z}{\gamma_r} + y\right)\right)^{\gamma_r - 1} f\left(\frac{z}{\gamma_r} + y\right) dz\, dy \\
&\quad = \frac{c_{r-1}}{(r-1)!} \int_0^\infty \left(1 - F\left(\frac{x}{\gamma_r} + z\right)\right)^{\gamma_r - 1} f\left(\frac{x}{\gamma_r} + z\right) g_m^{r-1}(F(z))\, dz.
\end{aligned}$$

The latter representation remains valid for $r = 1$.

On the other hand, we have

$$\begin{aligned}
&1 - F^{D(r+1,n,m,k)}(x) \\
&\quad = \frac{c_{r-1}}{(r-1)!} \int_0^\infty (1-F(z))^m f(z) g_m^{r-1}(F(z)) \left(1 - F\left(\frac{x}{\gamma_{r+1}} + z\right)\right)^{\gamma_{r+1}} dz.
\end{aligned}$$

Since F is IFR, $\log(1-F)$ is concave, and thus we find for $m \geq -1$ that

$$\begin{aligned}
&\log\left(1 - F\left(\frac{x}{\gamma_r} + z\right)\right) \\
&\qquad = \log\left(1 - F\left(\frac{(m+1)z}{\gamma_r} + \frac{\gamma_{r+1}}{\gamma_r}\left(\frac{x}{\gamma_{r+1}} + z\right)\right)\right) \\
&\qquad \geq \frac{m+1}{\gamma_r} \log(1 - F(z)) + \frac{\gamma_{r+1}}{\gamma_r} \log\left(1 - F\left(\frac{x}{\gamma_{r+1}} + z\right)\right)
\end{aligned}$$

implies

$$\left(1-F\left(\frac{x}{\gamma_r}+z\right)\right)^{\gamma_r} \geq (1-F(z))^{m+1}\left(1-F\left(\frac{x}{\gamma_{r+1}}+z\right)\right)^{\gamma_{r+1}}.$$

Since $r(z)-r(x/\gamma_r+z)\leq 0$, $x,z>0$, we conclude from

$$\begin{aligned}
0 &= ED(r+1,n,m,k)-ED(r,n,m,k)\\
&= \frac{c_{r-1}}{(r-1)!}\int_0^\infty\int_0^\infty g_m^{r-1}(F(z))\Bigg((1-F(z))^m f(z)\left(1-F\left(\frac{x}{\gamma_{r+1}}+z\right)\right)^{\gamma_{r+1}}\\
&\quad - \left(1-F\left(\frac{x}{\gamma_r}+z\right)\right)^{\gamma_r} r\left(\frac{x}{\gamma_r}+z\right)\Bigg)dz\,dx\\
&\leq \frac{c_{r-1}}{(r-1)!}\int_0^\infty\int_0^\infty (1-F(z))^{m+1}\left(1-F\left(\frac{x}{\gamma_{r+1}}+z\right)\right)^{\gamma_{r+1}}\\
&\quad \times g_m^{r-1}(F(z))\left(r(z)-r\left(\frac{x}{\gamma_r}+z\right)\right)dz\,dx \leq 0
\end{aligned}$$

that $r(z)=r(x/\gamma_r+z)$ for all $x,z>0$, which implies the assertion.

Under an NBU/NWU assumption, characterizations of exponential distributions can also be obtained by identical expectations of $X_{s,n}-X_{r,n}$ and $X_{s-r,n-r}$ as well as by the corresponding identity for record values as shown in Iwińska (1986) and Gajek & Gather (1989). The following theorem provides an extension of these results to generalized order statistics in the case $s=r+1$.

THEOREM 3. *Let F be absolutely continuous with density function f, $F(0)=0$, and suppose that F is strictly increasing on $(0,\infty)$, and either is NBU or NWU. Moreover, let the expected values involved be finite. Then $F\equiv \mathrm{Exp}(\lambda)$ for some $\lambda>0$ iff there exist integers r and n, $1\leq r\leq n-1$, such that $EX(r+1,n,m,k)-EX(r,n,m,k)=EX(1,n-r,m,k)$.*

Proof. Making use of the representations

$$\begin{aligned}
E(X(r+1,n,m,k)&-X(r,n,m,k))\\
&= \int_0^\infty\int_x^\infty f^{W_{r,r+1,n}}(w)\,dw\,dx\\
&= \frac{c_r}{(r-1)!}\int_0^\infty\int_x^\infty\int_0^\infty (1-F(y))^m f(y) g_m^{r-1}(F(y))\\
&\quad \times (1-F(y+w))^{\gamma_{r+1}-1} f(y+w)\,dy\,dw\,dx
\end{aligned}$$

$$= \frac{c_r}{(r-1)!\gamma_{r+1}} \int_0^\infty \int_0^\infty (1-F(y))^m f(y) g_m^{r-1}(F(y))(1-F(x+y))^{\gamma_{r+1}}\, dy\, dx$$

$$= \int_0^\infty \int_0^\infty f^{X(r,n,m,k)}(y)((1-F(x+y))^{\gamma_{r+1}}/(1-F(y))^{\gamma_{r+1}})\, dy\, dx$$

and

$$\begin{aligned} EX(1,n-r,m,k) &= \gamma_{r+1} \int_0^\infty \int_x^\infty (1-F(y))^{\gamma_{r+1}-1} f(y)\, dy\, dx \\ &= \int_0^\infty (1-F(x))^{\gamma_{r+1}}\, dx \\ &= \int_0^\infty \int_0^\infty f^{X(r,n,m,k)}(y)(1-F(x))^{\gamma_{r+1}}\, dy\, dx \end{aligned}$$

we obtain

$$\begin{aligned} &EW_{r,r+1,n} = EX(1,n-r,m,k) \\ &\quad\Leftrightarrow \int_0^\infty \int_0^\infty f^{X(r,n,m,k)}(y)((1-F(x+y))^{\gamma_{r+1}}/(1-F(y))^{\gamma_{r+1}} \\ &\qquad\qquad -(1-F(x))^{\gamma_{r+1}})\, dy\, dx = 0, \end{aligned}$$

which implies the assertion.

Remark 2. Without any further assumption, the equation

$$EX(r+1,n,m,k) - EX(r,n,m,k) = EX(1,n-r,m,k)$$

for just one pair (r,n), $1 \le r \le n-1$, does not characterize exponential distributions. For every choice of r,n and $m \neq -1$ there are distributions different from exponentials with the above property as shown in Kamps (1995, p. 128). E.g., the distributions given by

$$F(x) = 1-(1+cx^d)^{-1/(m+1)} \begin{cases} c>0,\ x \in (0,\infty), & m > -1, \\ c<0,\ x \in (0,(-1/c)^{1/d}), & m < -1, \end{cases}$$

with

$$d = \frac{k+(n-1)(m+1)}{k+(n-2)(m+1)} \qquad \left(=\frac{\gamma_1}{\gamma_2}\right)$$

satisfy the moment condition for $r=1$.

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