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OPTION PRICING IN THE CRR MODEL WITH PROPORTIONAL TRANSACTION COSTS: A CONE TRANSFORMATION APPROACH

Abstract. Option pricing in the Cox–Ross–Rubinstein model with transaction costs is studied. Using a cone transformation approach a complete characterization of perfectly hedged options is given.

1. Introduction. Let us consider a market with two assets: a risky one called the stock and a riskless one called the bond, which are traded in a discrete time. The price s_n of the stock at time n is subject to random changes. We shall assume that for $n = 0, 1, 2, \ldots$,

$$s_{n+1} = (1 + \varrho_n)s_n \tag{1}$$

where ϱ_n is a sequence of i.i.d. random variables which take as their values with a positive probability only a and b, where $a < b$ are given real numbers greater than -1. The bond earns interest with a constant rate r such that $a < r < b$. We also assume that both the stock and bond are infinitely divisible, so that the possession of a part of share invested in the stock or a part of the bond is allowed. At any time $n = 0, 1, 2, \ldots,$ we can transfer an amount of money invested in stocks to bonds paying proportional transaction costs with a rate $\mu > 0$. We also admit a transfer in the opposite direction, from bonds to stocks with proportional transaction costs with a rate $\lambda/(1 + \lambda)$, $\lambda > 0$. Let us denote by x_n, y_n the amounts of money invested in bonds and stocks respectively, at time n. Let l_n, m_n be the amounts of money for which we buy or sell respectively, shares of the stock at time n. Clearly l_n and m_n depend on $x_0, \ldots, x_n$, $y_0, \ldots, y_n$, $s_0, \ldots, s_n$ only.

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Taking into account transaction costs we have for $n = 0, 1, 2, \ldots,$

$$x_{n+1} = (1+r)(x_n - (1+\lambda)l_n + (1-\mu)m_n),$$
$$y_{n+1} = (1+\varrho_n)(y_n + l_n - m_n). \tag{2}$$

Consider now a financial instrument called a *contingent claim* that is a pair $(f_1(s_T), f_2(s_T))$ where f_1, f_2 are measurable functions and s_T stands for the price of the stock at a fixed time T called *maturity*. Given initial investments (x_0, y_0) in bonds and stocks respectively we look for a trading strategy $(l_n, m_n)_{n=0,1,\ldots,T-1}$ for which after possible transfers at time T, the amounts of money invested in bonds and in stocks exceed respectively $f_1(s_T)$ and $f_2(s_T)$. In that case we say that (l_n, m_n) is a *hedging strategy* against the contingent claim $(f_1(s_T), f_2(s_T))$ at maturity T.

Let

$$C = \left\{(x,y) \in \mathbb{R}^2 :\ y \geq \max\left\{-\frac{1}{1+\lambda}x, -\frac{1}{1-\mu}x\right\}\right\}$$

and

$$G_T(s) = (f_1(s), f_2(s)) + C \tag{3}$$

where the above sum means that $(f_1(s), f_2(s))$ is added to each element of C. Clearly C and $G_T(s)$ are cones. The hedging requirement can now be written as

$$(x_T, y_T) \in G_T(s_T). \tag{4}$$

We can easily show that

$$G_T(s) = \left\{(x,y) : y \geq \max\left\{-\frac{1}{1+\lambda}x + c_1(s), -\frac{1}{1-\mu}x + c_2(s)\right\}\right\} \tag{5}$$

where

$$c_1(s) = \frac{f_1(s)}{1+\lambda} + f_2(s), \qquad c_2(s) = \frac{f_1(s)}{1-\mu} + f_2(s). \tag{6}$$

Therefore we have a hedging when the system of inequalities

$$y_T \geq -\frac{1}{1+\lambda}x_T + c_1(s_T),$$
$$y_T \geq -\frac{1}{1-\mu}x_T + c_2(s_T), \tag{7}$$

is satisfied.

We say that a trading strategy (l_n, m_n) is *replicating* if (x_T, y_T) lies on the boundary of $G_T(s_T)$, or equivalently (7) holds and either $y_T = -\frac{1}{1+\lambda}x_T + c_1(s_T)$ or $y_T = -\frac{1}{1-\mu}x_T + c_2(s_T)$.

The *price* for the contingent claim (option) $(f_1(s), f_2(s))$ is the minimal value of $x_0 + (1-\mu)y_0$ for which there exists a hedging strategy against

$(f_1(s), f_2(s))$ with initial investments (x_0, y_0) in bonds and stocks respectively. The price for $(f_1(s), f_2(s))$ is called a *perfect hedging* or a *replicating cost* if a hedging strategy against $(f_1(s), f_2(s))$ corresponding to the minimal value of $x_0 + (1-\mu)y_0$ is replicating. The problem is to determine all cases for which perfect hedging is possible and then characterize replicating strategies.

Let $G_{T-1}(s)$ denote the set of all investments in bonds and stocks respectively at time $T-1$ such that given the stock price at $T-1$ equal to s, there is a strategy (l, m) for which we have a hedging at time T. Then

$$\begin{aligned}
(8)\qquad G_{T-1}(s) = \Big\{&(x,y) : \exists_{l,m\ge 0}\forall_{\varrho\in\{a,b\}} \\
&(1+\varrho)(y+l-m) \ge -\frac{1}{1+\lambda}(1+r)(x-(1+\lambda)l+(1-\mu)m) \\
&\qquad\qquad + c_1((1+\varrho)s), \\
&(1+\varrho)(y+l-m) \ge -\frac{1}{1-\mu}(1+r)(x-(1+\lambda)l+(1-\mu)m) \\
&\qquad\qquad + c_2((1+\varrho)s)\Big\}.
\end{aligned}$$

Clearly $G_{T-1}(s)$ is a polyhedron, but it may not be a cone. We show that if $G_{T-1}(s)$ is a cone then it is of the form (5) with suitably chosen functions $c_1(s), c_2(s)$ and it corresponds to a perfect hedging in one step.

By backward induction we can define the polyhedrons $G_{T-i}(s)$ for $i = 1, \ldots, T(s)$ as follows:

$$\begin{aligned}
(9)\qquad G_{T-i}(s) := \{(x,y) : \exists_{l,m}\forall_{\varrho\in\{a,b\}}\ &((1+r)(x-(1+\lambda)l+(1-\mu)m), \\
&(1+\varrho)(y+l-m)) \in G_{T-i+1}((1+\varrho)s)\}.
\end{aligned}$$

If for a given initial price s_0 of the stock the polyhedrons $G_0(s_0), G_1(s_1), \ldots, G_{T-1}(s_{T-1})$ are cones, then, as we show below, there exists a perfect hedging, and a replicating strategy that corresponds to that hedging is to buy or sell shares of the stock at time i so as to reach the vertex of the cone $G_i(s_i)$.

The option pricing model based on the binomial distribution of the price (1) of the stock was introduced first without transaction costs in [CRR]. The model was then considered in a number of papers (see [SKKM], [TZ], [MS] for more recent references). A version of this model with transaction costs was studied in various papers usually in the context of European call or put options (see [BV], [BLPS], [ENU], [MV], [R]), and sufficient conditions for perfect hedging were shown. In this paper we present complete characterizations of option pricing models with transaction costs for which contingent claims are functions of the price of the stock at maturity. Namely, by a detailed analysis of the behaviour of a certain system of controlled linear equations we obtain neccessary and sufficient conditions for perfect hedging. Our approach is based on a cone transformation that was considered in the case of diffusion models in [CK] and [SSC]. The study of discrete time models with transaction costs is particularly important because it was shown in [SSC], confirming the conjecture of Davis and Clark (see [DC]), that there is no nontrivial perfect hedging strategy for a continuous time lognormal model with proportional transaction costs.

2. Basic lemmas and notation. For simplicity of presentation we first introduce two sequences of equations of lines in $\mathbb{R}^2$. The first one, $(E1)$, $(E2)$, $(E3)$, $(E4)$, appears in the definition (8) of $G_{T-1}(s)$. In what follows for simplicity of notation we shall identify lines with their equations. Setting

$$z := z(l,m) = (1+\lambda)l - (1-\mu)m$$

we have

$$(E1)\qquad y = -\frac{1+r}{(1-\mu)(1+a)}x + \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z + m\frac{\lambda+\mu}{1+\lambda} + \frac{1}{1+a}c_2((1+a)s),$$

$$(E2)\qquad y = -\frac{1+r}{(1+\lambda)(1+a)}x + \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right)z + m\frac{\lambda+\mu}{1+\lambda} + \frac{1}{1+a}c_1((1+a)s),$$

$$(E3)\qquad y = -\frac{1+r}{(1-\mu)(1+b)}x + \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z + m\frac{\lambda+\mu}{1+\lambda} + \frac{1}{1+b}c_2((1+b)s),$$

$$(E4)\qquad y = -\frac{1+r}{(1+\lambda)(1+b)}x + \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{+\lambda}\right)z + m\frac{\lambda+\mu}{1+\lambda} + \frac{1}{1+b}c_1((1+b)s).$$

It will be convenient later to have the sequence $(F1), (F2), (F3), (F4)$ of equations of lines in $\mathbb{R}^2$ which are obtained from $(E1)$–$(E4)$ by the substitution $m = -z/(1-\mu)$. We have

$$(F1)\qquad y = -\frac{1+r}{(1-\mu)(1+a)}x + \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right)z + \frac{1}{1+a}c_2((1+a)s),$$

$$(F2)\qquad y = -\frac{1+r}{(1+\lambda)(1+a)}x + \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1-\mu}\right)z + \frac{1}{1+a}c_1((1+a)s),$$

$$(F3)\qquad y = -\frac{1+r}{(1-\mu)(1+b)}x + \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1-\mu}\right)z + \frac{1}{1+b}c_2((1+b)s),$$

$$(F4)\qquad y = -\frac{1+r}{(1+\lambda)(1+b)}x + \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z + \frac{1}{1+b}c_1((1+b)s).$$

The following values $z_1(s), \ldots, z_6(s)$ depending on the stock price s will be important in the construction of $G_{T-1}(s)$:

$$z_1(s) = \frac{(1-\mu)(1+\lambda)[(1+b)c_2((1+a)s) - (1+a)c_1((1+b)s)]}{(1+r)[(1+\lambda)(1+b) - (1-\mu)(1+a)]},$$

$$z_2(s) = \frac{1-\mu}{(1+r)(b-a)}[(1+b)c_2((1+a)s) - (1+a)c_2((1+b)s)],$$

$$z_3(s) = \frac{(1-\mu)(1+\lambda)}{(1+r)(\mu+\lambda)}[c_2((1+a)s) - c_1((1+a)s)],$$

$$z_4(s) = \frac{1+\lambda}{(1+r)(b-a)}[(1+b)c_1((1+a)s) - (1+a)c_1((1+b)s)],$$

$$z_5(s) = \frac{(1-\mu)(1+\lambda)}{(1+r)(\lambda+\mu)}[c_2((1+b)s) - c_1((1+b)s)],$$

$$z_6(s) = \frac{(1-\mu)(1+\lambda)[(1+a)c_2((1+b)s) - (1+b)c_1((1+a)s)]}{(1+r)[(1+\lambda)(1+a) - (1-\mu)(1+b)]}.$$

Notice that whenever both transaction costs (i.e. from stocks to bonds and from bonds to stocks) are equal, we have $1-\mu = 1/(1+\lambda)$, which simplifies the formulae for $z_1(s), z_3(s), z_5(s), z_6(s)$.

Using the notation $(Ei) \geq (Ek)$ when the graph of the line (Ei) is above (Ek) in the coordinate plane (x, y), by a trivial verification we obtain

LEMMA 1. *We have*

$$\begin{aligned}
(E1) \geq (E4) &\quad \textit{iff} \quad x \leq z + z_1,\\
(E1) \geq (E3) &\quad \textit{iff} \quad x \leq z + z_2,\\
(E1) \geq (E2) &\quad \textit{iff} \quad x \leq z + z_3,\\
(E4) \geq (E2) &\quad \textit{iff} \quad x \geq z + z_4,\\
(E4) \geq (E3) &\quad \textit{iff} \quad x \geq z + z_5.
\end{aligned}$$

Moreover,

- *if* $\dfrac{1+a}{1+b} > \dfrac{1-\mu}{1+\lambda}$, *then* $(E2) \geq (E3)$ *iff* $x \geq z + z_6$,
- *if* $\dfrac{1+a}{1+b} < \dfrac{1-\mu}{1+\lambda}$, *then* $(E2) \geq (E3)$ *iff* $x \leq z + z_6$,
- *if* $\dfrac{1+a}{1+b} = \dfrac{1-\mu}{1+\lambda}$, *then* $(E2) \geq (E3)$ *iff* $\dfrac{1}{1+a} c_1((1+a)s) \geq \dfrac{1}{1+b} c_2((1+b)s)$. ■

Define the indicators $(I1)(s)$, $(I2)(s), \ldots, (I5)(s)$ by

$$\begin{aligned}
(I1)(s) &:= c_2((1+a)s)(1-\mu)(b-a) + c_1((1+a)s)[(1-\mu)(1+a) \\
&\quad - (1+\lambda)(1+b)] + (1+a)(\mu+\lambda)c_1((1+b)s), \\
(I2)(s) &:= c_1((1+b)s)(1+\lambda)(b-a) + c_2((1+b)s)[(1-\mu)(1+a) \\
&\quad - (1+\lambda)(1+b)] + (1+b)(\mu+\lambda)c_2((1+a)s), \\
(I3)(s) &:= c_1((1+a)s)(1+\lambda)(b-a) + c_2((1+a)s)[(1+\lambda)(1+a) \\
&\quad - (1-\mu)(1+b)] - (1+a)(\mu+\lambda)c_2((1+b)s), \\
(I4)(s) &:= c_2((1+b)s)(1-\mu)(b-a) + c_1((1+b)s)[(1+\lambda)(1+a) \\
&\quad - (1-\mu)(1+b)] - (1+b)(\mu+\lambda)c_1((1+a)s), \\
(I5)(s) &:= [(1+\lambda)(1+a) - (1-\mu)(1+b)][(1+b)c_2((1+a)s) \\
&\quad - (1+a)c_1((1+b)s)] - [(1+\lambda)(1+b) - (1-\mu)(1+a)] \\
&\quad \times [(1+a)c_2((1+b)s) - (1+b)c_1((1+a)s)].
\end{aligned}$$

Let

$$\Delta(s) := c_2((1+a)s) + c_1((1+b)s) - c_1((1+a)s) - c_2((1+b)s). \tag{10}$$

Adding or subtracting suitable indicators, we obtain

Lemma 2.

(i) $(I1)(s) + (I2)(s) = [(1+\lambda)(1+b) - (1-\mu)(1+a)]\Delta(s)$,
(ii) $(I2)(s) + (I4)(s) = (1+b)(\mu+\lambda)\Delta(s)$,
(iii) $(I3)(s) + (I4)(s) = [(1+\lambda)(1+a) - (1-\mu)(1+b)]\Delta(s)$,
(iv) $(I1)(s) + (I3)(s) = (1+a)(\mu+\lambda)\Delta(s)$,
(v) $(I1)(s) - (I4)(s) = (1-\mu)(b-a)\Delta(s)$,
(vi) $(I2)(s) - (I3)(s) = (1+\lambda)(b-a)\Delta(s)$,
(vii) $(1+b)(I3)(s) - (1+a)(I4)(s) = (I5)(s)$,
(viii) $(1+a)(I2)(s) - (1+b)(I1)(s) = (I5)(s)$. ■

Using the indicators $(I1)(s)$–$(I5)(s)$ we can determine the allocation of the values $z_1(s), \ldots, z_6(s)$.

LEMMA 3. *We have*

$$\begin{aligned}
z_1(s) \le z_2(s) \quad &\textit{iff} \quad (I2)(s) \ge 0,\\
z_1(s) \le z_3(s) \quad &\textit{iff} \quad (I1)(s) \ge 0,\\
z_1(s) \ge z_4(s) \quad &\textit{iff} \quad (I1)(s) \ge 0,\\
z_1(s) \ge z_5(s) \quad &\textit{iff} \quad (I2)(s) \ge 0,\\
z_2(s) \ge z_3(s) \quad &\textit{iff} \quad (I3)(s) \ge 0,\\
z_4(s) \le z_5(s) \quad &\textit{iff} \quad (I4)(s) \ge 0,\\
z_3(s) = z_4(s) \quad &\textit{iff} \quad (I1)(s) = 0,\\
z_2(s) = z_5(s) \quad &\textit{iff} \quad (I2)(s) = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
z_1(s) \ge z_6(s) \quad \textit{iff} \quad & (I5)(s) \ge 0 \textit{ and } \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda}, \textit{ or}\\
& (I5)(s) \le 0 \textit{ and } \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda},\\
z_1(s) \le z_6(s) \quad \textit{iff} \quad & (I5)(s) \ge 0 \textit{ and } \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}, \textit{ or}\\
& (I5)(s) \le 0 \textit{ and } \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda},\\
z_2(s) \le z_6(s) \quad \textit{iff} \quad & (I3)(s) \ge 0 \textit{ and } \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}, \textit{ or}\\
& (I3)(s) \le 0 \textit{ and } \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda},\\
z_2(s) \ge z_6(s) \quad \textit{iff} \quad & (I3)(s) \ge 0 \textit{ and } \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda}, \textit{ or}\\
& (I3)(s) \le 0 \textit{ and } \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda},\\
z_3(s) \ge z_6(s) \quad \textit{iff} \quad & (I3)(s) \ge 0 \textit{ and } \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda}, \textit{ or}\\
& (I3)(s) \le 0 \textit{ and } \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda},\\
z_3(s) \le z_6(s) \quad \textit{iff} \quad & (I3)(s) \ge 0 \textit{ and } \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda} \textit{ or}\\
& (I3)(s) \le 0 \textit{ and } \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda},\\
z_4(s) \ge z_6(s) \quad \textit{iff} \quad & (I4)(s) \ge 0 \textit{ and } \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}, \textit{ or}\\
& (I4)(s) \le 0 \textit{ and } \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda},
\end{aligned}$$

$$
\begin{aligned}
z_4(s) \le z_6(s) \quad \textit{iff} \quad & (I4)(s) \ge 0 \ \textit{and}\ \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda},\ \textit{or}\\
& (I4)(s) \le 0 \ \textit{and}\ \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda},\\
z_5(s) \ge z_6(s) \quad \textit{iff} \quad & (I4)(s) \ge 0 \ \textit{and}\ \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda},\ \textit{or}\\
& (I4)(s) \le 0 \ \textit{and}\ \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda},\\
z_5(s) \le z_6(s) \quad \textit{iff} \quad & (I4)(s) \ge 0 \ \textit{and}\ \frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda},\ \textit{or}\\
& (I4)(s) \le 0 \ \textit{and}\ \frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}. \ \blacksquare
\end{aligned}
$$

Finally, for a real number h we define a transformation T_h of the real line as follows:

$$T_h x = (1+h)x \quad \text{for } x \in \mathbb{R}, \tag{11}$$

and then an operator $\mathcal{T}_h$ on $\mathbb{R}^2$ by

$$\mathcal{T}_h(x,y) = (T_r x, T_h y). \tag{12}$$

3. Construction of the cones G_{T-1} with the use of the indicators $(I1)$ and $(I2)$. In this section we study the cases $(I1)(s) \ge 0$ and $(I2)(s) \ge 0$, $(I1)(s) \le 0$ and $(I2)(s) \ge 0$, $(I1)(s) \ge 0$ and $(I2)(s) \le 0$. The remaining case $(I1)(s) \le 0$ and $(I2)(s) \le 0$ has to be split up into subcases in which other indicators are needed.

3(a) *Case* $(I1)(s) \ge 0$, $(I2)(s) \ge 0$. Various versions of European long call and put options are covered by the above case. We start with four examples.

Example 1 (European long call option with delivery). A holder of the option is entitled to buy one share of stock at a price q. We then have

$$f_1(s) = -q\,1_{s \ge q}, \quad f_2(s) = s\,1_{s \ge q},$$

and consequently (see (6))

$$c_1(s) = \left(s - \frac{q}{1+\lambda}\right)1_{s \ge q}, \quad c_2(s) = \left(s - \frac{q}{1-\mu}\right)1_{s \ge q}.$$

Example 2 (European long call option with delivery and cash settlement). As in Example 1 a holder is entitled to buy one share of stock at the price q, but his decision to exercise the option is made when the possible cash settlement is nonnegative. We have

$$f_1(s) = -q\,1_{s \ge q/(1-\mu)}, \quad f_2(s) = s\,1_{s \ge q/(1-\mu)},$$

and by (6),

$$c_1(s) = \left(s - \frac{q}{1+\lambda}\right) 1_{s \geq q/(1-\mu)}, \qquad c_2(s) = \left(s - \frac{q}{1-\mu}\right)^+.$$

EXAMPLE 3 (European long call option with delivery and settlement in shares of stock). The only change compared to Examples 1 and 2 is in the decision to exercise the option. The holder of the option is eager to owe the stock, and therefore he makes the decision to exercise the option when the settlement in shares of stock is nonnegative. In this case we have

$$f_1(s) = -q\, 1_{s \geq q/(1+\lambda)}, \qquad f_2(s) = s\, 1_{s \geq q/(1+\lambda)},$$

and (see (6))

$$c_1(s) = \left(s - \frac{q}{1+\lambda}\right)^+, \quad c_2(s) = \left(s - \frac{q}{1-\mu}\right) 1_{s \geq q/(1+\lambda)}.$$

EXAMPLE 4 (European long put option). A holder of the option is entitled to sell one share of stock at a price q. Then we can have the contingent claim functions

$$f_1(s) = q\, 1_{s \leq q}, \qquad f_2(s) = -s\, 1_{s \leq q},$$

and

$$c_1(s) = \left(-s + \frac{q}{1+\lambda}\right) 1_{s \leq q}, \qquad c_2(s) = \left(-s + \frac{q}{1-\mu}\right) 1_{s \leq q}.$$

One can show that for the contingent claims defined in Examples 1–4 we have $(I1)(s) \geq 0$ and $(I2)(s) \geq 0$. Furthermore, for s sufficiently large, $(I1)(s) = (I2)(s) = 0$. Moreover, in the examples considered above the contingent claim was considered from the so-called long position, i.e. the position of the buyer of an option. Consequently, the price of the option was the minimal one that compensated the seller's loss.

The main result of the section can be formulated as follows:

THEOREM 1. *Under* $(I1)(s) \geq 0$, $(I2)(s) \geq 0$ *we have*

$$\begin{aligned} G_{T-1}(s) &= \Big\{(x,y) : y \geq -\frac{1}{1+\lambda}x + c_1^{(1)}(s) \text{ for } x \geq z_1(s), \\ &\qquad\qquad y \geq -\frac{1}{1-\mu}x + c_2^{(1)}(s) \text{ for } x \leq z_1(s)\Big\} \\ &= (z_1(s), H_1 z_1(s)) + C \end{aligned}$$

where

$$
(13)\qquad
\begin{aligned}
c_1^{(1)}(s) &= -\left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1+\lambda}\right) z_1(s) + \frac{1}{1+b} c_1((1+b)s),\\
c_2^{(1)}(s) &= -\left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right) z_1(s) + \frac{1}{1+a} c_2((1+a)s),
\end{aligned}
$$

and

$$
(14)\qquad
\begin{aligned}
H_1 z_1(s) &= -\frac{1+r}{(1+\lambda)(1+b)} z_1(s) + \frac{1}{1+b} c_1((1+b)s)\\
&= -\frac{1+r}{(1-\mu)(1+a)} z_1(s) + \frac{1}{1+a} c_2((1+a)s).
\end{aligned}
$$

Moreover, *we have a perfect hedging in one step with replicating trading strategies*

$$
\begin{aligned}
l &= \frac{1}{1+\lambda}(x - z_1(s)),\ m = 0 \quad \textit{for } x \ge z_1(s),\\
l &= 0,\ m = -\frac{1}{1-\mu}(x - z_1(s)) \quad \textit{for } x \le z_1(s).
\end{aligned}
$$

In addition, *if* $(I1)((1+a)s) \ge 0$, $(I1)((1+b)s) \ge 0$, $(I2)((1+a)s) \ge 0$, $(I2)((1+b)s) \ge 0$, *then* $(I^{(1)}1)(s) \ge 0$ *and* $(I^{(1)}2)(s) \ge 0$ *where* $(I^{(1)}1)$ *and* $(I^{(1)}2)$ *are* $(I1)$, $(I2)$ *with* c_1, c_2 *replaced by* $c_1^{(1)}, c_2^{(1)}$.

Furthermore, *if for a given initial price* s_0 *of the stock we have*

$$
(15)\qquad (I1)(s_0(1+a)^i(1+b)^j) \ge 0, \qquad (I2)(s_0(1+a)^i(1+b)^j) \ge 0
$$

for nonnegative integers i, j *such that* $i + j = T - 1$, *then we have a perfect hedging with replicating strategy* (l_n, m_n) *that at each time* n *shifts* (x_n, y_n) *to the vertex of the cone* $G_n(s_n)$.

Proof. We first find the form of the polyhedron $G_{T-1}(s)$. By Lemma 3 we have

$$
(16)\qquad \max\{z_4(s), z_5(s)\} \le z_1(s) \le \min\{z_2(s), z_3(s)\}.
$$

Therefore, by Lemma 1,

$$
(17)\qquad
\begin{aligned}
&\text{for } x \le z_1(s) + z, \quad (E1) \ge \max\{(E2), (E3), (E4)\},\\
&\text{for } x \ge z_1(s) + z, \quad (E4) \ge \max\{(E1), (E2), (E3)\}.
\end{aligned}
$$

Since according to the definition of $G_{T-1}(s)$ we are looking for points $(x, y) \in \mathbb{R}^2$ which for some $l, m \ge 0$ dominate the lines $(E1)$–$(E4)$, to determine the boundary of $G_{T-1}(s)$ we shall consider only the cases when one of the control values l or m is 0.

Consider first the case when $x \le z_1(s)$. If moreover $x \le z_1(s) + z$, then either $z \in [x - z_1(s), 0]$ and $m = -z/(1-\mu)$, $l = 0$, or $z \in [0, \infty)$ and $m = 0$ (recall that $z := (1+\lambda)l - (1-\mu)m$).

If $z \in [x - z_1(s), 0]$ and $m = -z/(1-\mu)$, then the line $(E1)$ is above $(E2), (E3), (E4)$ and is of the form $(F1)$. Since we then have a family of lines $(F1)$ parametrized by $z \in [x - z_1(s), 0]$ and $\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu} > 0$, the lowest line in this family corresponds to $z = x - z_1(s)$, and its equation is

$$(\alpha 1) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right)z_1(s) + \frac{1}{1+a}c_2((1+a)s).$$

If $z \in [0, \infty)$ and $m = 0$, then the line $(E1)$ which is still above $(E2), (E3), (E4)$ takes its lowest position for $z = 0$ $\left(\text{since } \frac{1+r}{(1-\mu)(1+a)} > \frac{1}{1+\lambda}\right)$, and therefore does not lie below $(\alpha 1)$. If additionally to $x \le z_1(s)$ we have $x \ge z_1(s) + z$, then clearly $z \le x - z_1(s) \le 0$ and so $m = -z/(1-\mu)$. In this case $(E4)$ dominates $(E1), (E2), (E3)$ and is of the form $(F4)$. The lowest line $(F4)$ for the range $z \le x - z_1(s)$ corresponds to $z = x - z_1(s)$ and is of the form

$$(\alpha 2) \quad y = -\frac{1}{1-\mu} - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_1(s) + \frac{1}{1+b}c_1((1+b)s).$$

It follows from the definition of $z_1(s)$ that the lines $(\alpha 1)$ and $(\alpha 2)$ coincide. Therefore for $x \le z_1(s)$ the line $(\alpha 1) = (\alpha 2)$ forms the boundary of $G_{T-1}(s)$.

Let now $x \ge z_1(s)$. We again have two cases: either $x \le z_1(s) + z$, i.e. $z \in [x - z_1(s), \infty)$, and $m = 0$, or $x \ge z_1(s) + z$, and then for $z \in [0, x - z_1(s)]$ we put $m = 0$, while for $z \in (-\infty, 0]$ we let $m = -z/(1-\mu)$.

If $x \le z_1(s) + z$, i.e. $z \in [x - z_1(s), \infty)$, then $m = 0$, the line $(E1)$ lies above $(E2), (E3), (E4)$ and its lowest position corresponds to $z = x - z_1(s)$, and is of the form

$$(\beta 1) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_1(s) + \frac{1}{1+a}c_2((1+a)s).$$

If $x \ge z_1(s) + z$ and $z \in [0, x - z_1(s)]$, then $m = 0$ and the line $(E4)$ dominates $(E1), (E2), (E3)$. The lowest position of $(E4)$ corresponds then to the value $z = x - z_1(s)$, and that line is of the form

$$(\beta 2) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1+\lambda}\right)z_1(s) + \frac{1}{1+b}c_1((1+b)s).$$

If $x \ge z_1(s) + z$ and $z \in (-\infty, 0]$, then $m = -z/(1-\mu)$ and the line $(E4)$ which is again above $(E1), (E2), (E3)$ is of the form $(F4)$ with the lowest position for $z = 0$. Since the parameter $z = 0$ was considered in the minimization problem for which the minimal line was $(\beta 2)$, we conclude that the line $(\beta 2)$ is minimal for $x \ge z_1(s) + z$.

By the definition of $z_1(s)$ we know that $(\beta 1)$ and $(\beta 2)$ coincide. Therefore for $x \ge z_1(s)$ the boundary of $G_{T-1}(s)$ is $(\beta 1) = (\beta 2)$.

Notice that by the construction of $G_{T-1}(s)$ to reach the boundary we used the strategy $l = \frac{1}{1+\lambda}(x - z_1(s))$, $m = 0$ for $x \geq z_1(s)$ and $l = 0$, $m = -\frac{1}{1-\mu}(x - z_1(s))$ for $x \leq z_1(s)$. In other words, we shifted the pair (x, y) to the vertex of $G_{T-1}(s)$, which has coordinates $(z_1(s), H_1 z_1(s))$.

Since

$$T_a H_1 z_1(s) = -\frac{1}{1-\mu} T_r z_1(s) + c_2(T_a s)$$

and

$$T_b H_1 z_1(s) = -\frac{1}{1+\lambda} T_r z_1(s) + c_1(T_b s).$$

after the transformations $\mathcal{T}_a, \mathcal{T}_b$ the point $(z_1(s), H_1 z_1(s))$ lies on the boundary of $G_T(T_a s)$, $G_T(T_b s)$ respectively, and we have a perfect hedging in one step.

A direct algebraic calculation shows that $(I^{(1)}1)(s) \geq 0$ and $(I^{(1)}2)c(s) \geq 0$ provided $(I1)((1+a)s) \geq 0$, $(I1)((1+b)s) \geq 0$, $(I2)((1+a)s) \geq 0$ and $(I2)((1+b)s) \geq 0$.

Therefore under (15) the polyhedrons $G_n(s_n)$ are cones of the form (5) with suitably chosen functions c_1 and c_2 and the strategy to shift (x_n, y_n) to the vertex of $G_n(s_n)$ for $n = 0, 1, \ldots, T-1$ guarantees a perfect hedging. ∎

3(b) *Case* $(I1)(s) \leq 0, (I2)(s) \geq 0$. Under the above assumptions we obtain a perfect hedging in one step only in particular cases. We have

THEOREM 2. *If* $(I1)(s) \leq 0$ *and* $(I2)(s) \geq 0$, *then in the case when*

(a) $\dfrac{1+r}{1+a} < \dfrac{1+\lambda}{1-\mu}$

we have

$$\begin{aligned} G_{T-1}(s) = \Big\{(x,y) : {} & y \geq -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right) z_3(s) \\ & + \frac{1}{1+a} c_2((1+a)s) \ \textit{for } x \leq z_3(s), \\ & y \geq -1 + r \leq (1+\lambda)(1+a)x + \frac{1}{1+a} c_1((1+a)s) \\ & \textit{for } z_3(s) \leq x \leq z_4(s), \\ & y \geq -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right) z_4(s) \\ & + \frac{1}{1+b} c_1((1+b)s) \ \textit{for } x \leq z_4(s)\Big\} \end{aligned}$$

$$= \Big\{(x,y) : z_3(s) \le x \le z_4(s),$$
$$y = -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s)\Big\} + C$$

with hedging strategies

$$l = 0,\ m = \frac{z_3(s)-x}{1-\mu} \quad \text{for } x \le z_3(s),$$
$$m = l = 0 \quad \text{for } z_3(s) \le x \le z_4(s),$$
$$l = \frac{x-z_4(s)}{1+\lambda},\ m = 0 \quad \text{for } x \ge z_4(s),$$

and unless $z_3(s) = z_4(s)$ *we do not have a perfect hedging*; *while if*

(b) $\dfrac{1+r}{1+a} \ge \dfrac{1+\lambda}{1-\mu}$

we have

$$G_{T-1}(s) = \Big\{(x,y) : y \ge -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_4(s)$$
$$+\frac{1}{1+b}c_1((1+b)s) \text{ for } x \le z_4(s),$$
$$y \ge -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right)z_4(s)$$
$$+\frac{1}{1+a}c_1((1+a)s) \text{ for } x \ge z_4(s)\Big\}$$
$$= (z_4(s), H_2 z_4(s)) + C$$

with

$$H_2 z_4(s) = -\frac{1+r}{(1+\lambda)(1+b)}z_4(s) + \frac{1}{1+b}c_1((1+b)s) \tag{18}$$
$$= -\frac{1+r}{(1+\lambda)(1+a)}z_4(s) + \frac{1}{1+a}c_1((1+a)s)$$

and we have a perfect hedging in one step with replicating strategies

$$l = 0,\ m = \frac{z_4(s)-x}{1-\mu} \quad \text{for } x \le z_4(s),$$
$$l = \frac{x-z_4(s)}{1+\lambda},\ m = 0 \quad \text{for } x \ge z_4(s).$$

Proof. By Lemma 3 we have

$$\max\{z_3(s), z_5(s)\} \le z_1(s) \le \min\{z_2(s), z_4(s)\}. \tag{19}$$

Therefore from Lemma 1,

$$(20)\qquad \begin{aligned} &(E1) \geq \max\{(E2),(E3),(E4)\} && \text{for } x \leq z_3(s)+z,\\ &(E2) \geq \max\{(E1),(E3),(E4)\} && \text{for } z_3(s)+z \leq x \leq z_4(s)+z,\\ &(E4) \geq \max\{(E1),(E2),(E3)\} && \text{for } x \geq z_4(s)+z. \end{aligned}$$

The construction of $G_{T-1}(s)$ is split into three steps. Note that the labels $(\alpha 1),(\beta 1)$ etc. have other meanings than in Theorem 1.

Step I: $x \leq z_3(s)$. We have the following subcases:

1. Suppose $x \leq z_3(s)+z$. If $z \in [x-z_3(s),0]$ we let $m=-z/(1-\mu)$ and $(E1)$ which dominates $(E2),(E3),(E4)$ is of the form $(F1)$ and the lowest line corresponds to $z=x-z_3(s)$:

$$(\alpha 1)\qquad y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1-\mu)(1+a)}-\frac{1}{1-\mu}\right)z_3+\frac{1}{1+a}c_2((1+a)s).$$

If $z\in[0,\infty)$ we have $m=0$; therefore $(\alpha 1)$ is the minimal line.

2. If $z_3(s)+z \leq x \leq z_4(s)+z$, i.e. $z\in[x-z_4(s),x-z_3(s)]$, then $m=-z/(1-\mu)$ and $(E2)$ which is above $(E1)$, $(E2)$, $(E4)$ has the form $(F2)$ and attains its lowest position if $\frac{1+r}{1+a} \leq \frac{1+\lambda}{1-\mu}$ for $z=x-z_3(s)$, i.e.

$$(\alpha 21)\qquad y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1+\lambda)(1+a)}-\frac{1}{1-\mu}\right)z_3(s)+\frac{1}{1+a}c_1((1+a)s),$$

and if $\frac{1+r}{1+a} \geq \frac{1+\lambda}{1-\mu}$ for $z=x-z_4(s)$, i.e.

$$(\alpha 22)\qquad y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1+\lambda)(1+a)}-\frac{1}{1-\mu}\right)z_4(s)+\frac{1}{1+a}c_1((1+a)s).$$

3. If $z_4(s)+z \leq x$, then $z \leq x-z_4(s) < 0$, $m=-z/(1-\mu)$ and $(E4)$ dominates $(E1)$, $(E2)$, $(E3)$ and is of the form $(F4)$; the lowest position is attained for $z=x-z_4(s)$, i.e.

$$(\alpha 3)\qquad y=-\frac{1}{1-\mu}x+\left(\frac{1+r}{(1+\lambda)(1+b)}-\frac{1}{1-\mu}\right)z_4(s)+\frac{1}{1+b}c_1((1+b)s).$$

By the definitions of $z_3(s)$ and $z_4(s)$ we conclude that the lines $(\alpha 1)$, $(\alpha 21)$ and $(\alpha 3)$, $(\alpha 22)$ respectively coincide. Using the fact that $(I1)(s) \leq 0$ we also see that $(\alpha 1) \leq (\alpha 3)$ for $\frac{1+r}{1+a} \leq \frac{1+\lambda}{1-\mu}$, while $(\alpha 3) \geq (\alpha 1)$ for $\frac{1+r}{1+a} \geq \frac{1+\lambda}{1-\mu}$.

Step II: $z_3(s) \leq x \leq z_4(s)$. We again have three subcases:

1. If $x \leq z_3(s)+z$, then $z \geq x-z_3(s) > 0$, $m=0$, and $(E1)$ that is above $(E2)$, $(E3)$, $(E4)$ is in its lowest position for $z=x-z_3(s)$ and is then of the form

$$(\beta 1)\qquad y=-\frac{1}{1+\lambda}x+\left(\frac{1+r}{(1-\mu)(1+a)}-\frac{1}{1+\lambda}\right)z_3(s)+\frac{1}{1+a}c_2((1+a)s).$$

2. Suppose $z_3(s) + z \le x \le z_4(s) + z$. If $z \in [x - z_4(s), 0]$, then $m = -z/(1-\mu)$ and $(E2)$ has the form $(F2)$ and the lowest position in the case when $\frac{1+r}{1+a} \le \frac{1+\lambda}{1-\mu}$ is attained for $z = 0$ with

$$(\beta 21) \qquad y = -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s),$$

and when $\frac{1+r}{1+a} \ge \frac{1+\lambda}{1-\mu}$ for $z = x - z_4(s)$ with

$$(\beta 22) \qquad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1-\mu}\right)z_4(s) + \frac{1}{1+a}c_1((1+a)s).$$

If $z \in [0, x - z_3(s)]$ then $m = 0$ and the lowest position of the line $(E2)$ corresponds to $z = 0$ and this line either coincides with or lies above $(\beta 21)$, $(\beta 22)$.

3. If $z_4(s) + z \le x$, then $m = -z/(1-\mu)$, $(E4)$ has the form $(F4)$ and attains the lowest position for $z = x - z_4(s)$, which is

$$(\beta 3) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_4(s) + \frac{1}{1+b}c_1((1+b)s).$$

Clearly $(\beta 3) = (\beta 22) = (\alpha 3)$, moreover $(\beta 21)$ intersects $(\beta 1)$ and $(\alpha 1)$ for $x = z_3(s)$, and therefore lies below $(\beta 1)$ for $x \in [z_3(s), z_4(s)]$. Furthermore, $(\beta 3)$ intersects $(\beta 21)$ for $x = z_4(s)$. Therefore for $x \in [z_3(s), z_4(s)]$, the boundary of $G_{T-1}(s)$ is formed by the line $(\beta 21)$ when $\frac{1+r}{1+a} < \frac{1+\lambda}{1-\mu}$ and by $(\beta 3) = (\alpha 3)$ when $\frac{1+r}{1+a} \ge \frac{1+\lambda}{1-\mu}$.

Step III: $z_4(s) \le x$. We consider three subcases:

1. If $x \le z_3(s) + z$, then $z \ge x - z_3(s) > 0$, $m = 0$ and $(E1)$ attains its lowest position for $z = x - z_3(s)$ and has the form

$$(\gamma 1) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_3(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. If $z_3(s) + z \le x \le z_4(s) + z$, i.e. $z \in [x - z_4(s), x - z_3(s)]$, then $m = 0$, and $(E2)$ is minimal for $z = x - z_4(s)$ with the equation

$$(\gamma 2) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right)z_4(s) + \frac{1}{1+a}c_1((1+a)s).$$

3. Suppose $z_4(s) + z \le x$, i.e. $z \in (-\infty, x - z_4(s)]$. If $z \in (-\infty, 0]$ we have $m = -z/(1-\mu)$ and $(E4)$ attains its lowest position for $z = 0$, while if $z \in [0, x - z_4(s)]$, we have $m = 0$, and $(E4)$ is minimal for $z = x - z_4(s)$ and of the form

$$(\gamma 3) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1+\lambda}\right)z_4(s) + \frac{1}{1+b}c_1((1+b)s).$$

Clearly for $z = 0$, $(E4)$ is above $(\gamma 3)$. Since $(\gamma 2) = (\gamma 3)$ and $(\gamma 1) = (\beta 1)$ and $(\beta 21)$ intersects $(\gamma 3)$ for $x = z_4(s)$ we conclude that for $x \geq z_4(s)$ the boundary of $G_{T-1}(s)$ is $(\gamma 3)$.

This way we determined the form of $G_{T-1}(s)$. It remains to study the aspect of perfect hedging in one step.

In the case when $\frac{1+r}{1+a} < \frac{1+\lambda}{1-\mu}$ the boundary of $G_{T-1}(s)$ for $z_3(s) \leq x \leq z_4(s)$ is formed by the line segment

$$y = -\frac{1+r}{(1+\lambda)(1+a)} x + \frac{1}{1+a} c_1((1+a)s).$$

Therefore

$$T_a y = -\frac{1}{1+\lambda} T_r x + c_1(T_a s)$$

and $(T_r x, T_a y)$ lies on the boundary of $G_T(T_a s)$. On the other hand, when $z_3(s) < z_4(s)$,

$$T_b y = -\frac{1+b}{(1+a)(1+\lambda)} T_r x + \frac{1+b}{1+a} c_1((1+a)s)$$

and the point $(T_r x, T_b y)$ is on the boundary of $G_T(T_b s)$ only when

$$\frac{1+b}{(1+a)(1+\lambda)} = \frac{1}{1-\mu} \quad \text{and} \quad \frac{1+b}{1+a} c_1((1+a)s) = c_2((1+b)s),$$

which implies $(I5)(s) = 0$ and by Lemma 2(viii), $(I2)(s) = (I1)(s) = 0$ and consequently $z_3(s) = z_4(s)$ by Lemma 3.

In the case when $z_3(s) = z_4(s)$ we have

$$T_a y = -\frac{1}{1-\mu} T_r z_3(s) + c_2(T_a s),$$
$$T_b y = -\frac{1}{1+\lambda} T_r z_3(s) + c_1(T_b s),$$

and therefore a perfect hedging holds.

It remains to consider the case $\frac{1+r}{1+a} \geq \frac{1+\lambda}{1-\mu}$. By (18),

$$T_a H_2 z_4(s) = -\frac{1}{1+\lambda} T_r z_4(s) + c_1(T_a s),$$
$$T_b H_2 z_4(s) = -\frac{1}{1+\lambda} T_r z_4(s) + c_1(T_b s),$$

and we have a perfect hedging in one step.

The proof of Theorem is therefore complete. ∎

3(c) *Case* $(I1)(s) \geq 0$, $(I2)(s) \leq 0$. We now consider the case opposite to 3(b). A perfect hedging can again be obtained in a particular case only.

THEOREM 3. *Under* $(I1)(s) \geq 0$, $(I2)(s) \leq 0$, *if*

(a) $\dfrac{1+r}{1+b} \leq \dfrac{1-\mu}{1+\lambda}$

we have

$$G_{T-1}(s)=\Big\{(x,y): y\geq -\frac{1}{1-\mu}x-\left(\frac{1+r}{(1-\mu)(1+a)}-\frac{1}{1-\mu}\right)z_2(s)$$
$$+\frac{1}{1+a}c_2((1+a)s) \text{ for } x\leq z_2(s),$$
$$y\geq -1\leq 1+\lambda x-\left(\frac{1+r}{(1-\mu)(1+b)}-\frac{1}{1+\lambda}\right)z_2(s)$$
$$+\frac{1}{1+b}c_2((1+b)s) \text{ for } x\geq z_2(s)\Big\}$$
$$=(z_2(s),H_3z_2(s))+C$$

with

$$\begin{aligned} H_3z_2(s)&=-\frac{1+r}{(1-\mu)(1+a)}z_2(s)+\frac{1}{1+a}c_2((1+a)s)\\ &=-\frac{1+r}{(1-\mu)(1+b)}z_2(s)+\frac{1}{1+b}c_2((1+b)s), \end{aligned} \tag{21}$$

and for the replicating strategies

$$l=0,\ m=\frac{z_2(s)-x}{1-\mu} \qquad \text{if } x\leq z_2(s),$$
$$l=\frac{x-z_2(s)}{1+\lambda},\ m=0 \qquad \text{if } x\geq z_2(s),$$

a perfect hedging in one step is attained; *while if*

(b) $\dfrac{1+r}{1+b}>\dfrac{1-\mu}{1+\lambda}$

we have

$$G_{T-1}(s)=\Big\{(x,y): y\geq -\frac{1}{1-\mu}x-\left(\frac{1+r}{(1-\mu)(1+a)}-\frac{1}{1-\mu}\right)z_2(s)$$
$$+\frac{1}{1+a}c_2((1+a)s) \text{ for } x\leq z_2(s),$$
$$y\geq -\left(\frac{1+r}{(1-\mu)(1+b)}x\right)+\frac{1}{1+b}c_2((1+b)s)$$
$$\text{for } z_2(s)\leq x\leq z_5(s),$$
$$y\geq -\frac{1}{1+\lambda}x-\left(\frac{1+r}{(1+\lambda)(1+b)}-\frac{1}{1+\lambda}\right)z_5(s)$$
$$+\frac{1}{1+b}c_2((1+b)s) \text{ for } x\geq z_5(s)\Big\}$$

$$= \Big\{(x,y) : z_2(s) \le x \le z_5(s),$$
$$y = -\frac{1+r}{(1-\mu)(1+b)}x + \frac{1}{1+b}c_2((1+b)s)\Big\} + C$$

with hedging strategies

$$\begin{aligned} l = 0,\ m = \frac{z_2(s)-x}{1-\mu} \quad & \textit{for } x \le z_2(s),\\ l = m = 0 \quad & \textit{for } z_2(s) \le x \le z_5(s),\\ l = \frac{x-z_5(s)}{1+\lambda},\ m = 0 \quad & \textit{for } x \ge z_5(s), \end{aligned}$$

and unless $z_2(s) = z_5(s)$ we do not have a perfect hedging in one step.

Proof. Since the proof is similar to that of Theorem 1 or Theorem 2 we point out the main steps only.

By Lemma 3 we have

$$\max\{z_2(s), z_4(s)\} \le z_1(s) \le \min\{z_3(s), z_5(s)\} \tag{22}$$

and therefore by Lemma 1 the line dominating other lines is

$$\begin{aligned} &(E1) \quad \text{for } x \le z_2(s) + z,\\ &(E3) \quad \text{for } z_2(s) + z \le x \le z_5(s) + z,\\ &(E4) \quad \text{for } x \ge z_5(s) + z. \end{aligned} \tag{23}$$

Step I: $x \le z_2(s)$.

1. Suppose $x \le z_2(s) + z$. If $z \in [x - z_2(s), 0]$, then $m = -z/(1-\mu)$, the lowest position of $(E1)$ corresponds to $z = x - z_2(s)$ and has the form

$$(\alpha 1) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right)z_2(s) + \frac{1}{1+a}c_2((1+a)s).$$

The case $z \ge 0$, $m = 0$ leads to a line above $(\alpha 1)$.

2. If $z_2(s)+z \le x \le z_5(s)+z$, then $m = -z/(1-\mu)$, the lowest position of $(E3)$ corresponds to $z = x - z_2(s)$ and is of the form

$$(\alpha 2) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1-\mu}\right)z_2(s) + \frac{1}{1+b}c_2((1+b)s).$$

3. If $x \ge z_5(s) + z$, then $m = -z/(1-\mu)$ and the minimal location of $(E4)$ is for $z = x - z_5(s)$ and has the equation

$$(\alpha 3) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_5(s) + \frac{1}{1+b}c_1((1+b)s).$$

We clearly have $(\alpha 1) = (\alpha 2)$. Moreover, since $(I2)(s) \le 0$ we can show that $(\alpha 2) \le (\alpha 3)$.

S t e p II: $z_2(s) \le x \le z_5(s)$.

1. If $x \le z_2(s) + z$, then $m = 0$, the lowest position of $(E1)$ is for $z = x - z_2(s)$ and has the form

$$(\beta 1) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_2(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. Suppose $z_2(s)+z \le x \le z_5(s)+z$. If $z \in [0, x-z_2(s)]$, then $m = 0$ and in the case when $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ the minimal position of $(E3)$ is for $z = x-z_2(s)$ and has the form

$$(\beta 21) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_2(s) + \frac{1}{1+b}c_2((1+b)s);$$

when $\frac{1+r}{1+b} \ge \frac{1-\mu}{1+\lambda}$ the lowest position of $(E3)$ is for $z = 0$ and

$$(\beta 22) \qquad y = -\frac{1+r}{(1-\mu)(1+b)}x + \frac{1}{1+b}c_2((1+b)s).$$

If $z \in [x - z_5(s), 0]$, then $m = -z/(1-\mu)$ and the minimal location of $(E3)$ corresponds to $z = 0$ and coincides with $(\beta 22)$.

3. If $z_5(s) + z \le x$, then $m = -z/(1-\mu)$, the lowest location of $(E4)$ is for $z = x - z_5(s)$ and is of the form

$$(\beta 3) \qquad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_5(s) + \frac{1}{1+b}c_1((1+b)s).$$

We now easily see that $(\alpha 3) = (\beta 3)$, $(\beta 21) = (\beta 1)$, and $(\beta 21)$ intersects $(\beta 22)$ and $(\beta 3)$ at points with first coordinates $z_2(s)$ and $z_5(s)$ respectively. Therefore if $\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda}$ then the boundary of $G_{T-1}(s)$ is $(\beta 22)$, while for $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ the boundary of $G_{T-1}(s)$ is $(\beta 21)$.

S t e p III: $x \ge z_5(s)$.

1. If $x \le z_2(s) + z$, then $m = 0$, the minimal location of $(E1)$ is for $z = x - z_2(s)$ and is of the form

$$(\gamma 1) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_2(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. If $z_2(s) + z \le x \le z_5(s) + z$, then $m = 0$, the lowest position of $(E3)$ is in the case $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ for $z = x - z_2(s)$ with

$$(\gamma 21) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_2(s) + \frac{1}{1+b}c_2((1+b)s).$$

and in the case $\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda}$ for $z = x - z_5(s)$ with

$$(\gamma 22) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_5(s) + \frac{1}{1+b}c_2((1+b)s).$$

3. Suppose $x \ge z_5(s)+z$. If $z \in [0, x-z_5]$, then $m=0$, and the lowest position of $(E4)$ is for $z = x - z_5(s)$ with

$$(\gamma 3) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1+\lambda}\right) z_5(s) + \frac{1}{1+b} c_1((1+b)s).$$

If $z \in (-\infty, 0]$, then $m = -z/(1-\mu)$, and therefore $(E4)$ is above $(\gamma 3)$.

Notice now that $(\gamma 1) = (\gamma 21)$, $(\gamma 22) = (\gamma 3)$, $(\beta 21) = (\gamma 1)$ and under $(I2) \le 0$ for $\frac{1+r}{1+b} \ge \frac{1-\mu}{1+\lambda}$ we have $(\gamma 1) \ge (\gamma 3)$ while for $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$, $(\gamma 3) \ge (\gamma 1)$. The form of $G_{T-1}(S)$ is thus established.

If $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ we have

$$T_a H_3 z_2(s) = -\frac{1}{1-\mu} T_r z_2(s) + c_2(T_a s),$$
$$T_b H_3 z_2(s) = -\frac{1}{1-\mu} T_r z_2(s) + c_2(T_b s),$$

from which a perfect hedging follows.

If $\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda}$ then by a consideration similar to that of Theorem 2 we see that we have a perfect hedging only when $z_2(s) = z_5(s)$. The proof is complete. ∎

4. Construction of the cone $G_{T-1}(s)$ under $(I1)(s) \le 0$ and $(I2)(s) \le 0$. The study of the case $(I1)(s) \le 0$ and $(I2)(s) \le 0$ requires the additional indicators $(I3)(s)$ and $(I4)(s)$. Taking into account all possible signs of $(I3)(s)$ and $(I4)(s)$ we consider four subcases.

4(a) *Case* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \le 0$, $(I4)(s) \le 0$. Our main result in this case can be stated as follows:

THEOREM 4. *Under* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \le 0$, $(I4)(s) \le 0$, *in the case*

(a) $\dfrac{1+a}{1+b} > \dfrac{1-\mu}{1+\lambda}$

we have

$$\begin{aligned} G_{T-1}(s) = \Big\{(x,y) : y \ge\ & -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right) z_2(s) \\ & + \frac{1}{1+a} c_2((1+a)s) \text{ for } x \le z_2(s), \\ y \ge\ & -\frac{1+r}{(1-\mu)(1+b)}x + \frac{1}{1+b} c_2((1+b)s) \\ & \text{for } z_2(s) \le x \le z_6(s), \end{aligned}$$

$$
\begin{aligned}
y \geq\ & -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a} + c_1((1+a)s) \\
& \quad \textit{for } z_6(s) \leq x \leq z_4(s), \\
y \geq\ & -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1+\lambda}\right) z_4(s) \\
& + \frac{1}{1+b} c_1((1+b)s) \ \textit{for } x \geq z_4(s) \Big\}
\end{aligned}
$$

with hedging strategies

$$
\begin{aligned}
l = 0,\ m = \frac{z_2(s) - x}{1-\mu} & \quad \textit{for } x \leq z_2(s), \\
l = m = 0 & \quad \textit{for } z_2(s) \leq x \leq z_4(s), \\
l = \frac{x - z_4(s)}{1+\lambda},\ m = 0 & \quad \textit{for } x \geq z_4(s),
\end{aligned}
$$

and unless $z_2(s) = z_6(s) = z_4(s)$ *which is equivalent to* $(I1)(s) = (I2)(s) = (I3)(s) = (I4)(s) = 0$, *we do not have a perfect hedging.*

In the case

(b) $\dfrac{1+a}{1+b} = \dfrac{1-\mu}{1+\lambda}$

we have

$$
\begin{aligned}
G_{T-1}(s) = \Big\{ (x,y) : y \geq & -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1-\mu}\right) z_2(s) \\
& + \frac{1}{1+a} c_2((1+a)s) \ \textit{for } x \leq z_2(s), \\
y \geq & -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a} c_1((1+a)s) \\
& \textit{for } z_2(s) \leq x \leq z_4(s), \\
y \geq & -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1+\lambda}\right) z_4(s) \\
& + \frac{1}{1+b} c_1((1+b)s) \ \textit{for } x \geq z_4(s) \Big\} \\
= \Big\{ (x,y) : z_2(s) \leq & \ x \leq z_4(s), \\
y = & -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a} c_1((1+a)s) \Big\} + C
\end{aligned}
$$

and under the trading strategies

$$l=0,\ m=\frac{z_2(s)-x}{1-\mu} \qquad \textit{for } x\le z_2(s),$$
$$m=l=0 \qquad \textit{for } z_2(s)\le x\le z_4(s),$$
$$l=\frac{x-z_4(s)}{1+\lambda},\ m=0 \qquad \textit{for } x\ge z_4(s)$$

we obtain a perfect hedging.

Moreover, the case $\frac{1+a}{1+b}<\frac{1-\mu}{1+\lambda}$ *is impossible.*

Proof. By Lemma 2(ii), $(I2)(s)+(I4)(s)=(1+b)(\mu+\lambda)\Delta(s)\le 0$. Therefore $\Delta(s)\le 0$. Since $(I3)(s)+(I4)(s)\le 0$ by Lemma 2(iii) we have $(1+\lambda)(1+a)\ge(1-\mu)(1+b)$. Therefore the case $\frac{1+a}{1+b}<\frac{1-\mu}{1+\lambda}$ is excluded. Using Lemma 3 we have

$$z_2(s)\le z_3(s)\le z_1(s)\le z_5(s)\le z_4(s) \tag{24}$$

and under $\frac{1+a}{1+b}>\frac{1-\mu}{1+\lambda}$, if $(I5)(s)\ge 0$ then $z_6(s)\in[z_3(s),z_1(s)]$ while if $(I5)(s)\le 0$ then $z_6(s)\in[z_1(s),z_5(s)]$. By Lemma 1 we can determine the dominating lines for $\frac{1+a}{1+b}>\frac{1-\mu}{1+\lambda}$, namely they are

$$\begin{aligned}
&(E1) && \text{for } x\le z_2(s)+z,\\
&(E3) && \text{for } z_2(s)+z\le x\le z_6(s)+z,\\
&(E2) && \text{for } z_6(s)+z\le x\le z_4(s)+z,\\
&(E4) && \text{for } x\ge z_4(s)+z.
\end{aligned}$$

In the case when $\frac{1+a}{1+b}=\frac{1-\mu}{1+\lambda}$ by Lemma 2(iii) we obtain $(I3)(s)+(I4)(s)=0$ and therefore $(I3)(s)=(I4)(s)=0$. Consequently, $(I5)(s)=0$ and

$$\frac{1}{1+a}c_1((1+a)s)=\frac{1}{1+b}c_2((1+b)s), \tag{26}$$

which implies that $(E2)=(E3)$. Hence, the polyhedron $G_{T-1}(s)$ is determined by the following lines:

$$\begin{aligned}
&(E1) && \text{for } x\le z_2(s)+z,\\
(E2)=&(E3) && \text{for } z_2(s)+z\le x\le z_4(s)+z,\\
&(E4) && \text{for } x\ge z_4(s)+z.
\end{aligned} \tag{27}$$

Consider now the case $\frac{1+a}{1+b}>\frac{1-\mu}{1+\lambda}$. Since we follow similar arguments to the proofs of Theorems 1, 2 and 3, we only list below the values m,z and the equations of the lowest lines

Step I: $x\le z_2(s)$.

1. $x\le z_2(s)+z$.

(a) $z\in[x-z_2(s),0]$, $m=-z/(1-\mu)$; $z=x-z_2(s)$,

$$(\alpha 1)\qquad y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1-\mu)(1+a)}-\frac{1}{1-\mu}\right)z_2(s)+\frac{1}{1+a}c_2((1+a)s).$$

(b) $z\in[0,\infty)$, $m=0$; $z=0$ and the line is above $(\alpha 1)$.

2. $z_2(s)+z\le x\le z_6(s)+z$, $m=-z/(1-\mu)$; $z=x-z_2(s)$,

$$(\alpha 2)\qquad y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1-\mu)(1+b)}-\frac{1}{1-\mu}\right)z_2(s)+\frac{1}{1+b}c_2((1+b)s).$$

3. $z_6(s)+z\le x\le z_4(s)+z$, $m=-z/(1-\mu)$; $z=x-z_6(s)$,

$$(\alpha 3)\qquad y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1+\lambda)(1+a)}-\frac{1}{1-\mu}\right)z_6(s)+\frac{1}{1+a}c_1((1+a)s).$$

4. $x\ge z_2(s)+z$, $m=-z/(1-\mu)$; $z=x-z_4(s)$,

$$(\alpha 4)\qquad y-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1+\lambda)(1+b)}-\frac{1}{1-\mu}\right)z_4(s)+\frac{1}{1+b}c_1((1+b)s).$$

Clearly $(\alpha 1)=(\alpha 2)$. Moreover, one can show that if $(I3)(s)\le 0$ we have $(\alpha 1)\le(\alpha 3)$, while if $(I4)(s)\le 0$ we have $(\alpha 3)\le(\alpha 4)$. Therefore $(\alpha 1)$ is the boundary of $G_{T-1}(s)$.

Step II: $z_2(s)\le x\le z_6(s)$.

1. $x\le z_2(s)+z$, $m=0$; $z=x-z_2(s)$,

$$(\beta 1)\qquad y=-\frac{1}{1+\lambda}x-\left(\frac{1+r}{(1-\mu)(1+a)}-\frac{1}{1+\lambda}\right)z_2(s)+\frac{1}{1+a}c_2((1+a)s).$$

2. $z_2(s)+z\le x\le z_6(s)+z$.

(a) $z\in[x-z_6(s),0]$, $m=-z/(1-\mu)$; $z=0$,

$$(\beta 2)\qquad y=-\frac{1+r}{(1-\mu)(1+b)}x+\frac{1}{1+b}c_2((1+b)s).$$

(b) $z\in[0,x-z_2(s)]$, $m=0$; $z=0$ and the line coincides with $(\beta 2)$.

3. $z_6(s)+z\le x\le z_4(s)+z$, $m=-z/(1-\mu)$; $z=x-z_6(s)$,

$$(\beta 3)\qquad y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1+\lambda)(1+a)}-\frac{1}{1-\mu}\right)z_6(s)+\frac{1}{1+a}c_1((1+a)s).$$

4. $x\ge z_4(s)+z$, $m=-z/(1-\mu)$; $z=x-z_4(s)$,

$$(\beta 4)\qquad y-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1+\lambda)(1+b)}-\frac{1}{1-\mu}\right)z_4(s)+\frac{1}{1+b}c_1((1+b)s).$$

Notice that $(\alpha 3)=(\beta 3)$ and $(\alpha 4)=(\beta 4)$. Moreover, $(\beta 1)$ intersects $(\alpha 1)$ and $(\beta 2)$ for $x=z_2(s)$. Since for $x=z_6(s)$ the line $(\alpha 3)$ intersects $(\beta 2)$ we conclude that $(\beta 2)$ forms the boundary of $G_{T-1}(s)$.

Step III: $z_6(s) \le x \le z_4(s)$.

1. $x \le z_2(s) + z$, $m = 0$; $z = x - z_2(s)$,

$$(\gamma 1) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_2(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. $z_2(s) + z \le x \le z_6(s) + z$, $m = 0$; $z = x - z_6(s)$,

$$(\gamma 2) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_6(s) + \frac{1}{1+b}c_2((1+b)s).$$

3. $z_6(s) + z \le x \le z_4(s) + z$.

   (a) $z \in [x - z_4(s), 0]$, $m = -z/(1-\mu)$; $z = 0$,

$$(\gamma 3) \quad y = -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s).$$

   (b) $z \in [0, x - z_6(s)]$, $m = 0$; $z = 0$ and the line coincides with $(\gamma 3)$.

4. $x \ge z_4(s) + z$, $m = -z/(1-\mu)$; $z = x - z_4(s)$,

$$(\gamma 4) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_4(s) + \frac{1}{1+b}c_1((1+b)s).$$

Clearly $(\beta 4) = (\gamma 4)$ and $(\beta 1) = (\gamma 1)$. Moreover, the line $(\gamma 3)$ intersects $(\beta 2)$, $(\beta 3)$ and $(\gamma 2)$, $(\gamma 4)$ at points with first coordinate $z_6(s)$ and $z_4(s)$ respectively. Therefore the line $(\gamma 3)$ is the boundary of $G_{T-1}(s)$.

Step IV: $x \ge z_4(s)$.

1. $x \le z_2(s) + z$, $m = 0$; $z = x - z_2(s)$,

$$(\delta 1) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_2(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. $z_2(s) + z \le x \le z_6(s) + z$, $m = 0$; $z = x - z_6(s)$,

$$(\delta 2) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_6(s) + \frac{1}{1+b}c_2((1+b)s).$$

3. $z_6(s) + z \le x \le z_4(s) + z$, $m = 0$; $z = x - z_4(s)$,

$$(\delta 3) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right)z_4(s) + \frac{1}{1+a}c_1((1+a)s).$$

4. $x \ge z_4(s) + z$.

   (a) $z \in [0, x - z_4(s)]$, $m = 0$; $z = x - z_4(s)$,

$$(\delta\ 4) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1+\lambda}\right)z_4(s) + \frac{1}{1+b}c_1((1+b)s).$$

   (b) $z \in (-\infty, 0]$, $m = -z/(1-\mu)$; $z = 0$ and the line is above $(\delta 4)$.

Since $(\delta 1) = (\beta 1) = (\gamma 1)$, $(\delta 2) = (\gamma 2)$, $(\delta 3) = (\delta 4)$ and $(\gamma 3)$ intersects $(\delta 3)$ for $x = z_4(s)$, we see that the boundary of $G_{T-1}(s)$ is the line $(\delta 3) = (\delta 4)$.

As in Theorems 2 and 3 unless $z_2(s) = z_6(s) = z_4(s)$ we do not have a perfect hedging. If $z_2(s) = z_6(s) = z_4(s)$, then by Lemma 3, $(I3)(s) = (I4)(s) = 0$. Then from (iii) of Lemma 2 we have $\Delta(s) = 0$. Consequently, $(I1)(s) = (I2)(s) = 0$ and we have a perfect hedging as shown in Theorem 1.

Let now $\frac{1+a}{1+b} = \frac{1-\mu}{1+\lambda}$. By (27) we have three steps. As before we only list the values of m, z and the equations of the lines that are minimal.

Step I: $x \le z_2(s)$.

1. $x \le z_2(s) + z$, $m = -z/(1-\mu)$; $z = x - z_2(s)$, $(\alpha 1)$.
2. $z_2(s) + z \le x \le z_4(s) + z$, $m = -z/(1-\mu)$; $z = x - z_2(s)$, $(\alpha 1)$.
3. $x \ge z_4(s) + z$, $m = -z/(1-\mu)$; $z = x - z_4(s)$, $(\alpha 1)$.

The line $(\alpha 1)$ forms the boundary of $G_{T-1}(s)$.

Step I: $z_2(s) \le x \le z_4(s)$.

1. $x \le z_2(s) + z$, $m = 0$; $z = x - z_2(s)$, $(\beta 1)$.
2. $z_2(s) + z \le x \le z_4(s) + z$, $m = -z/(1-\mu)$; $z = 0$. $(\gamma 3)$
3. $x \ge z_4(s) + z$, $m = -z/(1-\mu)$; $z = x - z_4(s)$, $(\gamma 4)$.

Since at a point with first coordinate $z_2(s)$ we have $(\gamma\ 3) = (\beta 2) = (\alpha 1)$ and for $x = z_4(s)$, $(\alpha 4) = (\gamma 4) = (\gamma 3)$, we see that the boundary of $G_{T-1}(s)$ is $(\gamma 3)$.

Step III: $x \ge z_4(s)$.

1. $x \le z_2(s) + z$, $m = 0$; $z = x - z_2(s)$, $(\delta 1)$.
2. $z_2(s) + z \le x \le z_4(s) + z$, $m = 0$; $z = x - z_4(s)$, $(\delta 3)$.
3. $x \ge z_4(s) + z$, $m = 0$; $z = x - z_4(s)$, $(\delta 4)$.

Clearly as before $(\delta 3) = (\delta 4)$ and $(\delta 1) = (\beta 1)$. Since for $x = z_4(s)$ we have $(\delta 3) = (\gamma 3)$, the boundary of $G_{T-1}(s)$ is formed by $(\delta 3) = (\delta 4)$.

Having constructed the set $G_{T-1}(s)$ we now consider the aspect of hedging.

Under $\frac{1+a}{1+b} = \frac{1-\mu}{1+\lambda}$ we have by Lemma 2, $(I3)(s) + (I4)(s) = 0$ and $(I3)(s) = (I4)(s) = 0$. Consequently, $(I5)(s) = 0$ and

$$(1+b)c_1((1+a)s) = (1+a)c_2((1+b)s). \tag{28}$$

The boundary of $G_{T-1}(s)$ for $z_2(s) \le x \le z_4(s)$ is the line satisfying the following equivalent equations (see (28)):

$$y = -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s),$$
$$y = -\frac{1+r}{(1-\mu)(1+b)}x + \frac{1}{1+b}c_2((1+b)s).$$

Therefore

$$T_a y = -\frac{1}{1+\lambda} T_r x + c_1(T_a s), \qquad T_b y = -\frac{1}{1-\mu} T_r x + c_2(T_b s),$$

and we have a perfect hedging. ■

4(b) *Case* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \le 0$, $(I4)(s) \ge 0$. This case is similar to that of $(I1)(s) \ge 0$, $(I2)(s) \le 0$, and the statements of Theorems 3 and 5 below are almost identical.

THEOREM 5. *Under* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \le 0$, $(I4)(s) \ge 0$ *the form of the set* $G_{T-1}(s)$ *is the same as in Theorem* 3.

In the cases

$$\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda} \quad \text{or} \quad \frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda} \quad \text{with } z_2(s) = z_5(s),$$

which is equivalent to $(I1)(s) = (I2)(s) = (I3)(s) = (I4)(s) = 0$, *we have a perfect hedging with the same replicating strategies as in Theorem* 3.

If

$$\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda} \ne \frac{1+a}{1+b} \quad \text{and} \quad z_2(s) \ne z_5(s)$$

we do not have a perfect hedging, but for a hedging strategy one can choose the one defined in Theorem 3.

Finally, when

$$\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda} = \frac{1+a}{1+b} \quad \text{and} \quad z_2(s) \ne z_5(s)$$

we have a perfect hedging only when

$$\frac{1}{1+a} c_1((1+a)s) = \frac{1}{1+b} c_2((1+b)s)$$

and consequently $(I3)(s) = (I4)(s) = (I5)(s) = 0$, *with replicating strategies*

$$\begin{aligned} l = 0,\ m = \frac{z_2(s) - x}{1-\mu} \quad & \text{for } x \le z_2(s), \\ l = m = 0 \quad & \text{for } z_2(s) \le z \le z_5(s), \\ l = \frac{x - z_5(s)}{1+\lambda},\ m = 0 \quad & \text{for } x \ge z_5(s). \end{aligned}$$

Proof. By Lemma 2(vii) we have $(I5)(s) \le 0$. Using Lemma 3 we obtain

$$z_2(s) \le z_3(s) \le z_1(s) \le z_4(s) \le z_5(s) \tag{29}$$

and if $\frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}$ we have $z_6(s) \le z_2(s)$ while if $\frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda}$ it follows that $z_6(s) \ge z_5(s)$.

The case $\frac{1+a}{1+b} = \frac{1-\mu}{1+\lambda}$ holds only when (since then $(I5)(s) \le 0$)

$$\frac{1}{1+a} c_1((1+a)s) \le \frac{1}{1+b} c_2((1+b)s).$$

Therefore by Lemma 1 the following lines dominate in the respective intervals:

$$
\text{(30)}\qquad
\begin{aligned}
&(E1) \quad \text{for } x \le z_2(s) + z,\\
&(E3) \quad \text{for } z_2(s) + z \le x \le z_5(s) + z,\\
&(E4) \quad \text{for } x \ge z_5(s) + z.
\end{aligned}
$$

Notice that (30) is the same as (23). Since in the proof of Steps I–II in Theorem 3 to determine a minimal location of the lines we used the fact that $(I2)(s) \le 0$, which is satisfied in our case, the construction of the set $G_{T-1}(s)$ both in the case when $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ and $\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda}$ is identical to that of Theorem 3. We can also repeat the arguments concerning hedging for the cases $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ and $\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda}$ with $z_2(s) = z_5(s)$. Notice, however, that if $z_2(s) = z_5(s)$, then by Lemma 3, $(I2)(s) = 0$, and then since $(I5)(s) \le 0$, by Lemma 2(viii) we have $(I1)(s) = 0$. Therefore $(I5)(s) = 0$ and also $(I5)(s) = (I4)(s) = 0$ (by Lemma 2(vii)). In the case $\frac{1+r}{1+b} > \frac{1-\mu}{1+\lambda}$ with $z_2(s) \ne z_5(s)$ to have a perfect hedging the following equalities should be satisfied:

$$
\frac{1-\mu}{1+\lambda} = \frac{1+a}{1+b} \quad \text{and} \quad \frac{1}{1+a} c_1((1+a)s) = \frac{1}{1+b} c_2((1+b)s).
$$

Then $(I5)(s) = 0$ and consequently $(I3)(s) = (I4)(s) = 0$.

The proof of Theorem is thus complete. ∎

4(c) *Case* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \ge 0$, $(I4)(s) \le 0$. This case is very similar to that when $(I1)(s) \le 0$, $(I2)(s) \ge 0$. We show below that in both cases the sets $G_{T-1}(s)$ are identical.

THEOREM 6. *Under* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \ge 0$, $(I4)(s) \le 0$ *the set* $G_{T-1}(s)$ *is of the identical form as in Theorem* 2.

If

$$
\frac{1+r}{1+a} < \frac{1+\lambda}{1-\mu} \ne \frac{1+b}{1+a} \quad \textit{and} \quad z_3(s) \ne z_4(s)
$$

we do not have a perfect hedging. We have the same hedging strategy as in Theorem 2.

If

$$
\frac{1+r}{1+a} < \frac{1+\lambda}{1-\mu} = \frac{1+b}{1+a}
$$

we have a perfect hedging only when

$$
\frac{1+b}{1+a} c_1((1+a)s) = c_2((1+b)s)
$$

and then $(I3)(s) = (I4)(s) = (I5)(s) = 0$, *and the replicating strategies are*

$$l = 0,\ m = \frac{z_3(s) - x}{1-\mu} \quad \textit{for } x \le z_3(s),$$
$$m = l = 0 \quad \textit{for } z_3(s) \le x \le z_4(s),$$
$$l = \frac{x - z_4(s)}{1+\lambda},\ m = 0 \quad \textit{for } x \ge z_4(s).$$

If

$$\frac{1+r}{1+a} < \frac{1+\lambda}{1-\mu} \quad \textit{and} \quad z_3(s) = z_4(s)$$

(*equivalent to* $(I1)(s) = (I2)(s) = (I3)(s) = (I4)(s) = 0$), *or*

$$\frac{1+r}{1+a} \ge \frac{1+\lambda}{1-\mu},$$

then $G_{T-1}(s)$ *is a cone and we have a perfect hedging with replicating strategies*

$$l = 0,\ m = \frac{z_4(s) - x}{1-\mu} \quad \textit{for } x \le z_4(s),$$
$$l = \frac{x - z_4(s)}{1+\lambda},\ m = 0 \quad \textit{for } x \ge z_4(s).$$

Proof. By Lemma 2 we obtain $(I5)(s) \ge 0$. Then from Lemma 3,

$$z_3(s) \le z_2(s) \le z_1(s) \le z_5(s) \le z_4(s) \tag{31}$$

and when $\frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}$ we have $z_6(s) \ge z_4(s)$, while if $\frac{1+a}{1+b} > \frac{1-\mu}{1+\lambda}$, then $z_6(s) \le z_2(s)$. In the case when $\frac{1+a}{1+b} = \frac{1-\mu}{1+\lambda}$ since $(I5)(s) \ge 0$ we have $(1+b)c_1((1+a)s) \ge (1+a)c_2((1+b)s)$ and consequently $(E2) \ge (E3)$ (by Lemma 1).

Therefore we have the following dominating lines:

$$\begin{aligned} &(E1) \quad \text{for } x \le z_3(s) + z,\\ &(E2) \quad \text{for } z_3(s) + z \le x \le z_4(s) + z,\\ &(E4) \quad \text{for } x \ge z_4(s) + z. \end{aligned} \tag{32}$$

Notice now that (32) and (20) are identical. Since in the study of the location of the lines that formed the polyhedron $G_{T-1}(s)$, in the proof of Theorem 2, we used the fact that $(I1)(s) \le 0$ only, we can repeat the considerations of the proof of Theorem 2 to obtain the set $G_{T-1}(s)$.

The problem of perfect hedging can then be studied as in the proofs of Theorems 2 and 5 and therefore is left to the reader. Notice only that if $\frac{1+r}{1+a} < \frac{1+\lambda}{1-\mu}$ and $z_3(s) = z_4(s)$, then by Lemma 3, $(I1)(s) = 0$, and then by Lemma 2(iv), $\Delta(s) \ge 0$. Hence from Lemma 2(vi) we obtain $\Delta(s) = 0$ and consequently $(I1)(s) = (I2)(s) = (I3)(s) = (I4)(s) = 0$. ■

4(d) *Case* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \ge 0$, $(I4)(s) \ge 0$. This case is the most complicated; we have to split it into several subcases.

THEOREM 7. *Suppose* $(I1)(s) \le 0$, $(I2)(s) \le 0$, $(I3)(s) \ge 0$, $(I4)(s) \ge 0$. *If*

$$\frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}$$

then in the case

(a) $\dfrac{1+a}{1+r} \ge \dfrac{1-\mu}{1+\lambda}$ *and* $\dfrac{1+r}{1+b} \ge \dfrac{1-\mu}{1+\lambda}$

we have

$$\begin{aligned}G_{T-1}(s) = \Big\{(x,y) : y \ge{} & -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right)z_3(s)\\ & + \frac{1}{1+a}c_2((1+a)s) \textit{ for } x \le z_3(s),\\ y \ge{} & -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s)\\ & \textit{for } z_3(s) \le x \le z_6(s),\\ y \ge{} & -\frac{1+r}{(1-\mu)(1+b)}x + \frac{1}{1+b}c_2((1+b)s)\\ & \textit{for } z_6(s) \le x \le z_5(s),\\ y \ge{} & -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_5(s)\\ & + \frac{1}{1+b}c_2((1+b)s) \textit{ for } x \ge z_5(s)\Big\}\end{aligned}$$

with a perfect hedging only when $z_3(s) = z_6(s) = z_5(s)$, *which implies* $(I1)(s) = (I2)(s) = (I3)(s) = (I4)(s) = 0$, *and with a hedging strategy*

$$\begin{aligned} l = 0,\ m = \frac{z_3(s)-x}{1-\mu} \quad & \textit{for } x \le z_3(s),\\ m = l = 0 \quad & \textit{for } z_3(s) \le x \le z_5(s),\\ l = \frac{x - z_5(s)}{1+\lambda},\ m = 0 \quad & \textit{for } x \ge z_5(s);\end{aligned}$$

in the case

(b) $\dfrac{1+a}{1+r} \ge \dfrac{1-\mu}{1+\lambda}$ *and* $\dfrac{1+r}{1+b} \le \dfrac{1-\mu}{1+\lambda}$

we have

$$\begin{aligned}G_{T-1}(s) = \Big\{(x,y) : y \ge{} & -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right)z_3(s)\\ & + \frac{1}{1+a}c_2((1+a)s) \textit{ for } x \le z_3(s),\end{aligned}$$

$$y \geq -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s)$$
$$\text{for } z_3(s) \leq x \leq z_6(s),$$
$$y \geq -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right)z_6(s)$$
$$+ \frac{1}{1+a}c_1((1+a)s) \text{ for } x \geq z_6(s)\Big\}$$

with a perfect hedging only when $z_3(s) = z_6(s)$, *and a hedging strategy*

$$l = 0,\ m = \frac{z_3(s) - x}{1-\mu} \quad \text{for } x \leq z_3(s),$$
$$l = m = 0 \quad \text{for } z_3(s) \leq x \leq z_6(s),$$
$$l = \frac{x - z_6(s)}{1+\lambda},\ m = 0 \quad \text{for } x \geq z_6(s);$$

in the case

(c) $\dfrac{1+a}{1+r} \leq \dfrac{1-\mu}{1+\lambda}$ *and* $\dfrac{1+r}{1+b} \geq \dfrac{1-\mu}{1+\lambda}$

we have

$$G_{T-1}(s) = \Big\{(x,y) : y \geq -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1-\mu}\right)z_6(s)$$
$$+ \frac{1}{1+b}c_2((1+b)s) \text{ for } x \leq z_6(s),$$
$$y \geq -\frac{1+r}{(1-\mu)(1+b)}x + \frac{1}{1+b}c_2((1+b)s)$$
$$\text{for } z_6(s) \leq x \leq z_5(s),$$
$$y \geq -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_5(s)$$
$$+ \frac{1}{1+b}c_2((1+b)s) \text{ for } x \geq z_5(s)\Big\}$$

with a perfect hedging only when $z_5(s) = z_6(s)$, *and a hedging strategy*

$$l = 0,\ m = \frac{z_6(s) - x}{1-\mu} \quad \text{for } x \leq z_6(s),$$
$$m = l = 0 \quad \text{for } z_6(s) \leq x \leq z_5(s),$$
$$l = \frac{x - z_5(s)}{1+\lambda},\ m = 0 \quad \text{for } x \geq z_5(s);$$

and in the case

(d) $\dfrac{1+a}{1+r} \leq \dfrac{1-\mu}{1+\lambda}$ *and* $\dfrac{1+r}{1+b} \leq \dfrac{1-\mu}{1+\lambda}$

we have

$$\begin{aligned}G_{T-1}(s) = \Big\{(x,y) : y \geq & -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1-\mu}\right) z_6(s)\\ &+ \frac{1}{1+b}c_2((1+b)s) \textit{ for } x \leq z_6(s),\\ y \geq & -1 \leq 1+\lambda x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right) z_6(s)\\ &+ \frac{1}{1+a}c_1((1+b)s) \textit{ for } x \geq z_6(s)\Big\}\\ = (z_6(s), H_4 z_6(s)) + C &\end{aligned}$$

with

$$\begin{aligned}H_4 z_6(s) &= -\frac{1+r}{(1-\mu)(1+b)} z_6(s) + \frac{1}{1+b}c_2((1+b)s)\\ &= -\frac{1+r}{(1+\lambda)(1+a)} z_6(s) + \frac{1}{1+a}c_1((1+a)s)\end{aligned}$$

and we have a perfect hedging in one step with replicating strategies

$$\begin{aligned}l = 0,\ m &= \frac{z_6(s) - x}{1-\mu} \quad \textit{for } x \leq z_6(s),\\ l = \frac{x - z_6(s)}{1+\lambda},\ m &= 0 \quad \textit{for } x \geq z_6(s).\end{aligned}$$

If

$$\frac{1+a}{1+b} = \frac{1-\mu}{1+\lambda}$$

we have

$$\begin{aligned}G_{T-1}(s) = \Big\{(x,y) : y \geq & -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1-\mu}\right) z_3(s)\\ &+ \frac{1}{1+a}c_2((1+a)s) \textit{ for } x \leq z_3(s),\\ y \geq & -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s)\\ &\textit{for } z_3(s) \leq x \leq z_5(s),\\ y \geq & -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right) z_5(s)\\ &+ \frac{1}{1+b}c_2((1+b)s) \textit{ for } x \geq z_5(s)\Big\}\end{aligned}$$

with a perfect hedging and a replicating strategy

$$l=0,\ m=\frac{z_3(s)-x}{1-\mu}\qquad \textit{for } x\le z_3(s),$$
$$l=m=0\qquad \textit{for } z_3(s)\le x\le z_5(s),$$
$$l=\frac{x-z_5(s)}{1+\lambda},\ m=0\qquad \textit{for } x\ge z_5(s).$$

The case

$$\frac{1+a}{1+b}>\frac{1-\mu}{1+\lambda}$$

is impossible.

Proof. By Lemma 2(i), (iii), $\Delta(s)\le 0$ and consequently we have $\frac{1+a}{1+b}\le\frac{1-\mu}{1+\lambda}$. If $\frac{1+a}{1+b}=\frac{1-\mu}{1+\lambda}$ then $(I3)(s)=(I4)(s)=0$ and therefore $(I5)(s)=0$,

$$\frac{1}{1+a}c_1((1+a)s)=\frac{1}{1+b}c_2((1+b)s)$$

and $(E2)=(E3)$.

From Lemma 3 we then have

$$z_3(s)\le z_2(s)\le z_1(s)\le z_4(s)\le z_5(s)\tag{33}$$

and if $\frac{1+a}{1+b}<\frac{1-\mu}{1+\lambda}$, then $z_1(s)\ge z_6(s)$ for $(I5)(s)\le 0$ and $z_1(s)\le z_6(s)$ for $(I5)(s)\ge 0$.

Therefore using Lemma 1 we obtain the following dominating lines for $\frac{1+a}{1+b}<\frac{1-\mu}{1+\lambda}$:

$$\begin{aligned}&(E1)\quad \text{for } x\le z_3(s)+z,\\ &(E2)\quad \text{for } z_3(s)+z\le x\le z_6(s)+z,\\ &(E3)\quad \text{for } z_6(s)+z\le x\le z_5(s)+z,\\ &(E4)\quad \text{for } x\ge z_5(s)+z,\end{aligned}\tag{34}$$

and for $\frac{1+a}{1+b}=\frac{1-\mu}{1+\lambda}$:

$$\begin{aligned}&(E1)\quad \text{for } x\le z_3(s)+z,\\ &(E2)\quad \text{for } z_3(s)+z\le x\le z_5(s)+z,\\ &(E4)\quad \text{for } x\ge z_5(s)+z.\end{aligned}\tag{35}$$

We list below the values of m,z and the equations of the lowest lines for the case $\frac{1+a}{1+b}<\frac{1-\mu}{1+\lambda}$.

Step I: $x\le z_3(s)$.

1. $x\le z_3(s)+z$.

(a) $z\in[x-z_3(s),0]$, $m=-z/(1-\mu)$; $z=x-z_3$,

$$y=-\frac{1}{1-\mu}x-\left(\frac{1+r}{(1-\mu)(1+a)}-\frac{1}{1-\mu}\right)z_3(s)+\frac{1}{1+a}c_2((1+a)s).\tag{$\alpha1$}$$

(b) $z \geq 0$, $m = 0$; $z = 0$ and the line is above ($\alpha 1$).

2. $z_3(s) + z \leq x \leq z_6(s) + z$, $m = -z/(1-\mu)$.

(a) If $\frac{1+a}{1+r} \leq \frac{1-\mu}{1+\lambda}$, then $z = x - z_6(s)$,

$$(\alpha 21) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1-\mu}\right)z_6(s) + \frac{1}{1+a}c_1((1+a)s).$$

(b) If $\frac{1+a}{1+r} \geq \frac{1-\mu}{1+\lambda}$, then $z = x - z_3(s)$,

$$(\alpha 22) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1-\mu}\right)z_3(s) + \frac{1}{1+a}c_1((1+a)s).$$

3. $z_6(s) + z \leq x \leq z_5(s) + z$, $m = -z/(1-\mu)$; $z = x - z_6(s)$,

$$(\alpha 3) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1-\mu}\right)z_6(s) + \frac{1}{1+b}c_2((1+b)s).$$

4. $x \geq z_5(s) + z$, $m = -z/(1-\mu)$; $z = x - z_5(s)$,

$$(\alpha 4) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_5(s) + \frac{1}{1+b}c_1((1+b)s).$$

As $(I4)(s) \geq 0$ we obtain $(\alpha 3) \leq (\alpha 4)$. Similarly from $(I3)(s) \geq 0$ we see that $(\alpha 1) \geq (\alpha 3)$ for $\frac{1+a}{1+r} \leq \frac{1-\mu}{1+\lambda}$, while $(\alpha 1) \leq (\alpha 3)$ for $\frac{1+a}{1+r} \geq \frac{1-\mu}{1+\lambda}$. Therefore the lowest lines are $(\alpha 3)$ if $\frac{1+a}{1+r} \leq \frac{1-\mu}{1+\lambda}$, and $(\alpha 1)$ if $\frac{1+a}{1+r} \geq \frac{1-\mu}{1+\lambda}$.

Step II: $z_3(s) \leq x \leq z_6(s)$.

1. $x \leq z_3(s) + z$, $m = 0$; $z = x - z_3(s)$,

$$(\beta 1) \quad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_3(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. $z_3(s) + z \leq x \leq z_6(s) + z$.

(a) $z \in [x - z_6(s), 0]$, $m = -z/(1-\mu)$; and if $\frac{1+a}{1+r} \leq \frac{1-\mu}{1+\lambda}$, then $z = x - z_6(s)$,

$$(\beta 21) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1-\mu}\right)z_6(s) + \frac{1}{1+a}c_1((1+a)s),$$

while if $\frac{1+a}{1+r} \geq \frac{1-\mu}{1+\lambda}$, then $z = 0$,

$$(\beta 22) \quad y = -\frac{1+r}{(1+\lambda)(1+a)}x + \frac{1}{1+a}c_1((1+a)s).$$

(b) $z \in [0, x - z_3(s)]$, $m = 0$; $z = 0$ and the line coincides with ($\beta 22$).

3. $z_6(s) + z \leq x \leq z_5(s) + z$, $m = -z/(1-\mu)$; $z = x - z_6(s)$,

$$(\beta 3) \quad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1-\mu}\right)z_6(s) + \frac{1}{1+b}c_2((1+b)s).$$

4. $x \geq z_5(s)+z$, $m=-z/(1-\mu)$; $z = x - z_5$,

$$(\beta 4) \qquad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_5(s) + \frac{1}{1+b}c_1((1+b)s).$$

Notice that $(\alpha 3) = (\beta 3) = (\beta 21)$ and $(\alpha 4) = (\beta 4)$. Moreover, for $x = z_3(s)$ we have $(\alpha 1) = (\beta 1)$, while for $x = z_6(s)$, $(\alpha 3) = (\beta 22)$. In addition, from $(I3)(s) \geq 0$ we obtain $(\alpha 3) \leq (\beta 1)$ for $x = z_6(s)$. Therefore if $\frac{1+a}{1+r} \geq \frac{1-\mu}{1+\lambda}$ then the lowest line is $(\beta 22)$ and if $\frac{1+a}{1+r} \leq \frac{1-\mu}{1+\lambda}$ then the lowest line is $(\alpha 3) = (\beta 3)$.

Step III: $z_6(s) \leq x \leq z_5(s)$.

1. $x \leq z_3(s) + z$, $m = 0$; $z = x - z_3(s)$,

$$(\gamma 1) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_3(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. $z_3(s) + z \leq x \leq z_6(s) + z$, $m = 0$; $z = x - z_6(s)$,

$$(\gamma 2) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1+\lambda)(1+a)} - \frac{1}{1+\lambda}\right)z_6(s) + \frac{1}{1+a}c_1((1+a)s).$$

3. $z_6(s) + z \leq x \leq z_5(s) + z$.

(a) $z \in [x - z_5(s), 0]$, $m = -z/(1-\mu)$; $z = 0$,

$$(\gamma 31) \qquad y = -\frac{1+r}{(1-\mu)(1+b)}x + \frac{1}{1+b}c_2((1+b)s).$$

(b) $z \in [0, x - z_6(s)]$, $m = 0$; if $\frac{1+r}{1+b} \geq \frac{1-\mu}{1+\lambda}$, then $z = 0$ and the line coincides with $(\gamma 31)$, and if $\frac{1+r}{1+b} \leq \frac{1-\mu}{1+\lambda}$, then $z = x - z_6(s)$,

$$(\gamma 32) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+b)} - \frac{1}{1+\lambda}\right)z_6(s) + \frac{1}{1+b}c_2((1+b)s).$$

4. $x \geq z_5(s) + z$, $m = -z/(1-\mu)$; $z = x - z_5(s)$,

$$(\gamma 4) \qquad y = -\frac{1}{1-\mu}x - \left(\frac{1+r}{(1+\lambda)(1+b)} - \frac{1}{1-\mu}\right)z_5(s) + \frac{1}{1+b}c_1((1+b)s).$$

We have $(\gamma 4) = (\beta 4)$, $(\gamma 1) = (\beta 1)$, $(\gamma 2) = (\gamma 32)$ and since $(I3)(s) \geq 0$, $(\gamma 32) \leq (\gamma 1)$.

Moreover, for $x = z_6(s)$, $(\gamma 31) = (\gamma 2) = (\beta 22)$, and $(\gamma 31) = (\gamma 4)$ for $x = z_5(s)$. Therefore the boundary of $G_{T-1}(s)$ is $(\gamma 31)$ if $\frac{1+r}{1+b} \geq \frac{1-\mu}{1+\lambda}$ and $(\gamma 32)$ when $\frac{1+r}{1+b} \leq \frac{1-\mu}{1+\lambda}$.

Step IV: $x \geq z_5(s)$.

1. $x \leq z_3(s) + z$, $m = 0$; $z = x - z_3(s)$,

$$(\delta 1) \qquad y = -\frac{1}{1+\lambda}x - \left(\frac{1+r}{(1-\mu)(1+a)} - \frac{1}{1+\lambda}\right)z_3(s) + \frac{1}{1+a}c_2((1+a)s).$$

2. $z_3(s)+z \le x \le z_6(s)+z$, $m=0$; $z=x-z_6(s)$,

$$(\delta 2) \qquad y=-\frac{1}{1+\lambda}x-\left(\frac{1+r}{(1+\lambda)(1+a)}-\frac{1}{1+\lambda}\right)z_6(s)+\frac{1}{1+a}c_1((1+a)s).$$

3. $z_6(s)+z \le x \le z_5(s)+z$, $m=0$; if $\frac{1+r}{1+b} \ge \frac{1-\mu}{1+\lambda}$, then $z=x-z_5(s)$,

$$(\delta 31) \qquad y=-\frac{1}{1+\lambda}x-\left(\frac{1+r}{(1-\mu)(1+b)}-\frac{1}{1+\lambda}\right)z_5(s)+\frac{1}{1+b}c_2((1+b)s),$$

while if $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$, then $z=x-z_6(s)$,

$$(\delta 32) \qquad y=-\frac{1}{1+\lambda}x-\left(\frac{1+r}{(1-\mu)(1+b)}-\frac{1}{1+\lambda}\right)z_6(s)+\frac{1}{1+b}c_2((1+b)s).$$

4. $x \ge z_5(s)+z$.

(a) $z \in (-\infty,0]$, $m=-z/(1-\mu)$; $z=0$ so that the line is also considered in the case (b).

(b) $z \in [0,x-z_5(s)]$, $m=0$; $z=x-z_5(s)$,

$$(\delta 4) \qquad y=-\frac{1}{1+\lambda}x-\left(\frac{1+r}{(1+\lambda)(1+b)}-\frac{1}{1+\lambda}\right)z_5(s)+\frac{1}{1+b}c_1((1+b)s).$$

Notice that $(\delta 1)=(\beta 1)=(\gamma 1)$, $(\delta 31)=(\delta 4)$ and $(\delta 32)=(\delta 2)=(\gamma 2)$. Since $(I4)(s) \ge 0$, if $\frac{1+r}{1+b} \ge \frac{1-\mu}{1+\lambda}$ we have $(\delta 32) \ge (\delta 31)$ while if $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ then $(\delta 32) \le (\delta 31)$. Moreover, for $x=z_5(s)$ we have $(\gamma 31)=(\delta 31)$. Therefore the line $(\delta 31)$ if $\frac{1+r}{1+b} \ge \frac{1-\mu}{1+\lambda}$ and the line $(\delta 32)$ if $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$ each form the boundary of $G_{T-1}(s)$.

The construction of $G_{T-1}(s)$ when $\frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}$ is thus completed.

We list below the results necessary to find $G_{T-1}(s)$ when $\frac{1+a}{1+b} = \frac{1-\mu}{1+\lambda}$.

Step I: $x \le z_3(s)$.

1. $x \le z_3(s)+z$; $(\alpha 1)$.
2. $z_3(s)+z \le x \le z_5(s)+z$, $m=-z/(1-\mu)$; $z=x-z_3(s)$ and $(\alpha 22)$.
3. $x \ge z_5(s)+z$; $(\alpha 4)$.

We have $(\alpha 1)=(\alpha 22) \le (\alpha 4)$.

Step II: $z_3(s) \le x \le z_5(s)$.

1. $x \le z_3(s)+z$; $(\beta 1)$.
2. $z_3(s)+z \le x \le z_5(s)+z$.

(a) $z \in [x-z_5(s),0]$, $m=-z/(1-\mu)$; $z=0$ and $(\beta 22)$.

(b) $z \in [0,x-z_3(s)]$, $m=0$; $z=0$ and $(\beta 22)$.

3. $x \ge z_5(s)+z$ and $(\beta 4)$.

Clearly the line $(\beta 22)$ forms the boundary for $G_{T-1}(s)$.

Step III: $x \geq z_5(s)$.

1. $x \leq z_3(s) + z$; $(\delta 1)$.
2. $z_3(s) + z \leq x \leq z_5(s) + z$, $m = 0$; $z = x - z_5(s)$ and $(\delta 31)$.
3. $x \geq z_5(s) + z$, $m = 0$; $z = x - z_5(s)$ and $(\delta 4)$.

Since $(\delta 4) = (\delta 31) \leq (\delta 1)$, $(\delta 31)$ is the boundary of $G_{T-1}(s)$.

The aspect of perfect hedging can be studied as in the previous theorems. We only point out that under $\frac{1+a}{1+b} < \frac{1-\mu}{1+\lambda}$, $z_3(s) = z_6(s) = z_5(s)$ if and only if $(I3)(s) = (I4)(s) = 0$. Then using Lemma 2, we have $\Delta(s) = 0$ and consequently $(I1)(s) = (I2)(s) = 0$.

The remaining part of the proof is left to the reader. ■

5. Conclusions and examples. We are now in a position to combine Theorems 1–7. Notice first that a perfect hedging in one step usually occurs when $G_{T-1}(s)$ is a cone. The only exception to this rule is when $\frac{1+a}{1+b} = \frac{1-\mu}{1+\lambda}$. Excluding this case under the assumption that $\frac{1+r}{1+a} = \frac{1+b}{1+r}$ we easily obtain the following necessary and sufficient condition for a perfect hedging.

THEOREM 8. *Let*

$$\frac{1+r}{1+a} = \frac{1+b}{1+r} \tag{36}$$

and

$$\frac{1+a}{1+b} \neq \frac{1-\mu}{1+\lambda}. \tag{37}$$

We have a perfect hedging in one step if and only if either $(I1)(s) \geq 0$ *and* $(I2)(s) \geq 0$, *or*

$$\frac{1-\mu}{1+\lambda} \geq \frac{1+a}{1+r}. \tag{38}$$

Moreover, *under* (38) *all polyhedrons* $G_t(s)$, $t = 0, 1, \ldots, T-1$, $s \geq 0$, *are cones.* ■

If we drop the assumption (36) we still obtain a sufficient condition for hedging that generalizes Theorem 4 of [B-V].

THEOREM 9. *Under* (37) *and*

$$\min\left\{\frac{1+r}{1+a}, \frac{1+b}{1+r}\right\} \geq \frac{1+\lambda}{1-\mu} \tag{39}$$

we have a perfect hedging. ■

Notice that in Theorem 9 no assumptions on the contingent claim $(f_1(s_T), f_2(s_T))$ are imposed.

From Theorem 9 we obtain Theorem 3.2 of [BLPS]:

COROLLARY 1. *Under* (36) *and* (38) *we have a perfect hedging.*

Proof. It remains to notice that

$$\frac{1+r}{1+a}\cdot\frac{1+b}{1+r}=\frac{1+b}{1+a}\geq\left(\frac{1+\lambda}{1-\mu}\right)^2>\frac{1+\lambda}{1-\mu},$$

from which (37) follows, and then use Theorem 9. ■

Before we formulate the next result we introduce the following condition:

(As) the polyhedrons $G_0(s)$, $G_1((1+a)^{i_1}(1+b)^{j_1}s)$, $\ldots$, $G_{T-1}((1+a)^{i_{T-1}} \times (1+b)^{j_{T-1}}s)$, with nonnegative integers i_k, j_k, $k=1,\ldots,T-1$, such that $i_k+j_k=k$, are cones.

THEOREM 10. *Under* (37) *for a given initial price s_0 of the stock we have a perfect hedging if and only if (As_0) is satisfied.*

Assuming (As_0) only, the polyhedrons $G_k(s_k)$ for $k=0,1,\ldots,T-1$ are cones and there is a perfect hedging with a replicating strategy $(\overline{l}_n,\overline{m}_n)$ that shifts (x_n,y_n) at time n to the vertex of the cone $G_n(s_n)$.

Proof. The proof is based on an analysis of Theorems 1–7 and the definition (8), (9) of the polyhedrons $G_n(s_n)$. ■

It will be convenient to have the following equivalent conditions for the signs of the indicators $(I1)$–$(I4)$, which can be obtained by an easy verification.

LEMMA 4. (i) $(I1)(s)\geq 0$ *iff*

$$f_1((1+a)s)-f_1((1+b)s)\leq\frac{1+\lambda}{1+a}[(1+a)f_2((1+b)s)-(1+b)f_2((1+a)s)].$$

(ii) $(I2)(s)\geq 0$ *iff*

$$f_1((1+a)s)-f_1((1+b)s)\geq\frac{1-\mu}{1+b}[(1+a)f_2((1+b)s)-(1+b)f_2((1+a)s)].$$

(iii) $(I3)(s)\geq 0$ *iff*

$$f_1((1+a)s)-f_1((1+b)s)\geq\frac{1-\mu}{1+a}[(1+a)f_2((1+b)s)-(1+b)f_2((1+a)s)].$$

(iv) $(I4)(s)\geq 0$ *iff*

$$f_1((1+a)s)-f_1((1+b)s)\leq\frac{1+\lambda}{1+b}[(1+a)f_2((1+b)s)-(1+b)f_2((1+a)s)].\ ■$$

Combining (i) and (ii) we obtain

COROLLARY 2. *If* $(I1)(s)\geq 0$ *and* $(I2)(s)\geq 0$ *then*

$$(1+a)f_2((1+b)s)\geq(1+b)f_2((1+a)s).$$

If $(I1)(s)\leq 0$ *and* $(I2)(s)\leq 0$ *then*

$$(1+a)f_2((1+b)s)\leq(1+b)f_2((1+a)s).\ ■$$

Consider now the following examples of options.

Example 5 (European call option with cash settlement). In contrast to Example 2 we do not have delivery. We assume that $f_1(s) = (s - q)^+$ and $f_2(s) = 0$. By Lemma 4(i), (ii) we easily see that $(I1)(s) \geq 0$ and $(I2)(s) \leq 0$. Consequently, by Theorem 3 we have a perfect hedging in one step, for each $s \geq 0$, only when $\frac{1+r}{1+b} \leq \frac{1-\mu}{1+\lambda}$.

Example 6 (European short call option). We consider the European call option from the so-called *short position*, i.e. from the position of the seller of the option. His contingent claim is then

$$f_1(s) = q\, 1_{s\geq q}, \qquad f_2(s) = -s\, 1_{s\geq q}.$$

It is easy to check (using e.g. Lemma 4) that

$$\begin{aligned} (I1)(s) \leq 0,\ (I2)(s) \leq 0,\ (I3)(s) \geq 0 \quad & \text{for } s \geq \frac{q}{(1+b)(1-\mu)}, \\ (I3)(s) \leq 0 \quad & \text{for } s \leq \frac{q}{(1+b)(1-\mu)}, \\ (I4)(s) \geq 0 \quad & \text{for } s \leq \frac{q}{(1+a)(1+\lambda)}, \\ (I4)(s) \leq 0 \quad & \text{for } s \geq \frac{q}{(1+a)(1+\lambda)}, \end{aligned}$$

with a perfect hedging only in particular cases, e.g. when (37) and (39) are satisfied.

Another family of examples come from exotic options:

Example 7 (Long collar). The collar is an example of so-called *packages* which are combinations of options, assets and cash. Let $q_2 > q_1$. The payoff for the long position in the collar is

$$f_1(s) = q_1\, 1_{s<q_1} + q_2\, 1_{s>q_2}, \qquad f_2(s) = s\, 1_{q_1\leq s\leq q_2}.$$

Using Lemma 4 one can verify that $(I1)(s) < 0$, $(I2)(s) < 0$ for s such that

$$\frac{q_1}{1+a} \leq s \leq \frac{q_2}{1+a} \quad \text{and} \quad \frac{q_2}{1+b} < s$$

and for such s the signs of $(I3)(s)$ and $(I4)(s)$ are not uniquely determined. For other $s \geq 0$ we have $(I1)(s) \geq 0$ and $(I2)(s) \geq 0$. Consequently, we have a perfect hedging independent of s only when (37) and (39) are satisfied.

Example 8 (Long spread). This option is the sum of long European call and put options with various striking prices q_1 and q_2, where e.g. $q_1 \leq q_2$. The contingent claim is

$$f_1(s) = -q_1\, 1_{s\leq q_1} + q_2\, 1_{s\leq q_2}, \qquad f_2(s) = -s\, 1_{q_1\leq s\leq q_2}.$$

By Lemma 4 we have $(I1)(s) \le 0$ and $(I2)(s) \ge 0$ and consequently by Theorem 2 we have a perfect hedging in one step only when $\frac{1+r}{1+a} \ge \frac{1+\lambda}{1-\mu}$.

EXAMPLE 9 (Binary options). A large family of contracts with payoff depending in a discontinuous way on the price of the underlying asset at maturity are called *binary options*. The simplest examples of such options are cash or nothing and asset or nothing options. Considering the long position of call options in the first case (long call cash or nothing) we have

$$f_1(s) = X\,1_{s>K}, \quad f_2(s) = 0,$$

while in the second (long call asset or nothing)

$$f_1(s) = 0, \quad f_2(s) = s\,1_{s<K},$$

with X and K being fixed constants. Using Lemma 4 again we find that in the case of the long call cash or nothing option we have $(I1)(s) \ge 0$ and $(I2)(s) \le 0$ and by Theorem 3 there is a perfect hedging in one step when $\frac{1+r}{1+b} \le \frac{1-\mu}{1+\lambda}$, while in the case of the long call asset or nothing $(I1)(s) \le 0$ and $(I2)(s) \ge 0$, so by Theorem 2 there is a perfect hedging in one step independent of $s \ge 0$ only when $\frac{1+r}{1+a} \ge \frac{1+\lambda}{1-\mu}$.

Remark. In the examples we were mainly interested in the so-called long position, i.e. the position of the buyer of an option, and the price was the one that guaranteed replication of the potential loss of the seller. One can consider, similarly to Example 6, options from the short position, i.e. the position of the seller, and then using the results of Sections 3–5 characterize perfect hedging situations. Moreover, each option can have various forms of realization: in assets or in cash (see Examples 1–3 and 5), according to the preferences of the buyer.

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