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STATISTICAL ESTIMATION OF HIGHER-ORDER SPECTRAL DENSITIES BY MEANS OF GENERAL TAPERING

Abstract. Given a realization on a finite interval of a continuous-time stationary process, we construct estimators for higher order spectral densities. Tapering and shift-in-time methods are used to build estimators which are asymptotically unbiased and consistent for all admissible values of the argument. Asymptotic results for the fourth-order densities are given. Detailed attention is paid to the nth order case.

1. Introduction. Higher-order spectra are of importance in many applications: geophysics, astronomy, turbulence, plasmas and other topics (see [3], [7], [12], [21], [22]). Estimation of higher-order spectral densities is also of considerable interest in resolving problems about stochastic processes. To realize such an estimation, we need more general processes (e.g., nongaussian, nonlinear) than second order processes (see [2], [5], [8], [25]). A large class of statistics have been proposed for this estimation in [2], [5], [8], [14], [16], [20], [27]. The statistics in [8] have an essential shortcoming: they do not allow us to construct estimators of higher-order spectral densities of all (admissible) values of the arguments $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 + \ldots + \lambda_n = 0$. These statistics do not give any answer on subsets where certain subgroups of arguments satisfy the same relation $\lambda_{k_1} + \ldots + \lambda_{k_p} = 0$, $1 \le k_i \le n$, $i = 1, \ldots, p$, $p = 1, \ldots, n$. On these manifolds, estimation of higher-order spectral densities is disturbed because the cumulant spectral densities and Fourier transforms of product moments of the same order do not agree. Such a problem does not appear when the cumulant spectral density and the moment spectral density are the same, as is the case for

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order two and three spectra for zero mean stationary processes (see [6], [13], [23], [24], [26]).

In the present work, we construct a statistic for higher-order spectral densities of zero mean continuous-time stationary processes. To this end we first estimate the product moment spectral densities and then use an identity between cumulant and product moment spectral densities.

This statistic does not have the inconveniences previously indicated; we use general tapering and shift-in-time methods. The tapering method consists in multiplying the observation $X(t)$ by a suitably chosen function $h_T(t)$ and in studying statistical properties by means of the tapered process $h_T(t)X(t)$. This method was suggested in Cooley and Tukey [9]; it reduces considerably the impact of dependence on remote frequencies, but leads to a noticeable increase in the variance level and the mean-square deviation for large size observations. Another application of this method is to reduce leakage when the spectrum has strong peaks (see [4], [23]) and it is used in situations of missing observations; here $h_T(t)$ is taken to be 0 or 1 (see [4], [11], [15]). To reduce the difficulty caused by the tapering method, we use moreover the shift-in-time method. We show that tapering allows us not only to reduce the bias but also to reduce the estimator variance (see, for example, [2], [6], [10], [17], [23] for further discussion of tapering).

In Zhurbenko [27, Chapter VI] a statistic for higher-order spectral densities is constructed and studied when the process is discrete, and a shift in time is used; that statistic avoids the preceding defects of [8]. Isakova [14] studied the same statistic as in [27] and in addition used gaussian tapering of a continuous-time process without restricting the set of admissible arguments; however, her estimator seems difficult to compute effectively, for it requires the observation of the process on $\mathbb{R}$ and we must know $X(t)$ for all t, even when gaussian tapering allows eliminating large values of t.

In this paper we avoid this problem by using tapering functions with finite support. In Section 2, we give definitions and notations. In Section 3, we define the estimators and explain the method used. In Section 4, examples of processes and tapering functions are given; in Sections 5 and 6, we study the asymptotic mean and variance of our estimators. First we give asymptotic results for the fourth-order case, and then we completely study the nth order case.

2. Definitions and notations. As explained in the introduction, we consider a class of processes which is more general than the class of second-order processes. We use a class Δ which is similar to $\Delta^{(k)}$ proposed by Kolmogorov (see [25]). Brillinger [5] proposed a class $\psi^{(k)}$ generalizing the class $\Delta^{(k)}$ and showed that this class is suitable for higher-order spectra. Before defining the class Δ, we give some definitions and notations.

1. $x = (x_1, \dots, x_n)$ represents a point or vector in a Euclidean space $\mathbb{R}^n$ of dimension n, with canonical scalar product, $x \cdot y = x_1 y_1 + \dots + x_n y_n$.

2. Let $\overline{x}_1, \dots, \overline{x}_p$ be a partition of the set of coordinates of the vector x into unordered subsets $\overline{x}_k = (x_{k_1}, \dots, x_{k_{n_k}})$, $k = 1, \dots, p$, $\overline{x}_k \in \mathbb{R}^{n_k}$; $\overline{x}_k$ stands for the projection of $x \in \mathbb{R}^n$ onto $\mathbb{R}^{n_k}$, with $n_1 + \dots + n_p = n$. In that case we write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_p}$ and $x = \overline{x}_1 \dot{+} \dots \dot{+} \overline{x}_p$. By $\lambda', \lambda'' \in \mathbb{R}^{2n}$ we shall denote vectors whose first, or respectively last, n coordinates coincide with $\lambda \in \mathbb{R}^n$ while the others are zero, i.e., $\lambda' = (\lambda_1, \dots, \lambda_n, 0, \dots, 0)$ and $\lambda'' = (0, \dots, 0, \lambda_1, \dots, \lambda_n)$.

3. For $x \in \mathbb{R}^n$, $|x| = x_1 + \dots + x_n$, $dx = dx_1 \dots dx_n$, $d\breve{x} = dx_1 \dots dx_{n-1}$ and $\delta|x| = \delta_{x_1 + \dots + x_n = 0}$, where δ is the Dirac function.

Let us notice that some of these definitions were used in [27, Chapter VI].

4. $X = \{X(t), -\infty < t < \infty\}$ stands for a complex- or real-valued stochastic process.

5. The *product moment* $m(t) = m(t_1, \dots, t_k)$ of order k is given by

$$m(t) = \mathbb{E}X(t_1) \dots X(t_k). \tag{1}$$

6. The *cumulant* $c(t) = c(t_1, \dots, t_n)$ of order n is defined through product moments by

$$c(t) = \sum (-1)^{p-1} (p-1)!\, m(\overline{t}_1) \dots m(\overline{t}_p), \tag{2}$$

where the summation is over all partitions of $(t_1, \dots, t_n)$ into unordered subsets $\overline{t}_k = (t_{k_1}, \dots, t_{k_{n_k}})$, $1 \le k_j \le n$, $j = 1, \dots, n_k$, $k = 1, \dots, p$.

We now define the class Δ. It contains the real-valued, continuous-time stochastic processes of zero mean satisfying the following conditions.

(a) The product moments $m(t)$, for every $t \in \mathbb{R}^k$, exist up to order n.

(b) For $u \in \mathbb{R}$, $k = 2, \dots, n$ and $(t_1, \dots, t_k) \in \mathbb{R}^k$,

$$m(t_1 + u, \dots, t_k + u) = m(t_1, \dots, t_k).$$

(c) For $k = 2, \dots, n$ there exists a measure $\delta|\lambda| F(d\lambda)$ absolutely continuous with respect to the Lebesgue measure on the manifolds $\lambda_1 + \dots + \lambda_k = 0$ such that

$$c(t) = \int\limits_{\mathbb{R}^{k-1}} \delta|\lambda| \exp\{i(t_1 \lambda_1 + \dots + t_k \lambda_k)\} F(d\lambda), \tag{3}$$

where $t = (t_1, \dots, t_k) \in \mathbb{R}^k$.

Throughout this work, X is a stochastic process belonging to the class Δ. A number of comments may be made about Δ.

(i) In view of (c), there exist functions $f(\lambda)$ such that $\delta|\lambda|F(d\lambda) = \delta|\lambda|f(\lambda)d\lambda$ for $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $|\lambda| = 0$. The function $f(\lambda)$ is called the *cumulant spectral density*.

(ii) In the same manner, we can define the moment spectral density g, as a Fourier transform of the product moment m. The cumulant and the moment spectral densities are connected with each other by the relations

$$\begin{aligned} \delta|x|\, f(x) &= \sum (-1)^{p-1}(p-1)!\ \delta|\overline{x}_1|\, g(\overline{x}_1)\dots\delta|\,\overline{x}_p|\, g(\overline{x}_p), \\ \delta|x|\, g(x) &= \sum \delta|\overline{x}_1|\, f(\overline{x}_1)\dots\delta|\overline{x}_p|\, f(\overline{x}_p), \end{aligned} \tag{4}$$

where the summation is over all partitions of $(x_1, \dots, x_n)$ into unordered subsets $\overline{x}_k = (x_{k_1}, \dots, x_{k_{n_k}})$, $1 \le k_j \le n$, $j = 1, \dots, n_k$.

(iii) Conditions (a) and (b) imply the stationarity of the process X.

(iv) Moreover, if the cumulant function $c(\cdot)$ and the cumulant spectral density $f(\cdot)$ are absolutely integrable, for a process X satisfying conditions (a) and (b) above, then X belongs to the class Δ. In that case, the inverse Fourier transform of (3) exists and is uniformly continuous.

For more details, see [5] and [2, Chapter 1].

3. Estimation. To construct estimators of the higher-order spectral density $f(\lambda)$, and to eliminate the difficulties caused by lower order moment spectral densities, we first estimate the moment spectral densities $g(\overline{\lambda}_k)$ by means of suitable estimators $\widehat{g}(\overline{\lambda}_k)$, for all elements $\overline{\lambda}_k$ in the set of admissible arguments: $\overline{\lambda}_k = (\lambda_{k_1}, \dots, \lambda_{k_{n_k}}) \in \mathbb{R}^{n_k}$, $|\overline{\lambda}_k| = 0$, for the vector $\lambda = (\lambda_1, \dots, \lambda_n)$, and then in (4) we set $\widehat{g}(\overline{\lambda}_k)$ in place of $g(\overline{\lambda}_k)$ to obtain the estimator $\widehat{f}(\lambda)$ of $f(\lambda)$, given by

$$\widehat{f}(\lambda) = \sum\nolimits^{0} (-1)^{p-1}(p-1)!\, \widehat{g}(\overline{\lambda}_1)\dots\widehat{g}(\overline{\lambda}_p), \tag{5}$$

where the summation is over all unordered partitions of coordinates of $\lambda \in \mathbb{R}^n$, with $|\lambda| = 0$, $n \ge 2$, $\overline{\lambda}_k = (\lambda_{k_1}, \dots, \lambda_{k_{n_k}})$, $|\overline{\lambda}_k| = 0$, $k = 1, \dots, p$, $1 \le p \le m$, $1 \le m \le n$ and $\overline{\lambda}_1 \dotplus \dots \dotplus \overline{\lambda}_p = \lambda$. The integer $m = m(\lambda)$ is called the characteristic number of the vector λ and was introduced by Zhurbenko [27]:

Definition 3.1. The *characteristic number* $m = m(\lambda) > 0$ of the vector λ such that $|\lambda| = 0$ is the maximum integer for which the equations

$$\lambda = \overline{\lambda}_1 \dotplus \dots \dotplus \overline{\lambda}_m, \qquad |\overline{\lambda}_k| = 0, \qquad k = 1, \dots, m,\ 1 \le m \le n,$$

are satisfied and the coordinate spaces of the vectors $\overline{\lambda}_k$, $k = 1, \dots, m$, have nonzero dimension.

In the case $m = 1$, the statistics $\widehat{f}(\lambda)$ and $\widehat{g}(\lambda)$ are the same; and for $m > 1$, the statistic $\widehat{f}(\lambda)$ will have, according to the definition, nonzero complementary terms.

3.1. *Moment density estimates.* We construct the estimator after observing the process X on an interval $[-T, T+LU]$, $T, L, U > 0$. The estimation method used here is based upon tapering and shift in time simultaneously: first smoothing the observed path with a suitable function h_T, then making calculations over shifted intervals, and averaging the result.

For any subset $\overline{\lambda}_k$ of λ in (5), we consider the following statistic (the existence of which is supposed):

$$I^{(u)}(\overline{\lambda}_k) = \frac{1}{(2\pi)^{n_k(n-1)/n}(H_n^T)^{n_k/n}} \int_{\mathbb{R}^{n_k}} X(\overline{t}_k) h_T(\overline{t}_k - L\mathbf{u}) e^{-i\overline{t}_k\cdot\overline{\lambda}_k}\, d\overline{t}_k, \tag{6}$$

where $\overline{\lambda}_k = (\lambda_{k_1}, \ldots, \lambda_{k_{n_k}})$, $\overline{t}_k = (t_{k_1}, \ldots, t_{k_{n_k}})$, $\mathbf{u} = (u, \ldots, u)$ are in $\mathbb{R}^{n_k}$, $k = 1, \ldots, p$, $1 \le p \le m, 1 \le m \le n$, $L > 0$ and H_n^T $(= H_n^T(0))$ is the Fourier transform of $h_T^n(\cdot)$ at the origin and is assumed to be nonzero.

We shall construct the estimate of the moment spectral density $g(\overline{\lambda}_k)$ by

$$\widehat{g}(\overline{\lambda}_k) = \frac{1}{U}\int_0^U I^{(u)}(\overline{\lambda}_k)\, du. \tag{7}$$

Let us note that Le Fe Do [18] proved, in the nontapered case, that the statistic (6) with $n_k = n = 4$ is strongly consistent.

One further approximation is possible when the tapering function can be written in the form $h_T(t) = h(t/T)$; this definition holds for most of the taper functions considered in practice. We choose $h(\cdot)$ to be a real, positive, even function, which is zero outside the interval $[-1, 1]$. These hypotheses about $h(\cdot)$ are referred to as $\mathcal{H}0$ in the sequel.

Let $H(\cdot)$ and $H^T(\cdot)$ denote the Fourier transform of $h(\cdot)$ and $h_T(\cdot)$ respectively. The precise relationship between $H^T(\cdot)$ and $H(\cdot)$ is given by

$$H^T(x) = TH(Tx). \tag{8}$$

For every nonzero integer k and every real x, we define

$$H_k(x) = \int_{-1}^{1} h^k(t) e^{itx}\, dt.$$

In the same way we define $H_k^T(\cdot)$ as the Fourier transform of $h_T^k(\cdot)$ and we have a relation similar to (8), i.e.,

$$H_k^T(x) = TH_k(Tx). \tag{9}$$

In the sequel we write H_k for $H_k(0)$, H_k^j for $(H_k(0))^j$ for every integer j, k and H_k^T for $H_k^T(0)$. For every $x \in \mathbb{R}$ and $k = 1$, we have $H_1(x) = H(x)$ and $H_1^T(x) = H^T(x)$.

For every $u \in \mathbb{R}^+$, we define

$$ {}_uH_k(x) = \int_{Lu-1}^{Lu+1} h^k(t - Lu)e^{itx}\,dt. \tag{10} $$

Let us write $H_k(x)$ for ${}_0H_k(x)$; it is evident that ${}_uH_k(x) = H_k(x)e^{iLux}$.

If $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, we choose $h(\cdot)$ such that $h(t) = h(t_1)\ldots h(t_n)$. The following properties are immediate:

(a) The functions $H_k^T(x)$, $x \in \mathbb{R}^n$, $n \geq 1$, are uniformly continuous for every $T > 0$, $k \geq 1$.

(b) $\sup_{x\in\mathbb{R}} |H_k(x)| \leq H_k \leq 2\sup_t h^k(t) < \infty$, $k \geq 1$.

(c) For all $k, l \in \mathbb{N}$ and $\alpha, \gamma \in \mathbb{R}$, we get

$$ \int_{\mathbb{R}} H_k^T(\alpha - \beta)H_l^T(\beta - \gamma)\,d\beta = 2\pi H_{k+l}(\alpha - \gamma). \tag{11} $$

(d) For every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $n \geq 2$, we get

$$ \int_{\mathbb{R}^{n-1}} \delta|x|\,H^T(x)\,d\breve{x} = (2\pi)^{n-1}H_n^T(-|\lambda|), \tag{12} $$

and

$$ \int_{\mathbb{R}^{2n-1}} \delta|x|\,H^T(x - \lambda' + \lambda'')\,d\breve{x} = (2\pi)^{2n-1}H_{2n}^T, \tag{13} $$

where $d\breve{x}$, λ' and λ'' are defined in Section 2.

Let us note that property (c) follows from the convolution theorem, and property (d) comes from (c).

3.2. *Assumptions.* Throughout this work, we suppose that the following conditions are satisfied.

• The cumulant spectral densities $f(\lambda)$, $\lambda \in \mathbb{R}^k$, $k \geq 2$, exist for X up to order n and satisfy the following hypothesis. There exists α such that, for every $\lambda \in \mathbb{R}^k$, every $x \in \mathbb{R}^k$, $2 \leq k \leq n$, and some $C_0 > 0$,

$$ \begin{aligned} &|f(x+\lambda) - f(\lambda)| \leq C_0\|x\|^\alpha && \text{if } 0 < \alpha \leq 1, \\ &\Big|f(x+\lambda) - f(\lambda) - \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i} f(\lambda)x_i\Big| \leq C_0\|x\|^\alpha && \text{if } 1 < \alpha \leq 2. \end{aligned} \tag{14} $$

• $h(\cdot)$ satisfies condition $\mathcal{H}0$ and its Fourier transform $H(\cdot)$ satisfies the following conditions:

$\mathcal{H}1$. $TH_k(Tx)$, $k \geq 1$, converges to zero as $T \to \infty$, uniformly on the set $\{x \in \mathbb{R} : |x| > \eta > 0\}$ (here $|x|$ is the absolute value of the real x).

$\mathcal{H}2$. $\int_{\mathbb{R}^{k-1}} \delta|x|\,|H(x)|^2\,d\breve{x} < \infty$, $2 \leq k \leq n$.

$\mathcal{H}3$. $\int_{\mathbb{R}^{k-1}} \delta|x|\,\|x\|^\alpha |H(x)|^q\, d\check{x} < \infty$, $2 \le k \le n$, $q = 1, 2$, for some $\alpha \in]0,2]$, where $\|\cdot\|$ is the uniform norm.

Note that in this work we require $\mathcal{H}3$ to be valid only for α ($\in]0,2]$) defined by condition (14). In the following section we give some examples of functions $h(\cdot)$ that satisfy condition $\mathcal{H}0$ and have Fourier transforms satisfying $\mathcal{H}1$–$\mathcal{H}3$ for every $\alpha \in]0,2]$.

4. Examples

4.1. *Example of a process X.* Let

$$X(t) = \int v(s) Y(t-s)\, ds,$$

where v is a smooth real function and

$$Y(t) = e^2(t) - 1,$$

where $\{e(t)\}$ are independent, identically distributed normal variables of zero mean and variance one. The theoretical nth order spectral density $f(\lambda)$ is given by

$$f(\lambda_1, \ldots, \lambda_n) = 2^{n-1}(n-1)!\, V(\lambda_1)\ldots V(\lambda_n), \tag{15}$$

where V is the Fourier transform of v and $\lambda_1 + \ldots + \lambda_n = 0$. Now it suffices to choose v such that the right hand side in (14) satisfies condition (15). This example may be used for simulation.

4.2. *Example of a tapering function h.* First we consider the tapering functions which are differentiable up to order $r \ge 1$.

Example 1 (Polynomial tapering). The function

$$h_r(t) = (1-t)^r (1+t)^r \mathbf{1}_{\{|t|\le 1\}}$$

satisfies condition $\mathcal{H}0$ and it is of class C^{r-1} on $[-1,1]$, and of class C^{r+1} on $]-1,1[$. Furthermore, from successive integrations by parts, we get for $x \ne 0$,

$$|H^T_{k,r}(x)| \le K T^{-kr} |x|^{-kr-1}, \tag{16}$$

where $H^T_{k,r}(x)$ is the Fourier transform of $h^k_r(t)$; this proves condition $\mathcal{H}1$. The proof of the other conditions requires integral calculus and the inequality

$$\Big(\sum_{i=1}^{n-1} |x_i|\Big)^\alpha \le \begin{cases} \displaystyle\sum_{i=1}^{n-1} |x_i|^\alpha & \text{if } 0 < \alpha \le 1, \\ \displaystyle\sum_{i=1}^{n-2} 2^{i\alpha}|x_i|^\alpha + 2^{(n-2)\alpha}|x_{n-1}|^\alpha & \text{if } 1 < \alpha. \end{cases} \tag{17}$$

In the following example, we consider the tapering functions which are not necessarily differentiable.

EXAMPLE 2 (Bartlett's tapering). The function $h(t) = (1 - |t|)\mathbf{1}_{\{|t|\le 1\}}$ satisfies condition $\mathcal{H}0$, and using successive integrations by parts we obtain

$$|H_k^T(x)| \le 4k! \sum_{j=2}^{k+1} T^{-j+1}|x|^{-j}.$$

For example, if $k = 1$, then we have

$$H^T(x) = \frac{1}{T}\left(\frac{\sin(Tx/2)}{x/2}\right)^2,$$

which implies $|H^T(x)| \le 4T^{-1}|x|^{-2}$; this proves condition $\mathcal{H}1$. The other conditions require integral calculus and inequality (17) (see [2] for more details).

5. Fourth-order estimation. In the expression of $\widehat{f}(\lambda)$, we use the set of admissible arguments of the vector λ on which the summation $\sum^0$ is carried out. To write completely the fourth-order spectral estimate $\widehat{f}(\lambda)$ for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ we shall state the possible values of the characteristic number m and the set of admissible arguments according to the value of m. Since univariate spectral densities equal zero, the possible values of the characteristic number are 1 and 2. The case $m = 1$ gives us one admissible argument $\Lambda_1 = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\}$; for $m = 2$ we get three sets of admissible arguments: $\Lambda_2 = \{(\lambda_1, \lambda_2), (\lambda_3, \lambda_4)\}$, $\Lambda_3 = \{(\lambda_1, \lambda_3), (\lambda_2, \lambda_4)\}$ and $\Lambda_4 = \{(\lambda_1, \lambda_4), (\lambda_2, \lambda_3)\}$. Note that the term in Λ_1 is always admissible; we call it the *principal argument* of λ, in order to distinguish it from other admissible arguments.

Now, we write completely the expression of $\widehat{f}(\lambda)$ according to the value of m.

(i) Case $m = 1$, only the principal argument occurs for $\widehat{f}(\lambda)$; in this case we have

$$\widehat{f}(\lambda) = \widehat{g}(\lambda). \tag{18}$$

(ii) Case $m = 2$; we distinguish three cases.

(a) The six terms of Λ_2, Λ_3 and Λ_4 are admissible for $\widehat{f}(\lambda)$, i.e., $\lambda_i = -\lambda_j$, $i \ne j$ and $i, j = 1, \dots, 4$. It follows that $\lambda = (0, 0, 0, 0)$, from the definition of $\widehat{f}(\lambda)$, we get

$$\widehat{f}(\lambda) = \widehat{g}(\lambda) - 3(\widehat{g}(0, 0))^2. \tag{19}$$

(b) The terms of two sets Λ_i, Λ_j, $i \ne j$, $i, j = 2, 3, 4$, are admissible for $\widehat{f}(\lambda)$. Let for instance $i = 2$, $j = 3$ (for the other cases the expression of

$\widehat{f}(\lambda)$ remains the same). In this case we have $\lambda = (\lambda_1, -\lambda_1, -\lambda_1, \lambda_1)$, and from the definition of $\widehat{f}(\lambda)$ we get

$$\widehat{f}(\lambda) = \widehat{g}(\lambda) - 2\widehat{g}(\lambda_1, -\lambda_1)\widehat{g}(-\lambda_1, \lambda_1). \tag{20}$$

(c) The terms of one set Λ_i, $i = 2, 3, 4$, are admissible for $\widehat{f}(\lambda)$. For example, let $i = 2$; the other cases are similar. In this case we have $\lambda = (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2)$, and from the definition of $\widehat{f}(\lambda)$ we get

$$\widehat{f}(\lambda) = \widehat{g}(\lambda) - \widehat{g}(\lambda_1, -\lambda_1)\widehat{g}(-\lambda_2, \lambda_2). \tag{21}$$

5.1. *Preliminary results.* Consider the kernel

$$K_M(y) = \frac{1}{2\pi M} \cdot \frac{\sin^2(My/2)}{(y/2)^2}, \qquad y \in \mathbb{R},\ M > 0. \tag{22}$$

This kernel has the following properties:

(i) $\int_{\mathbb{R}} K_M(y)\, dy = 1$ for every $M > 0$,
(ii) $\lim_{M\to\infty} \int_{|y|>\eta} K_M(y)\, dy = 0$ for every $\eta > 0$.

Let $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a continuous function; we write $\phi(x)$ for $\phi(x, 0)$. The function ϕ will be called the *coordinate function*. We state the following lemmas whose proofs are given in the appendix.

LEMMA 5.1. *Let ψ be a function defined on $\mathbb{R}^n$ such that $\psi \circ \phi$ is integrable, and choose positive T and N such that $T \to \infty$, $N \to \infty$, $T/N \to 0$; then for $n \geq 1$ we get the asymptotic equality*

$$\int_{\mathbb{R}^n\times\mathbb{R}} \psi \circ \phi(x, z) K_{N/T}(z)\, dx\, dz = \int_{\mathbb{R}^n} \psi \circ \phi(x)\, dx + O(1/T) + O(T/N),$$

where $x = (x_1, \ldots, x_n)$.

LEMMA 5.2. *Let ξ be a bounded, continuous function on $\mathbb{R}$. For all positive T, N such that $T \to \infty$, $N \to \infty$, $T/N \to 0$, we get*

$$\int_{\mathbb{R}} K_{N/T}(x)\xi(y - x)\, dx = \xi(y) + O(1/T) + O(T/N).$$

LEMMA 5.3. *Suppose that the cumulant spectral densities of the process X exist up to order 4 and that condition (14) with $0 < \alpha \leq 2$ is satisfied; moreover, suppose that the function $H(\cdot)$ satisfies conditions $\mathcal{H}1$–$\mathcal{H}3$. Then for the statistic $\widehat{g}$ defined by (7), for every $\lambda = (\lambda, -\lambda)$, $\mu = (\mu, -\mu)$ in $\mathbb{R}^2$ and for T, N $(= LU)$ such that $N \to \infty$, $T \to \infty$ and $T/N \to 0$, we can write*

$$\mathbb{E}\widehat{g}(\lambda)\widehat{g}(\mu) = \mathbb{E}\widehat{g}(\lambda)\mathbb{E}\widehat{g}(\mu) + o(T^{3-\alpha}/N) + o(T^3/N).$$

5.2. *Asymptotic results.* We will now state asymptotic results for the mean and variance of fourth-order cumulant spectral estimates given by (18)–(21); their proofs are based on Lemmas 5.1–5.3 (see [1] for more details).

5.2.1. *Fourth-order mean estimate*

THEOREM 5.1. *Suppose that the cumulant spectral densities of the process* X *exist up to order* 4, *and satisfy condition* (14) *with* $0 < \alpha \le 2$, *and that the function* $H(\cdot)$ *satisfies conditions* $\mathcal{H}1$–$\mathcal{H}3$. *Then for the statistic* $\widehat{f}(\lambda)$ *defined by* (5), *for every* $\lambda \in \mathbb{R}^4$ *with* $m(\lambda) = 1, 2$ *and for* T, N $(= LU)$ *such that* $T \to \infty$, $N \to \infty$ *and* $T/N \to 0$, *we get*

$$\mathbb{E}\widehat{f}(\lambda) = f(\lambda) + O(T^{-\alpha}) + o(T^3/N) + o(T^{3-\alpha}/N).$$

5.2.2. *Fourth-order variance estimate*

THEOREM 5.2. *Suppose that the cumulant spectral densities of the process* X *exist up to order* 8, *and satisfy condition* (14) *with* $0 < \alpha \le 2$, *and that the function* $H(\cdot)$ *satisfies conditions* $\mathcal{H}1$–$\mathcal{H}3$. *Then for the statistic* $\widehat{f}(\lambda)$ *defined by* (5), *for every* $\lambda \in \mathbb{R}^4$ *with* $m(\lambda) = 1, 2$ *and for* T, N $(= LU)$ *such that* $T \to \infty$, $N \to \infty$ *and* $T/N \to 0$ *we get*

$$\operatorname{var} \widehat{f}(\lambda) = \frac{\Gamma_4^*(\lambda) f(\lambda_1) \ldots f(\lambda_4)}{(2\pi)^5 H_4^2} \cdot \frac{T^3}{N} \int_{\mathbb{R}^3} d\breve{x}\, \delta|x|\, H^2(x) + O(T^{3-\alpha}/N),$$

where $\Gamma_4^*(\lambda)$ *is the number of partitions of the set of* 8 *coordinates of the vectors* λ *and* $-\lambda$ *into pairs* (λ_i, λ_j) *such that* $\lambda_i = \lambda_j$ *and* $\lambda_i \in \{\lambda\}$, $\lambda_j \in \{-\lambda\}$.

6. n**th order estimation.** The asymptotic results on the mean and variance of nth order estimates for $n \ge 2$ and $1 \le m(\lambda) \le n$ are based on the following lemma whose proof is given in the appendix.

LEMMA 6.1. *Suppose that the cumulant spectral densities of the process* X *exist up to order* n, *and satisfy condition* (14) *with* $0 < \alpha \le 2$; *moreover, suppose that the function* $H(\cdot)$ *satisfies conditions* $\mathcal{H}1$–$\mathcal{H}3$. *Then for each term of the statistic* $\widehat{f}(\lambda)$, $\lambda \in \mathbb{R}^n$, $\lambda_1 + \ldots + \lambda_n = 0$, *for* N $(= LU) \to \infty$, $T \to \infty$, $T/N \to 0$, *we have*

$$\mathbb{E}\widehat{g}(\overline{\lambda}_1) \ldots \widehat{g}(\overline{\lambda}_p) = \mathbb{E}\widehat{g}(\overline{\lambda}_1) \ldots \mathbb{E}\widehat{g}(\overline{\lambda}_p) + o(T^{n-1}/N) + o(T^{n-1-\alpha}/N).$$

THEOREM 6.1. *Suppose that the cumulant spectral densities of the process* X *exist up to order* n, *and condition* (14) *is satisfied with* $0 < \alpha \le 2$; *moreover, suppose that the function* $H(\cdot)$ *satisfies conditions* $\mathcal{H}1$–$\mathcal{H}3$. *Then* *for the statistic* $\widehat{f}(\lambda)$ *defined by* (5), *for every* $\lambda = (\lambda_1, \ldots, \lambda_n)$, T, $N(= LU)$ *such that* $|\lambda| = 0$, $T \to \infty$, $N \to \infty$ *and* $T/N \to 0$, *we can write*

$$\mathbb{E}\widehat{f}(\lambda) = f(\lambda) + O(T^{-\alpha}) + o(T^{n-1}/N) + o(T^{n-1-\alpha}/N).$$

Proof. The definition of $\widehat{f}(\lambda)$ and Lemma 6.1 yield the relation

$$\mathbb{E}\widehat{f}(\lambda) = \sum\nolimits^0 (-1)^{(p-1)}(p-1)!\mathbb{E}\widehat{g}(\overline{\lambda}_1)\ldots\mathbb{E}\widehat{g}(\overline{\lambda}_p) + o(T^{n-1}/N) + o(T^{n-1-\alpha}/N),$$

where the summation is over the same sets of coordinate subspaces as in the definition of the statistic $\widehat{f}(\lambda)$.

From the definition of $\widehat{g}(\overline{\lambda}_k)$ and $H^T(\cdot)$ and taking into account the fact that $|\overline{\lambda}_k| = 0$, $k = 1, \ldots, p$, it follows that

$$\begin{aligned}\mathbb{E}\widehat{f}(\lambda) = \frac{1}{(2\pi)^{n-1}H_n^T} & \int\limits_{\mathbb{R}^n} dx H^T(x-\lambda) \\ & \times \Big[\sum\nolimits^0 (-1)^{p-1}(p-1)! g(\overline{x}_1)\delta|\overline{x}_1|\ldots g(\overline{x}_p)\delta|\overline{x}_p|\Big] \\ & + O(T^{n-1}/N) + o(T^{n-1-\alpha}/N),\end{aligned}$$

where $\lambda = \overline{\lambda}_1 \dot{+} \ldots \dot{+} \overline{\lambda}_p$, $x = \overline{x}_1 \dot{+} \ldots \dot{+} \overline{x}_p$ and $\overline{x}_k, \overline{\lambda}_k$ have the same coordinate space $\mathbb{R}^{n_k}$ with dimension n_k.

Expressing moment spectral densities via cumulant spectral densities by the second identity of (4) and reduction of similar terms, we transform the sum $\sum^0$ into the sum of $f(x)\delta|x|$ and $\sum^\phi$ which is taken over $\widetilde{x}_1 \in \mathbb{R}^{n_1}$, $\ldots$, $\widetilde{x}_q \in \mathbb{R}^{n_q}$ in the coordinate subspaces of vectors $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_q$ such that $\widetilde{\lambda}_1 \dot{+} \ldots \dot{+} \widetilde{\lambda}_q = \lambda$, $|\widetilde{\lambda}_j| \neq 0$, $j = 1, \ldots, q$, $1 < q \le n$. We have

$$\begin{aligned}\mathbb{E}\widehat{f}(\lambda) = & \frac{1}{(2\pi)^{n-1}H_n^T} \int\limits_{\mathbb{R}^n} H^T(x-\lambda) \\ & \times \Big\{f(x)\delta|x| + \sum\nolimits^\phi f(\widetilde{x}_1)\delta|\widetilde{x}_1|\ldots f(\widetilde{x}_q)\delta|\widetilde{x}_q|\Big\}\,dx \\ & + o(T^{n-1}/N) + o(T^{n-1-\alpha}/N).\end{aligned} \tag{23}$$

In fact, the summand of $\sum^\phi$ corresponding to the collection $\widetilde{\lambda}_1$, $\ldots$, $\widetilde{\lambda}_q$ can be obtained by applying (4) to the product $g(\overline{\lambda}_1)\delta|\overline{\lambda}_1|\ldots g(\overline{\lambda}_p)\delta|\overline{\lambda}_p|$. Substitution of the second identity of (4) for the identical partition into $\sum^0$ gives the identity $f(\lambda)\delta|\lambda| = f(\lambda)\delta|\lambda|$. Then after reduction of similar terms the sum comprises the summands involved in the decomposition of $g(\lambda)$ by formulae (4) and not occurring in further decompositions, all of these making up $\sum^\phi$. Let us write (23) as

$$\mathbb{E}\widehat{f}(\lambda) = \frac{1}{(2\pi)^{n-1}H_n^T}\Big\{\int\limits_{\mathbb{R}^{n-1}} d\breve{x}\, f(x)H^T(x-\lambda)\,\delta|x| + \sum\nolimits^{\phi}\prod_{j=1}^{q}\int\limits_{\mathbb{R}^{n_j-1}} d\breve{x}_j\,\delta|\widetilde{x}_j|\, f(\widetilde{x}_j)H^T(\widetilde{x}_j-\widetilde{\lambda}_j)\Big\} + o(T^{n-1}/N) + o(T^{n-1-\alpha}/N). \tag{24}$$

Let I_1 denote the first term on the right hand side of (24) and write it as a sum of two integrals

$$I_1 = \frac{1}{(2\pi)^{n-1}H_n^T} f(\lambda)\int\limits_{\mathbb{R}^{n-1}} d\breve{x}\,\delta|x|\, H^T(x-\lambda) + \frac{1}{(2\pi)^{n-1}H_n^T}\int\limits_{\mathbb{R}^{n-1}} d\breve{x}\,\delta|x|\,(f(x)-f(\lambda))H^T(x-\lambda). \tag{25}$$

It follows from (12), (9), (14), $\mathcal{H}3$ and $|\lambda| = 0$ that

$$I_1 = f(\lambda) + O(T^{-\alpha}).$$

For the evaluation of the second term on the right hand side of (24), denoted by I_2, we use (14), (12), $\mathcal{H}1$ and the fact that $|\widetilde{\lambda}_j| \neq 0$, $j = 1,\ldots,q$. Hence we get

$$I_2 = o(T^{-1}) + o(T^{-\alpha}).$$

The theorem is proved.

THEOREM 6.2. *Suppose that all cumulant spectral densities of the process X exist up to order $2n$, and satisfy condition* (14) *with $0 < \alpha \le 2$; moreover, suppose that the function $H(\cdot)$ satisfies conditions $\mathcal{H}1$–$\mathcal{H}3$. Then for the statistic $\widehat{f}(\lambda)$ defined by* (5), *for every $\lambda \in \mathbb{R}^n$, T, N $(= LU)$ such that $|\lambda| = 0$, $T\to\infty$, $N\to\infty$ and $T/N\to 0$, we can write*

$$\operatorname{var}\widehat{f}(\lambda) = \frac{\varGamma_n^*(\lambda)f(\lambda_1)\ldots f(\lambda_n)}{(2\pi)^{2n-3}H_n^2}\cdot\frac{T^{n-1}}{N}\int\limits_{\mathbb{R}^{n-1}} d\breve{x}\,\delta|x|\,(H(x))^2 + O(T^{n-1-\alpha}/N),$$

where $\varGamma_n^*(\lambda)$ is the number of partitions of the set of $2n$ coordinates of the vectors λ and $-\lambda$ into pairs (λ_i,λ_j) such that $\lambda_i = \lambda_j$ and $\lambda_i \in \{\lambda\}$, $\lambda_j \in \{-\lambda\}$.

Proof. It follows from the definition of $\widehat{f}(\lambda)$ that

$$\operatorname{var}\widehat{f}(\lambda) = \sum\nolimits^{0}(-1)^{p+q-2}(p-1)!(q-1)! \times \operatorname{cov}\{\widehat{g}(\lambda_1')\ldots\widehat{g}(\lambda_p'),\widehat{g}(\lambda_1'')\ldots\widehat{g}(\lambda_q'')\}, \tag{26}$$

where the sum $\sum^0$ is over all decompositions $\lambda_1',\ldots,\lambda_p',\lambda_1'',\ldots,\lambda_q''$, $|\lambda_j'| = 0$, $j = 1,\ldots,p$, $|\lambda_k''| = 0$, $k = 1,\ldots,q$, $\lambda_1' \dotplus \ldots \dotplus \lambda_p' = \lambda'$, $\lambda_1'' \dotplus \ldots \dotplus \lambda_q'' = \lambda''$, λ' and λ'' are defined in Section 2.

Using the definition of the statistic $\widehat{g}(\cdot)$, Lemma A.2 and the spectral representation of cumulants, we obtain

$$\begin{aligned}\operatorname{var}\widehat{f}(\lambda) = \sum\nolimits^0 & \frac{(-1)^{p+q}(p-1)!(q-1)!}{(2\pi)^{2n-2}(H_n^T)^2U^{p+q}} \\ & \times \sum\nolimits^* \int_0^U\!\!\ldots\!\int du'\,du'' \int_{\mathbb{R}^{4n-l}} d\widetilde{x}\,dt\,\delta|\widetilde{x}_1|\,f(\widetilde{x}_1)\ldots\delta|\widetilde{x}_l|\,f(\widetilde{x}_l) \\ & \times \prod_{j=1}^{p} h_T(t'_j - Lu'_j) \prod_{k=1}^{q} h_T(t''_k - Lu''_k)\exp\{it\cdot(\widetilde{x}+\lambda'-\lambda'')\},\end{aligned}$$

where the sum $\sum^*$ is over all indecomposable partitions relative to the vector $t = t' \dotplus t''$ and $u' = u'_1 \dotplus \ldots \dotplus u'_p$, $u'' = u''_1 \dotplus \ldots \dotplus u''_p$, $t' = t'_1 \dotplus \ldots \dotplus t'_p$, $t'' = t''_1 \dotplus \ldots \dotplus t''_p$, $\widetilde{x} = \widetilde{x}_1 \dotplus \ldots \dotplus \widetilde{x}_l$, u'_j, t'_j (resp. u''_j, t''_j) have the same coordinate space $\mathbb{R}^{n'_j}$ (resp. $\mathbb{R}^{n''_j}$) and $\widetilde{x}_j$ is in the coordinate space $\mathbb{R}^{\widetilde{n}_j}$ with $n'_1 \dotplus \ldots \dotplus n'_p = n''_1 \dotplus \ldots \dotplus n''_p = \widetilde{n}_1 + \ldots + \widetilde{n}_l = 2n$, $1 \le l \le 2n$.

By definition of $H^T(\cdot)$, the above expression becomes

$$\begin{aligned}\operatorname{var}\widehat{f}(\lambda) = \sum\nolimits^0 \sum\nolimits^* & \frac{(-1)^{p+q}(p-1)!(q-1)!}{(2\pi)^{2n-2}(H_n^T)^2U^{p+q}} \\ & \times \int_{\mathbb{R}^{2n-l}} d\widetilde{x}\,\delta|\widetilde{x}_1|\,f(\widetilde{x}_1)\ldots\delta|\widetilde{x}_l|\,f(\widetilde{x}_l)H^T(\widetilde{x}_1-\widetilde{\lambda}_1)\ldots H^T(\widetilde{x}_l-\widetilde{\lambda}_l) \\ & \times \int_0^U\!\!\ldots\!\int du'\,du''\exp\Big\{iL\sum_{k=1}^{l}\widetilde{u}_k\cdot(\widetilde{x}_k-\widetilde{\lambda}_k)\Big\},\end{aligned}$$

where $\widetilde{u}_1 \dotplus \ldots \dotplus \widetilde{u}_l = u' \dotplus u''$ and $\widetilde{\lambda}_1 \dotplus \ldots \dotplus \widetilde{\lambda}_l = \lambda' \dotplus (-\lambda'')$.

It is suitable to consider the indecomposable partitions $\{(\widetilde{u}_1,\widetilde{\lambda}_1),\ldots \ldots,(\widetilde{u}_l,\widetilde{\lambda}_l)\}$ relative to $(u',\lambda)\dotplus(u'',\lambda)$ where (u',λ) and (u'',λ) are the vectors of coordinates of the pairs (u'_{i_k},λ_{j_k}) and $(u''_{i_k},\lambda_{j_k})$, respectively, with $u'_{i_k} \in \{u'_1 \dotplus \ldots \dotplus u'_p\}$, $u''_{i_k} \in \{u''_1 \dotplus \ldots \dotplus u''_q\}$ and $\lambda_{j_k} \in \{\lambda\}$; these partitions can be classified into the following categories:

- $P_1 = \{(2,\ldots,2),\ n \text{ pairs}\}$ with $\varGamma_n(\lambda)$ terms.
- $P_s = P \setminus P_1$ (i.e., P_s is the complement of P_1 in P), where P is the set of all indecomposable partitions.

First we can see that all terms in P_s contribute a smaller order of magnitude to $\operatorname{var}\widehat{f}(\lambda)$; this can be derived from evaluation of (33). Thus in $\sum^*$ we can leave only those terms having coordinate spaces of the vector $\widetilde{x}_k$ of dimension 2. In view of (13) the coordinate subspaces must contain a vector $\widetilde{\lambda}_k = (\lambda_{i_k},\lambda_{j_k})$, $k = 1,\ldots,n$, lying in

- $\mathcal{A} = \{\widetilde{\lambda}_k : \lambda_{i_k} = \lambda_{j_k},\ \lambda_{i_k} \in \{\lambda'\},\ \lambda_{j_k} \in \{\lambda''\}\}$, or

- $\mathcal{B} = \{\widetilde{\lambda}_k : \lambda_{i_k} = -\lambda_{j_k},\ \lambda_{i_k}, \lambda_{j_k}$ are in the same set $\{\lambda'\}$ or $\{\lambda''\}\}$.

Thus the maximum order comes from the terms $\{(\widetilde{u}_1, \widetilde{\lambda}_1), \ldots, (\widetilde{u}_n, \widetilde{\lambda}_n)\}$ having the maximum number of $(\widetilde{u}_k, \widetilde{\lambda}_k)$ such that $u'_{i_1} = \ldots = u'_{i_n}$ and $u''_{i_1} = \ldots = u''_{i_n}$, where $\widetilde{u}_k = (u'_{i_k}, u''_{i_k})$ and $\{i_1, \ldots, i_n\}$ is an arbitrary permutation of the indices $(1, \ldots, n)$ and $\widetilde{\lambda}_k \in \mathcal{A}$; such properties hold for $p = q = 1$ and $\widetilde{\lambda}_k \in \mathcal{A}$, $k = 1, \ldots, n$. Let $\Gamma_n^*(\lambda)$ be the number of such terms. A typical term involving the maximum order is

$$A = \frac{1}{(2\pi)^{2n-3}(H_n^T)^2 N} \int_{\mathbb{R}^n} dx_1 \ldots dx_n \, K_N(x_1 + \ldots + x_n) \times f(x_1 + \lambda_1) \ldots f(x_n + \lambda_n) |H^T(x_1)|^2 \ldots |H^T(x_n)|^2. \tag{27}$$

Let us write

$$\begin{aligned} f(x_1 + \lambda_1) \ldots f(x_n + \lambda_n) &= f(\lambda_1) \ldots f(\lambda_n) \\ &\quad + \sum_{k=1}^{n-1} f(x_1 + \lambda_1) \ldots f(x_{k-1} + \lambda_{k-1}) \\ &\quad \times f(\lambda_{k+1}) \ldots f(\lambda_n)(f(x_k + \lambda_k) - f(\lambda_k)) \\ &\quad + f(x_1 + \lambda_1) \ldots f(x_{n-1} + \lambda_{n-1})(f(x_n + \lambda_n) - f(\lambda_n)), \end{aligned} \tag{28}$$

with the convention that $f(\lambda_{k+1}) = 1$ for $k \geq n$. Combining (28) with (27) and using Lemma 5.3 and condition $\mathcal{H}3$, we obtain

$$A = \frac{f(\lambda_1) \ldots f(\lambda_n)}{(2\pi)^{2n-3}(H_n)^2} \cdot \frac{T^{n-1}}{N} \int_{\mathbb{R}^{n-1}} d\breve{x} \, \delta|x| \, (H(x))^2 + O(T^{n-1-\alpha}/N).$$

The theorem is proved.

Let $\widehat{b}(\lambda)$ $(= \mathbb{E}\widehat{f}(\lambda) - f(\lambda))$ denote the bias of the nth order spectral estimate given by (5). We have the following results.

COROLLARY 6.1. *If the conditions of Theorem* 6.1 *are satisfied and if moreover* $T^{n-1}/N \to 0$, *then the nth order spectral estimate* $\widehat{f}(\lambda)$ *defined by* (5) *is asymptotically unbiased*, *and more precisely*,

$$\widehat{b}(\lambda) = O(T^{-\alpha}) + o(T^{n-1}/N) + o(T^{n-1-\alpha}/N).$$

COROLLARY 6.2. *If the conditions of Theorem* 6.2 *are satisfied*, *and moreover* $T^{n-1}/N \to 0$, *then the nth order spectral estimate* $\widehat{f}(\lambda)$ *defined by* (5) *is consistent*, *and more precisely*,

$$\operatorname{var} \widehat{f}(\lambda) = O(T^{n-1}/N) + O(T^{n-1-\alpha}/N).$$

7. Conclusion. Under the conditions of Theorems 6.1 and 6.2, and for every $\lambda \in \mathbb{R}^n$, $|\lambda| = 0$, $n \geq 2$, $T \to \infty$, $N\,(= LU) \to \infty$, and $T^{n-1}/N \to 0$, the statistic $\widehat{f}(\lambda)$ defined by (5) is asymptotically unbiased and consistent as an estimator of the cumulant spectral density $f(\lambda)$. Thus, this statistic can be used for constructing efficient estimators of higher-order cumulant spectral densities, and is valid for all admissible values of λ.

Appendix

A.1. *Cumulants*. We introduce some definitions and results on the joint cumulants of a number of nonelementary random variables.

Let $t_1, \ldots, t_r$ be a collection of vectors with $t_j = (t_1^{(j)}, \ldots, t_{n_j}^{(j)}) \in \mathbb{R}^{n_j}$, $j = 1, \ldots, r$, and $\overline{t}_1, \ldots, \overline{t}_p$ a partition of the set of coordinates of the vector $t_1 \dotplus \ldots \dotplus t_r$, according to Section 2.

(α) We shall say that the sets of coordinates of the vectors $\overline{t}_\nu$ and $\overline{t}_\mu$, $1 \leq \nu, \mu \leq p$, of the partition, are *hooked* if there exist t_{i_k} and t_{j_l} belonging to the sets of coordinates of $\overline{t}_\nu$ and $\overline{t}_\mu$ respectively such that $i_k = j_l$.

(β) We shall say that those sets *communicate* if there exists a subcollection of vectors $\overline{t}_\nu = \overline{t}_{\nu_1}, \ldots, \overline{t}_{\nu_s} = \overline{t}_\mu$ such that, for $k = 1, \ldots, s-1$, $\overline{t}_{\nu_k}$ and $\overline{t}_{\nu_{k+1}}$ are hooked.

Definition A.1. The partition $\overline{t}_1, \ldots, \overline{t}_p$ is said to be *indecomposable* relative to $(t_1, \ldots, t_r)$ if all sets of coordinates of $\overline{t}_1, \ldots, \overline{t}_p$ communicate.

Lemma A.1 (see [19]). *The partition $\overline{t}_1, \ldots, \overline{t}_p$ is indecomposable relative to $(t_1, \ldots, t_r)$ if there are no sets $\{k_1, \ldots, k_a\}$, $a < p$, and $\{j_1, \ldots, j_b\}$, $b < r$, such that $\overline{t}_{k_1} \dotplus \ldots \dotplus \overline{t}_{k_a} = t_{j_1} \dotplus \ldots \dotplus t_{j_b}$.*

The following lemma allows us to represent the cumulant as a sum of products of cumulants. The proof is given in [6, Theorem 2.3.2].

Lemma A.2. *Let $X(t_1), \ldots, X(t_r)$ be random variables with $X(t_j) = X(t_1^{(j)}) \ldots X(t_{n_j}^{(j)})$, for $j = 1, \ldots, r$. Then the cumulant* $\mathrm{cum}\{X(t_1), \ldots, X(t_r)\}$ *is given by*

$$\mathrm{cum}\{X(t_1), \ldots, X(t_r)\} = \sum\nolimits^{*} c(\overline{t}_1) \ldots c(\overline{t}_p),$$

where the summation is over all indecomposable partitions $\overline{t}_1, \ldots, \overline{t}_p$ relative to $(t_1, \ldots, t_r)$ and $c(\overline{t}_k) = \mathrm{cum}\{X(t_{k_1}), \ldots, X(t_{k_{n_k}})\}$ for $\overline{t}_k = (t_{k_1}, \ldots, t_{k_{n_k}})$ and $k = 1, \ldots, p$.

A.2. *Proof of the lemmas*

Proof of Lemma 5.1. Let

$$I = \int\limits_{\mathbb{R}^n \times \mathbb{R}} \psi \circ \phi(x, z) K_M(z)\, dx\, dz - \int\limits_{\mathbb{R}^n} \psi \circ \phi(x)\, dx,$$

with $M = N/T$. Using property (i) of K_M, I becomes

$$I = \int_{\mathbb{R}} K_M(z)\Big\{ \int_{\mathbb{R}^n} [\psi\circ\phi(x,z) - \psi\circ\phi(x)]\,dx\Big\}dz.$$

The set of continuous functions on $\mathbb{R}^n$ with compact support is dense in $L^1(\mathbb{R}^n)$; hence suppose, first, that $\psi\circ\phi$ is a continuous function with compact support. We write the integral I as a sum $I = I_1 + I_2$ of integrals over $\{|z| \le \eta\}$ and $\{|z| > \eta\}$, for arbitrary positive η. For every $\varepsilon > 0$, since $\psi\circ\phi$ is continuous with compact support, we can choose η such that $|\psi\circ\phi(x,z) - \psi\circ\phi(x)| < \varepsilon$ for $|z| \le \eta$ and all x in $\mathbb{R}^n$. Since $\psi\circ\phi$ has a compact support, the L^1-norm of $\psi\circ\phi(x,z) - \psi\circ\phi(x)$ is arbitrarily small; then I_1 is smaller than ε. We keep the same notation "ε" for small values. Now it is evident that I_2 is less than

$$2\|\psi\circ\phi\|_1 \int_{|z|>\eta} K_M(z)\,dz,$$

where $\|\cdot\|_1$ is the L^1-norm. Now with $Mz = y$, it is obvious that

$$\int_{|z|>\eta} K_M(z)dz \le \frac{4}{\pi}\int_{\eta M}^{\infty} \frac{1}{y^2}\,dy = \frac{4}{\pi\eta M}.$$

Choosing $\varepsilon = T^{-1}$, we obtain $I = O(T^{-1}) + O(TN^{-1})$.

Coming back to the general case, choose a continuous function φ with compact support such that $\|\psi\circ\phi - \varphi\|_1 \le \varepsilon$. Then

$$\begin{aligned} I = \int_{\mathbb{R}} K_M(z)\Big\{ &\int_{\mathbb{R}^n} [\psi\circ\phi(x,z) - \varphi(x,z)]\,dx \\ &+ \int_{\mathbb{R}^n} [\varphi(x,z) - \varphi(x)]\,dx + \int_{\mathbb{R}^n} [\varphi(x) - \psi\circ\phi(x)]\,dx\Big\}. \end{aligned}$$

In the same manner we get $|I| \le 3\varepsilon + 4/(\pi\eta M)$. The lemma is proved.

Lemma 5.2 follows immediately from Lemma 5.1. Notice that Lemmas 5.1 and 5.2 remain true if we replace K_M by any kernel (satisfying (i) and (ii)).

Proof of Lemma 5.3. We have the equality

$$\frac{1}{U^2}\iint_0^U du\,dv \exp\{iL(u-v)\beta\} = \frac{2\pi}{N}K_N(\beta),$$

where K_N is defined by (22). From (6), (7), (9) and Lemma A.2, we obtain

$$\begin{aligned} (29)\quad \operatorname{cov}\{\widehat{g}(\lambda), \widehat{g}(\mu)\} = \frac{1}{(2\pi)^2 H_4^T N}\Big\{ &\int_{\mathbb{R}^3} d\check{x}\,\delta|x|\,K_N(x_1+x_2)f(x)H^T(x_1-\lambda) \\ &\times H^T(x_2+\lambda)H^T(x_3+\mu)H^T(x_4-\mu) \end{aligned}$$

$$+ \int_{\mathbb{R}^2} dx\, dy\, K_N(x+y) f(x) f(y) H^T(x-\lambda) H^T(x-\mu)$$
$$\times H^T(y+\lambda) H^T(y+\mu)$$
$$+ \int_{\mathbb{R}^2} dx\, dy\, K_N(x+y) f(x) f(y) H^T(x-\lambda) H^T(x+\mu)$$
$$\times H^T(y+\lambda) H^T(y-\mu) \Big\}.$$

To prove the lemma, we evaluate each of the three summands in the above expression. Represent the first summand I_1 as a sum $I_1 = I_{11} + I_{12}$, with

$$I_{11} = \frac{1}{(2\pi)^2 H_4^T N} f(\lambda \dot{-} \mu) \int_{\mathbb{R}^3} d\breve{x}\, \delta|x|\, K_N(x_1+x_2) H^T(x_1-\lambda) H^T(x_2+\lambda)$$
$$\times H^T(x_3+\mu) H^T(x_4-\mu),$$
$$I_{12} = \frac{1}{(2\pi)^2 H_4^T N} \int_{\mathbb{R}^3} d\breve{x}\, \delta|x|\, (f(x) - f(\lambda \dot{-} \mu)) K_N(x_1+x_2) H^T(x_1-\lambda)$$
$$\times H^T(x_2+\lambda) H^T(x_3+\mu) H^T(x_4-\mu),$$

where $\lambda \dot{-} \mu = (\lambda, -\lambda, -\mu, \mu)$ and $x = (x_1, \dots, x_4)$. Using the definition of δ, (10), (11) and an appropriate change of variables, we get

$$I_{11} = \frac{T}{H_4 N} f(\lambda \dot{-} \mu) \int_{\mathbb{R}} du\, K_{N/T}(u) |H_2(u)|^2.$$

By Lemma 5.2, we get

$$I_{11} = \frac{T H_2^2}{H_4 N} f(\lambda \dot{-} \mu) + O(1/N) = O(T/N) + O(1/N).$$

In the same manner, but applying conditions (14), $\mathcal{H}3$ and Lemma 5.1 instead of Lemma 5.2, we get

$$I_{12} = O(T^{1-\alpha}/N) + O(T/N).$$

To evaluate the second summand I_2 in (29), we put the expression

$$f^2(\lambda) + f(x)[f(y) - f(\lambda)] + f(\lambda)[f(x) - f(\lambda)],$$

in place of $f(x)f(y)$, in order to write the second summand I_2 as a sum of four integrals similar to I_{11} and I_{12}. By similar arguments and since I_2 and the third summand I_3 in (29) are of the same type, we obtain

$$I_2 + I_3 = O(T^{2-\alpha}/N) + O(T^2/N).$$

Lemma 5.3 then follows by using the known identity

$$\mathbb{E}\widehat{g}(\lambda)\widehat{g}(\mu) - \mathbb{E}\widehat{g}(\lambda)\mathbb{E}\widehat{g}(\mu) = \operatorname{cov}\{\widehat{g}(\lambda), \widehat{g}(\mu)\},$$

and the fact that

$$O(T^{2-\alpha}/N) = o(T^{3-\alpha}/N) \quad \text{and} \quad O(T^2/N) = o(T^3/N).$$

Proof of Lemma 6.1. The following identity is obvious:

$$\mathbb{E}\widehat{g}(\overline{\lambda}_1)\dots\widehat{g}(\overline{\lambda}_p) \tag{30}$$
$$= \mathbb{E}\widehat{g}(\overline{\lambda}_1)\dots\mathbb{E}\widehat{g}(\overline{\lambda}_p) + \sum_{j=1}^{p-1}\operatorname{cov}\{\widehat{g}(\overline{\lambda}_1)\dots\widehat{g}(\overline{\lambda}_j),\, \widehat{g}(\overline{\lambda}_{j+1})\}\mathbb{E}\widehat{g}(\overline{\lambda}_{j+2})\dots\mathbb{E}\widehat{g}(\overline{\lambda}_p),$$

with the convention that $\mathbb{E}\widehat{g}(\overline{\lambda}_j) = 1$ for $j > p$.

Let us evaluate $I_j = \operatorname{cov}\{\widehat{g}(\overline{\lambda}_1)\dots\widehat{g}(\overline{\lambda}_j), \widehat{g}(\overline{\lambda}_{j+1})\}$. From the definition of the statistic $\widehat{g}(\overline{\lambda})$, it follows that

$$\begin{aligned} I_j = {} & K(T,N_j)U^{-j-1} \\ & \times \int_0^U\!\!\dots\!\int du_1\dots du_{j+1} \int_{\mathbb{R}^{N_j}} dt\,\operatorname{cov}\{X(\overline{t}_1)\dots X(\overline{t}_j), X(\overline{t}_{j+1})\} \\ & \times h_T(\overline{t}_1 - Lu_1)\dots h_T(\overline{t}_{j+1} - Lu_{j+1}) \\ & \times \exp\{-i(\overline{t}_1\cdot\overline{\lambda}_1 + \dots + \overline{t}_j\cdot\overline{\lambda}_j - \overline{t}_{j+1}\cdot\overline{\lambda}_{j+1})\}, \end{aligned} \tag{31}$$

where $K(T,N_j) = (2\pi)^{-(n-1)/nN_j}(H_n^T)^{-N_j/n}$, $N_j = n_1+\dots+n_{j+1}$ and n_k is the dimension of the coordinate space $\mathbb{R}^{n_k}$ for the vector $\overline{t}_k$, $k = 1,\dots,j+1$, with $\overline{t}_1 \dotplus \dots \dotplus \overline{t}_{j+1} = t$.

We use the representation of the covariance as a sum of product cumulants and the spectral representation of cumulants; the covariance in the above expression becomes

$$\begin{aligned} & \operatorname{cov}\{X(\overline{t}_1)\dots X(\overline{t}_j), X(\overline{t}_{j+1})\} \\ & \qquad = \sum\nolimits^{*} \int_{\mathbb{R}^{\tilde{n}_1+\dots+\tilde{n}_l-l}} d\widetilde{x}\, f(\widetilde{x}_1)\delta|\widetilde{x}_1|\dots f(\widetilde{x}_l)\,\delta|\widetilde{x}_l| \\ & \qquad\quad \times \exp\{i(\widetilde{t}_1\cdot\widetilde{x}_1 + \dots + \widetilde{t}_l\cdot\widetilde{x}_l)\}, \end{aligned} \tag{32}$$

where $\widetilde{x}_1 + \dots + \widetilde{x}_l = \widetilde{x}$, $\widetilde{x}_k = (x_1^{(k)},\dots,x_{n_k}^{(k)})$, $k = 1,\dots,l$, and the sum $\sum^*$ is over all indecomposable partitions $\widetilde{t}_1,\dots,\widetilde{t}_l$ relative to the pair $(\overline{t}_1 \dotplus \dots \dotplus \overline{t}_j, \overline{t}_{j+1})$, $1 \le l \le N_j$; $\mathbb{R}^{\tilde{n}_k}$ is the coordinate space of the vectors $\widetilde{x}_k$, $\widetilde{t}_k$ with dimension $\widetilde{n}_k$, with $\widetilde{n}_1 + \dots + \widetilde{n}_l = N_j$. Substituting (32) to (31) and using the definition of $H^T(\cdot)$, we obtain

$$\begin{aligned} I_j = {} & K(T,N_j)\sum\nolimits^{*} \int_{\mathbb{R}^{N_j-l}} d\widetilde{x}\, f(\widetilde{x}_1)\,\delta|\widetilde{x}_1|\dots\, f(\widetilde{x}_l)\,\delta|\widetilde{x}_l| \\ & \times \frac{1}{U^{j+1}}\int_0^U\!\!\dots\!\int d\widetilde{u}_{j+1}\, H^T(\widetilde{x}_1 - \widetilde{\lambda}_1)\dots H^T(\widetilde{x}_l - \widetilde{\lambda}_l) \\ & \times \exp\{iL[\widetilde{u}_1(\widetilde{x}_1 - \widetilde{\lambda}_1) + \dots + \widetilde{u}_l(\widetilde{x}_l - \widetilde{\lambda}_l)]\}, \end{aligned} \tag{33}$$

where $\widetilde{\lambda}_k$ (resp. $\widetilde{u}_k$), $k=1,\dots,l$, are the projections of the vector $\overline{\lambda}_1 \dot{+} \dots \dot{+} \overline{\lambda}_{j+1}$ (resp. $u_1 \dot{+} \dots \dot{+} u_{j+1}$) on the coordinate spaces $\mathbb{R}^{\widetilde{n}_k}$, $k=1,\dots,l$, and $d\widetilde{u}_{j+1} = du_1 \dots du_{j+1}$.

Remark A.1. 1. $\{\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_l : l=1,\dots,N_j\}$ is the set of indecomposable partitions relative to the pair $(\overline{\lambda}_1 \dot{+} \dots \dot{+} \overline{\lambda}_j, \overline{\lambda}_{j+1})$.

2. In $\widehat{f}(\lambda)$, the decomposition $\overline{\lambda}_1,\dots,\overline{\lambda}_p$ of the vector $\lambda=(\lambda_1,\dots,\lambda_n)$ is the maximal decomposition satisfying $|\overline{\lambda}_k| = 0$, $k=1,\dots,p$; hence, the decomposition $\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_l$ of $(\overline{\lambda}_1 \dot{+} \dots \dot{+} \overline{\lambda}_j, \overline{\lambda}_{j+1})$ is not an indecomposable partition; this means that there is at least one $k=1,\dots,l$ such that $|\widetilde{\lambda}_k| \neq 0$.

3. In the sequel, we consider the projection of the vector $\widetilde{x}_1 \dot{+} \dots \dot{+} \widetilde{x}_l$ on the coordinate space $\mathbb{R}^{n_k}$ of the admissible arguments $\overline{\lambda}_k$, $k=1,\dots,j+1$; let x_k be this projection. Then

(i) $\widetilde{x}_1 \dot{+} \dots \dot{+} \widetilde{x}_l = x_1 \dot{+} \dots \dot{+} x_{j+1}$, $l=1,\dots,N_j$,

(ii) $\{\widetilde{x}_1,\dots,\widetilde{x}_l\} \cap \{x_1,\dots,x_{j+1}\} \subsetneq \{x_1,\dots,x_{j+1}\}$,

(iii) $\displaystyle\sum_{i=1}^{l} \widetilde{u}_i \cdot (\widetilde{x}_i - \widetilde{\lambda}_i) = \sum_{i=1}^{j+1} u_i (x_i - \overline{\lambda}_i) = \sum_{i=1}^{j+1} u_i |x_i|.$

By (iii) the expression (33) becomes

$$I_j = K(T,N_j) \sum\nolimits^* \int\limits_{\mathbb{R}^{N_j-l}} d\widetilde{x}\, f(\widetilde{x}_1) H^T(\widetilde{x}_1 - \widetilde{\lambda}_1)\, \delta|\widetilde{x}_1| \dots \dots f(\widetilde{x}_l) H^T(\widetilde{x}_l - \widetilde{\lambda}_l)\, \delta|\widetilde{x}_l| \times \frac{1}{U^{j+1}} \int\limits_0^U \!\!\dots\! \int d\widetilde{u}_{j+1} \exp\{iL[u_1|x_1| + \dots + u_{j+1}|x_{j+1}|]\}. \tag{34}$$

Let ν be the number of vectors in the set $\{\widetilde{x}_1,\dots,\widetilde{x}_l\} \cap \{x_1,\dots,x_{j+1}\}$. Then $0 \le \nu \le \min(l, j+1)$. Without loss of generality, suppose that $\widetilde{x}_1 = x_1,\dots,\widetilde{x}_\nu = x_\nu$ and write $\widetilde{n}_1 + \dots + \widetilde{n}_\nu = \widetilde{N}_\nu$, $\widetilde{n}_{\nu+1} + \dots + \widetilde{n}_l = n_{\nu+1} + \dots + n_{j+1} = N_{l-\nu}$, $\widetilde{x}_1 \dot{+} \dots \dot{+} \widetilde{x}_\nu = \widetilde{x}'$, $\widetilde{x}_{\nu+1} + \dots + \widetilde{x}_l = \widetilde{x}''$, $x_1 \dot{+} \dots \dot{+} x_\nu = x'$ and $x_{\nu+1} + \dots + x_l = x''$. Then

$$I_j = K(T,N_j,N) \sum\nolimits^* \prod_{k=1}^{\nu} \int\limits_{\mathbb{R}^{\widetilde{n}_k-1}} dx_k\, f(x_k)\, \delta|x_k|\, H^T(x_k - \overline{\lambda}_k) \times \int\limits_{\mathbb{R}^{N_{l-\nu}-(l-\nu)}} d\widetilde{x}'' \prod_{k=\nu+1}^{l} f(\widetilde{x}_k)\, \delta|\widetilde{x}_k|\, H^T(\widetilde{x}_k - \widetilde{\lambda}_k) \times B_N(|x_{\nu+1}|) \dots B_N(|x_{j+1}|),$$

where $K(N_j,T,N)=K(N_j,T)N^{\nu-j-1}$, $N=LU$, $B_N(y)=\sin(Ny/2)/(y/2)$ for $y\neq 0$, and $B_N(0)=1$. Denote by $J_1(k)$, $J_2(k)$ the two integrals in the above expression:

$$J_1(k)=\int_{\mathbb{R}^{\tilde n_k-1}} dx_k\, f(x_k)\,\delta|x_k|\,H^T(x_k-\overline{\lambda}_k),$$

$$J_2(k)=\int_{\mathbb{R}^{N_{l-\nu}-(l-\nu)}} d\widetilde{x}''\prod_{k=\nu+1}^{l} f(\widetilde{x}_k)\,\delta|\widetilde{x}_k|\,H^T(\widetilde{x}_k-\widetilde{\lambda}_k)$$
$$\times B_N(|x_{\nu+1}|)\dots B_N(|x_{j+1}|).$$

Evaluation of J_1. We write $J_1(k)$ as a sum

$$\begin{aligned}J_1(k)&=f(\overline{\lambda}_k)\int_{\mathbb{R}^{\tilde n_k-1}} dx_k\,\delta|x_k|\,H^T(x_k-\overline{\lambda}_k)\\&\quad+\int_{\mathbb{R}^{\tilde n_k-1}} dx_k\,(f(x_k)-f(\overline{\lambda}_k))\,\delta|x_k|\,H^T(x_k-\overline{\lambda}_k),\\&=J_{11}(k)+J_{12}(k).\end{aligned}$$

Using (12), (10) and the fact that $\overline{\lambda}_k$, $k=1,\dots,\nu$, are admissible arguments, we obtain

$$J_{11}(k)=(2\pi)^{\tilde n_k-1}f(\overline{\lambda}_k)TH_{\tilde n_k}.$$

Therefore $J_{11}(k)=O(T)$. Using condition (14), for some $|\theta|<1$ we get

$$J_{12}(k)=C_0\theta\int_{\mathbb{R}^{\tilde n_k-1}} dx_k\,\|x_k\|^{\alpha}H^T(x_k)\,\delta|x_k-\overline{\lambda}_k|,$$

where C_0,α are given by condition (14). In view of (10) and $\overline{\lambda}_k$ being an admissible argument, we get

$$J_{12}(k)=C_0\theta T^{1-\alpha}\int_{\mathbb{R}^{\tilde n_k-1}} dx_k\,\|x_k\|^{\alpha}\,H(x_k)\,\delta|x_k|.$$

Therefore, by condition $\mathcal{H}3$, we get $J_{12}(k)=O(T^{1-\alpha})$. It follows that

$$J_1(k)=O(T)+O(T^{1-\alpha}). \tag{35}$$

Evaluation of $J_2(k)$. Let us write $J_2(k)$ as a sum $J_{21}(k)+J_{22}(k)$, with

$$J_{21}(k)=\prod_{k=\nu+1}^{l} f(\widetilde{\lambda}_k)\int_{\mathbb{R}^{N_{l-\nu}-(l-\nu)}} d\widetilde{x}''\prod_{k=\nu+1}^{l}\delta|\widetilde{x}_k|\,H^T(\widetilde{x}_k-\widetilde{\lambda}_k)$$
$$\times B_N(|x_{\nu+1}|)\dots B_N(|x_{j+1}|),$$

$$J_{22}(k) = \int\limits_{\mathbb{R}^{N_{l-\nu}-(l-\nu)}} d\widetilde{x}'' \prod_{k=\nu+1}^{l} \delta|\widetilde{x}_k| \, (f(\widetilde{\lambda}_k) - f(\widetilde{x}_k)) H^T(\widetilde{x}_k - \widetilde{\lambda}_k)$$
$$\times B_N(|x_{\nu+1}|) \ldots B_N(|x_{j+1}|).$$

From condition (14), and for some $|\theta| < 1$, $J_{22}(k)$ becomes

$$J_{22}(k) = C_0\theta \int\limits_{\mathbb{R}^{N_{l-\nu}-(l-\nu)}} d\widetilde{x}'' \prod_{k=\nu+1}^{l} \delta|\widetilde{x}_k| \, (f(\widetilde{\lambda}_k) - f(\widetilde{x}_k)) \, H^T(\widetilde{x}_k - \widetilde{\lambda}_k)$$
$$\times B_N(|x_{\nu+1}|) \ldots B_N(|x_{j+1}|). \tag{36}$$

Let us evaluate first $J_{21}(k)$. By definition of δ, we get

$$J_{21}(k) = K \int\limits_{\mathbb{R}^{N_{l-\nu}-(l-\nu)}} d\widetilde{x}'' \prod_{k=\nu+1}^{l} H^T(x_1^{(k)} - \lambda_1^{(k)}) \ldots$$
$$\times H^T(x_{\tilde{n}_k-1}^{(k)} - \lambda_{\tilde{n}_k-1}^{(k)}) \, H^T(-x_1^{(k)} - \ldots - x_{\tilde{n}_k-1}^{(k)} - \lambda_{\tilde{n}_k}^{(k)})$$
$$\times B_N(|\widehat{x}_{\nu+1}|) \ldots B_N(|\widehat{x}_{j+1}|), \tag{37}$$

where $K = \prod_{k=\nu+1}^{l} f(\widetilde{\lambda}_k)$, $\widehat{x}_k = (x_{k1}, \ldots, x_{k\widehat{n}_k})$ and $|\widehat{x}_k|$ is the sum of the remaining coordinates of the vector x_k, after using the definition of $\delta(\cdot)$. Note that $\{\widehat{x}_{\nu+1} \dotplus \ldots \dotplus \widehat{x}_{j+1}\}$ is included in $\{\delta|\widetilde{x}_{\nu+1}|\,\widetilde{x}_{\nu+1} \dotplus \ldots \dotplus \delta|\widetilde{x}_l|\,\widetilde{x}_l\}$ (where $\{x\}$ denotes the set of coordinates of the vector x), therefore for $k = \nu+1, \ldots, j+1$, there exist $i_1, \ldots, i_s$ in $\{\nu+1, \ldots, l\}$ such that $\{\widehat{x}_k\} \subseteq \{\delta|\widetilde{x}_{i_1}|\,\widetilde{x}_{i_1}\} \cup \ldots \cup \{\delta|\widetilde{x}_{i_s}|\,\widetilde{x}_{i_s}\}$. Suppose without loss of generality that the union $\bigcup_{i_1,\ldots,i_s}$ is only one set. Three cases are to be considered $j+1 < l$, $j+1 = l$ and $j+1 > l$. Let us evaluate $J_{21}(k)$ for $j+1 < l$; the other cases are similar (see [2] for more details).

Without loss of generality, we suppose that $\{\widehat{x}_k\} \subset \{\delta|\widetilde{x}_k|\,\widetilde{x}_k\}$, $k = \nu+1, \ldots, j+1$ and $x_{k1} = x_1^{(k)}, \ldots, x_{k\widehat{n}_k} = x_{\widehat{n}_k}^{(k)}$, with $\widehat{n}_k < \widetilde{n}$. Hence (37) becomes

$$J_{21}(k) \le K \prod_{k=\nu+1}^{j+1} \int\limits_{\mathbb{R}^{\tilde{n}_k-1}} d\widehat{x}_k \, B_N|\widehat{x}_k| \, H^T(x_{k1} - \lambda_1^{(k)}) \ldots H^T(x_{k\widehat{n}_k} - \lambda_{\widehat{n}_k}^{(k)})$$
$$\times H^T(x_{\widehat{n}_k+1}^{(k)} - \lambda_{\widehat{n}_k+1}^{(k)}) \ldots H^T(x_{\tilde{n}_k-1}^{(k)} - \lambda_{\tilde{n}_k-1}^{(k)})$$
$$\times H^T(-x_{k1} - \ldots - x_{k\widehat{n}_k-1} - x_{k\widehat{n}_k+1} - \ldots - x_{k\tilde{n}_k-1} - \lambda_{\widehat{n}_k}^{(k)})$$
$$\times \prod_{k=j+2}^{l} \int\limits_{\mathbb{R}^{\tilde{n}_k-1}} d\widetilde{x}_k H^T(x_1^{(k)} - \lambda_1^{(k)}) \ldots H^T(x_{\tilde{n}_k-1}^{(k)} - \lambda_{\tilde{n}_k-1}^{(k)})$$
$$\times H^T(-x_1^{(k)} - \ldots - x_{\tilde{n}_k-1}^{(k)} - \lambda_{\tilde{n}_k}^{(k)}).$$

By (12), with $x_{k1} + \ldots + x_{k\widehat{n}_k} = z_k$, $k = \nu+1, \ldots, j+1$, the above inequality becomes

$$J_{21}(k) \leq K \prod_{k=j+2}^{l} (2\pi)^{\tilde{n}_k - 1} H^T_{\tilde{n}_k}(|\widetilde{\lambda}_k|) \prod_{k=\nu+1}^{j+1} \int_{\mathbb{R}} dz_k\, B_N(z_k)$$
$$\times \int_{\mathbb{R}^{\tilde{n}_k-2}} d\breve{x}_k\, H^T(x_{k1} - \lambda_1^{(k)}) \ldots H^T(x_{k\widehat{n}_k-1} - \lambda^{(k)}_{\widehat{n}_k-1})$$
$$\times H^T(z_k - x_{k1} - \ldots - x_{k\widehat{n}_k-1} - \lambda^{(k)}_{\widehat{n}_k})$$
$$\times H^T(x^{(k)}_{\widehat{n}_k+1} - \lambda^{(k)}_{\widehat{n}_k+1}) \ldots H^T(x^{(k)}_{\tilde{n}_k-1} - \lambda^{(k)}_{\tilde{n}_k-1})$$
$$\times H^T(z_k - x^{(k)}_{\widehat{n}_k+1} - \ldots - x_{k\tilde{n}_k-1} - \lambda^{(k)}_{\tilde{n}_k}),$$

where $d\breve{x}_k = dx_{k1} \ldots dx_{k\widehat{n}_k-1} dx^{(k)}_{\widehat{n}_k+1} \ldots dx^{(k)}_{\tilde{n}_k-1}$. In view of (12), we get

$$J_{21}(k) \leq K \prod_{k=j+2}^{l} (2\pi)^{\tilde{n}_k-1} H^T_{\tilde{n}_k}(|\lambda_k|) \prod_{k=\nu+1}^{j+1} (2\pi)^{\tilde{n}_k-2}$$
$$\times \prod_{k=\nu+1}^{j+1} \int_{\mathbb{R}} dz_k B_N(z_k)\, H^T_{\widehat{n}_k}(z_k - |\lambda^{(k)}_{\widehat{n}_k}|)\, H^T_{\tilde{n}_k-\widehat{n}_k}(z_k - |\lambda^{(k)}_{\tilde{n}_k-\widehat{n}_k}|), \tag{38}$$

with $\widetilde{\lambda}_{\widehat{n}_k}$ and $\widetilde{\lambda}_{\tilde{n}_k-\widehat{n}_k}$ being such that $\widetilde{\lambda}_{\widehat{n}} \dot{+} \widetilde{\lambda}_{\tilde{n}_k-\widehat{n}_k} = \widetilde{\lambda}_k$. Let

$$D(k) = \int_{\mathbb{R}} dz_k\, B_N(z_k) H^T_{\widehat{n}_k}(z_k - |\lambda^{(k)}_{\widehat{n}_k}|) H^T_{\tilde{n}_k-\widehat{n}_k}(z_k - |\lambda^{(k)}_{\tilde{n}_k-\widehat{n}_k}|).$$

Using (9) and with $Nz_k = u_k$, we obtain

$$D(k) = T^2 \int_{\mathbb{R}} du_k\, B_1(u_k) H_{\widehat{n}_k}\left(\frac{T}{N}u_k - T|\lambda^{(k)}_{\widehat{n}_k}|\right) H_{\tilde{n}_k-\widehat{n}_k}\left(\frac{T}{N}u_k - T|\lambda^{(k)}_{\tilde{n}_k-\widehat{n}_k}|\right).$$

It follows from the inequality $|(\sin u_k)/u_k| \leq 1$ and (11) that

$$D(k) \leq 2\pi T^2 H_{\tilde{n}_k}(T|\widetilde{\lambda}_k|). \tag{39}$$

Hence (38) becomes

$$J_{21}(k) \leq K T^{l+j-2\nu+1} \prod_{k=\nu+1}^{l} (2\pi)^{\tilde{n}_k-1} H^T_{\tilde{n}_k}(|\widetilde{\lambda}_k|). \tag{40}$$

By using condition (14) and by similar analysis we get

$$J_{22}(k) = O(T^{(l-\nu)(1-\alpha)}). \tag{41}$$

From (35), (40) and (41), we get

$$I_j \leq Q(T, |\widetilde{\lambda}_{\nu+1}|, \ldots, |\widetilde{\lambda}_l|) T^{-N_j/n+l+j-\nu+1} N^{\nu-j-1}, \tag{42}$$

where $Q(T, |\widetilde{\lambda}_{\nu+1}|, \ldots, |\widetilde{\lambda}_l|) = K \prod_{k=\nu+1}^{l} H_{\tilde{n}_k}(T|\widetilde{\lambda}_k|)$ and K is a constant independent of T. From Remarks A.1, there is at least one vector $\widetilde{\lambda}_k$, $k = \nu+$

$1, \ldots, l$, such that $|\widetilde{\lambda}_k| \neq 0$; therefore by condition $\mathcal{H}1$, $Q(T, |\widetilde{\lambda}_{\nu+1}|, \ldots, |\widetilde{\lambda}_l|)$ goes to zero as $T \to \infty$, except for $\lambda_1 = \ldots = \lambda_n = 0$; obviously for $j+1 > l$, we shall obtain $N^{\nu-l}$ instead of $N^{\nu-j-1}$, hence

$$I_j = O(T^{-N_j/n+2l-\nu} N^{-\beta}), \tag{43}$$

where $\beta = \min(l-\nu,\, j+1-\nu)$.

Now let us evaluate the order of $\prod_{k=j+2}^p \mathbb{E}\widehat{g}(\overline{\lambda}_k)$. By definition of $\widehat{g}(\overline{\lambda}_k)$ and $H^T(\cdot)$, we get

$$\mathbb{E}\widehat{g}(\overline{\lambda}_k) = \frac{1}{(2\pi)^{n_k(n-1)/n}(H_n^T)^{n_k/n}} \int\limits_{\mathbb{R}^{n_k-1}} d\breve{x}_k\, \delta|x_k|\, g(x_k)\, H^T(x_k - \overline{\lambda}_k).$$

Expressing the moment spectral density through the cumulant spectral densities, we get

$$\mathbb{E}\widehat{g}(\lambda_k) = \frac{1}{(2\pi)^{n_k(n-1)/n}(H_n^T)^{n_k/n}} \sum \prod_{i=1}^q \int\limits_{\mathbb{R}^{n_i-1}} d\breve{x}_i\, \delta|\widetilde{x}_i|\, f(\widetilde{x}_i)\, H^T(\widetilde{x}_i - \widetilde{\lambda}_i),$$

where $\widetilde{x}_i$, $\widetilde{\lambda}_i$, $i = 1, \ldots, q$, have the same coordinate space and the sum is over all unordered partitions of the set of coordinates of the vectors x_k. Let us represent the last expression as a sum $I_1 + I_2$ with

$$I_1 = \frac{1}{(2\pi)^{n_k(n-1)/n}(H_n^T)^{n_k/n}} \sum \prod_{i=1}^q f(\widetilde{\lambda}_i) \int\limits_{\mathbb{R}^{n_i-1}} d\breve{x}_i\, \delta|\widetilde{x}_k|\, H^T(\widetilde{x}_i - \widetilde{\lambda}_i),$$

$$I_2 = \frac{1}{(2\pi)^{n_k(n-1)/n}(H_n^T)^{n_k/n}} \times \sum \prod_{i=1}^q \int\limits_{\mathbb{R}^{n_i-1}} \delta|\widetilde{x}_k|\, (f(\widetilde{x}_i) - f(\widetilde{\lambda}_i)) H^T(\widetilde{x}_i - \widetilde{\lambda}_i)\, d\breve{x}_i.$$

It follows from (12) and (11) that

$$I_1 \leq KT^{q-n_k/n} H_{\tilde{n}_i}(T|\widetilde{\lambda}_i|),$$

where K is a constant independent of T. To evaluate I_2 we use conditions (14) and $\mathcal{H}3$ to obtain

$$I_2 = KT^{q(1-\alpha)}.$$

By definition of $\widehat{f}(\lambda)$, the decomposition $\overline{\lambda}_1, \ldots, \overline{\lambda}_p$ of the vector λ is the maximal decomposition such that $|\overline{\lambda}_k| = 0$; therefore by condition $\mathcal{H}1$, $H_{n_i}(T|\widetilde{\lambda}_i|)$ converges to zero except for $q = 1$, where it is H_{n_k}. This implies that the maximal order is obtained for $q = 1$, hence

$$\prod_{k=j+2}^p \mathbb{E}\widehat{g}(\overline{\lambda}_k) = O(T^{p-j-1-(n_{j+2}+\ldots+n_p)/n}) + (T^{p-j-\alpha-(n_{j+2}+\ldots+n_p)/n}). \tag{44}$$

From (43) and (44) we get

$$\mathbb{E}\widehat{g}(\overline{\lambda}_1)\ldots\widehat{g}(\overline{\lambda}_p) - \mathbb{E}\widehat{g}(\overline{\lambda}_1)\ldots\mathbb{E}\widehat{g}(\overline{\lambda}_p) = o(T^{n-1}/N) + o(T^{n-1-\alpha}/N).$$

The o-terms depend on T and the maximal order is realized for $l = [(n_1 + \ldots + n_{j+1})/2]$, $j = p-1$ and $\nu = 0$. The lemma is proved.

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