APPLICATIONES MATHEMATICAE
24,3 (1997), pp. 325–334

J. KOTOWICZ (Białystok)

ON THE EXISTENCE OF A COMPACTLY SUPPORTED L^p-SOLUTION FOR TWO-DIMENSIONAL TWO-SCALE DILATION EQUATIONS

Abstract. Necessary and sufficient conditions for the existence of compactly supported L^p-solutions for the two-dimensional two-scale dilation equations are given.

1. Introduction. One of the fundamental problems in higher dimensional wavelet theory is to study the properties of solutions of the dilation equation

$$f(\mathbf{x}) = \sum_{k\in\mathbb{Z}^d} c_k f(\alpha\mathbf{x} - \beta_k), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

where $k \in A \subset \mathbb{Z}^d$, A is finite and $\mathbb{R} \ni \alpha > 1$.

Using the Fourier method the following fundamental theorem was obtained in [1]:

THEOREM 1.1. *Define* $P(\xi) = \frac{1}{\alpha^d}\sum_{k\in\mathbb{Z}^d} c_k e^{i\langle\beta_k,\xi\rangle}$, $\xi \in \mathbb{C}^d$ *and* $\Delta = P(0)$.

(a) *If* $|\Delta| \le 1$ *and* $\Delta \ne 1$, *then the only* L^1*-solution to* (1) *is trivial.*

(b) *If* $|\Delta| = 1$ *and* (1) *has a non-trivial* L^1*-solution* f, *then* f *is unique up to scale and* $\widehat{f}$ *is given by*

$$\widehat{f}(\xi) = f(0)\prod_{m=1}^{\infty} P(\xi/\alpha^m).$$

Moreover, f *is compactly supported and*

$$\operatorname{supp} f \subseteq \frac{K}{\alpha - 1}, \quad \textit{where} \quad K = \text{conv-hull}(\beta_k).$$

1991 *Mathematics Subject Classification*: 39A10, 42A05.
Key words and phrases: compactly supported L^p scaling function, dilation equation.

(c) *If* $|\Delta| > 1$, *then a necessary condition for* (1) *to have a non-trivial compactly supported* L^1*-solution is* $\Delta = \alpha^k$, *for some* $k \in \mathbb{Z}_+$. *In this case*

$$\widehat{f}(\xi) = h(\xi) \prod_{m=1}^{\infty} \frac{P(\xi/\alpha^m)}{\Delta},$$

where h *is a homogeneous polynomial of degree* k.

The non-zero solutions of (1) are called *scaling functions*.

Our aim in this paper is to study the L^p-integrability properties of the scaling functions in the case when $d = 2$, $\alpha = 2$ and $\beta_k = k \in A = \{(i,j) \in \mathbb{Z}^2 : 0 \le i, j \le N\}$.

In this case the equation (1) and the condition $|\Delta| = 1$ can be rewritten as

$$f(x,y) = \sum_{0 \le i,j \le N} c_{(i,j)} f(2(x,y) - (i,j)), \tag{2}$$

$$\sum_{0 \le i,j \le N} c_{(i,j)} = 4. \tag{3}$$

Let us note a simple consequence of Theorem 1.1.

Corollary 1.2. *Suppose that the condition* (3) *holds. If there exists a non-trivial* L^1*-solution* f *of* (2), *then it must be unique up to scale and* $\operatorname{supp} f \subseteq [0,N]^2$.

Such a special class of scaling functions is important because of its applications in the wavelet theory on $\mathbb{R}^2$, in subdivision schemes in approximation theory, and in practical image processing.

The L^p-integrability properties of the scaling function give information on the corresponding wavelet basis. A major problem is to determine the L^p-integrability properties from the values of c_k for $k \in A$. For solving this, we adopt the matrix implementation of the iteration method, which in the one-dimensional case was used in [2–4], [5–6], [7], [8–9].

2. Technical facts.

The following notations are used everywhere: $\|\cdot\|$ for any norm in $\mathbb{R}^N \times \mathbb{R}^N$, N is the same as in (2), $K = [0,1)^2$ and $B + x = \{a + x : a \in B\}$ for $B \subseteq \mathbb{R}^2$, $x \in \mathbb{R}^2$.

Let $g : \mathbb{R}^2 \to \mathbb{R}$ have $\operatorname{supp} g \subseteq [0,N]^2$. Define a matrix-valued function $\vec{g} : K \to \mathbb{R}^N \times \mathbb{R}^N$ by

$$(\vec{g}(x,y))_{i,j} = g((x,y) + (i,j))\chi_K(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2$$

where $0 \le i$, $j \le N-1$ and χ_K is the characteristic function of the set K.

Conversely, for any matrix-valued function $\vec{f}$ on K we define a function f on $\mathbb{R}^2$ by

$$f(x,y)=\begin{cases}\vec{f}_{i,j}(\widetilde{x},\widetilde{y}) & \text{for } (x,y)=(\widetilde{x}+i,\widetilde{y}+j) \text{ and } (\widetilde{x},\widetilde{y})\in K,\\ 0 & \text{for } (x,y)\notin[0,N]^2.\end{cases}$$

For $k,l\in\{0,1\}$, consider the linear operators $T^{(k,l)}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N\times\mathbb{R}^N$ with coefficients

$$T^{(k,l)}_{i_1,i_2;j_1,j_2}=c_{(2i_1-j_1+k,2i_2-j_2+l)}\quad\text{where } 0\le i_1,i_2,j_1,j_2\le N-1; \tag{4}$$

we use the convention that $c_{(i,j)}=0$ whenever $(i,j)\notin\{(k,l)\in\mathbb{Z}^2:0\le k,l\le N\}$.

The action of these operators on a matrix-valued function $\vec{g}:K\to\mathbb{R}^N\times\mathbb{R}^N$ is defined by

$$(T^{(k,l)}\cdot\vec{g})_{i_1,i_2}=\sum_{j_1,j_2}T^{(k,l)}_{i_1,i_2;j_1,j_2}\vec{g}_{j_1,j_2}.$$

Set

$$T=T^{(0,0)}+T^{(0,1)}+T^{(1,0)}+T^{(1,1)}, \tag{5}$$

and consider the following transformations of the plane:

$$\phi_{(i,j)}(x,y)=\left(\frac{1}{2}x+\frac{i}{2},\frac{1}{2}y+\frac{j}{2}\right)\quad\text{for } i,j\in\{0,1\}.$$

Then for any function g such that $\operatorname{supp} g\subseteq[0,N]^2$ define an operator $\mathbf{T}$ by

$$(\mathbf{T}\vec{g})(x,y)=\sum_{k,l\in\{0,1\}}T^{(k,l)}\vec{g}(\phi^{-1}_{(k,l)}(x,y)).$$

It can be rewritten explicitly as

$$(\mathbf{T}\vec{g})(x,y)=\begin{cases}T^{(0,0)}\vec{g}(2x,2y), & (x,y)\in[0,1/2)^2,\\ T^{(0,1)}\vec{g}(2x,2y-1), & (x,y)\in[0,1/2)\times[1/2,1),\\ T^{(1,0)}\vec{g}(2x-1,2y), & (x,y)\in[1/2,1)\times[0,1/2),\\ T^{(1,1)}\vec{g}(2x-1,2y-1), & (x,y)\in[1/2,1)^2,\\ 0, & (x,y)\notin K.\end{cases}$$

Let $A=\{(0,0),(0,1),(1,0),(1,1)\}$, J be a finite sequence of elements of A, $|J|$ be the length of J (we assume that $|J|$=0 if $J=\emptyset$), and $\Lambda=\{J=(j_1,\ldots,j_k):j_l\in A \text{ and } k\ge0\}$.

For $J=(j_1,\ldots,j_k)\in\Lambda$, define $\phi_J=\phi_{j_1}\circ\ldots\circ\phi_{j_k}$ (if $J=\emptyset$ then $\phi_J:=\mathrm{Id}$), $K_J=\phi_J(K)$ and $T_J=T^{j_1}\circ\ldots\circ T^{j_k}$. Notice that $K_J=\bigcup_{i,j\in\{0,1\}}K_{(J,(i,j))}$ and $K_{(J,J_1)}\subseteq K_J$ for $J,J_1\in\Lambda$.

Define an operator $\mathbf{S}$ by

$$(\mathbf{S}g)(x,y)=\sum_{0\le i,j\le N}c_{(i,j)}g(2(x,y)-(i,j)).$$

Remark 2.1. (i) Let f be a function such that $\operatorname{supp} f \subseteq [0,N]^2$. Then

$$\overrightarrow{\mathbf{S}f} = \mathbf{T}\vec{f}.$$

(ii) f is a non-trivial compactly supported L^p-solution of (2) if and only if $\vec{f} \in L^p(K, \mathbb{R}^N \times \mathbb{R}^N)$ and $\vec{f} = \mathbf{T}\vec{f}$.

Proof. The proof of the first part can be found in [1]. The second one follows from (i), Corollary 1.2 and the equation (2).

Now we present several lemmas which show properties and connections between the operator $\mathbf{T}$, an eigenvector of T corresponding to the eigenvalue 4 and the solution of the dilation equation.

LEMMA 2.2. *If* $\sum_{(i,j)} c_{(i,j)} = 4$, *then there exists an eigenvector* (*which is an* $N \times N$ *matrix*) *of* T *corresponding to the eigenvalue* 4.

Proof. Let $\vec{w} \in \mathbb{R}^N \times \mathbb{R}^N$ be such that $\vec{w}_{i,j} = 1$ for $0 \le i,\ j \le N-1$. Applying (4) and (5) we get

$$(\vec{w}^t T)_{k,l} = \sum_{0 \le i,j \le N} c_{(i,j)} = 4 \quad \text{whenever } 0 \le k,l \le N-1.$$

So $\vec{w}$ is a left eigenvector of T corresponding to the eigenvalue 4 and hence we get the assertion.

For a matrix-valued function $\vec{f}$ such that $\operatorname{supp} f \subseteq [0,N]^2$ we define its average matrix $\vec{v} \in \mathbb{R}^N \times \mathbb{R}^N$ on the unit square. The coordinates of $\vec{v}$ are

$$\vec{v}_{i,j} = f_{[i,i+1]\times[j,j+1]} \quad \text{for } 0 \le i,j \le N-1,$$

where $f_Q = \frac{1}{m(Q)} \int_Q f(x,y)\,dm(x,y)$ for any cube Q.

LEMMA 2.3. *Let* f *be a compactly supported* L^p*-solution of* (2) *and let* $\vec{v}$ *be its average matrix. Then* $\vec{v}$ *is an eigenvector of* T *corresponding to the eigenvalue* 4.

Proof. From Lemma 2.1 we get $\vec{f} = \mathbf{T}\vec{f}$. When we integrate separately both of this equation over the sets $[0,1/2)^2, [0,1/2) \times [1/2,1), [1/2,1) \times [0,1/2), [1/2,1)^2$ we observe that for $k,l \in \{0,1\}$, and $0 \le i,j \le N-1$ we have

$$(T^{(k,l)}\vec{v})_{i,j} = f_{[k/2,(k+1)/2)\times[l/2,(l+1)/2)+(i,j)}.$$

After taking into account that

$$4\vec{f}_{K+(i,j)} = \vec{f}_{[0,1/2)^2+(i,j)} + \vec{f}_{[0,1/2)\times[1/2,1)+(i,j)} + \vec{f}_{[1/2,1)\times[0,1/2)+(i,j)} + \vec{f}_{[1/2,1)^2+(i,j)}, \quad 0 \le i,j \le N-1,$$

we obtain the assertion.

LEMMA 2.4. *For $\vec{v} \in \mathbb{R}^N \times \mathbb{R}^N$ define functions*

$$\vec{f}_0(x,y) = \vec{v} \quad \textit{for } (x,y) \in K, \quad \textit{and} \quad \vec{f}_{k+1} = \mathbf{T}\vec{f}_k \quad \textit{for } k \geq 0.$$

Then:

(i) $\vec{f}_k(x,y) = T_J\vec{v}$ *for* $(x,y) \in K_J$, $|J| = k$.

(ii) *If f is a compactly supported L^p-solution of* (2) *and $\vec{v}$ is its average matrix*, *then*

$$(6) \qquad (\vec{f}_k(x,y))_{i,j} = f_{K_J+(i,j)}, \qquad 0 \leq i,j \leq N-1,\ |J| = k,\ (x,y) \in K_J,$$

and moreover $\vec{f}_k$ converges to $\vec{f}$ in L^p.

Proof. (i) is proved by induction with respect to k. For $k = 0$, (i) follows from the definition of $\vec{f}_0$. Suppose that (i) is true for $|J| = k$. Now if $|J| = k+1$, then one of the following holds:

$$J = ((0,0), J_1); \quad J = ((0,1), J_1); \quad J = ((1,0), J_1); \quad J = ((1,1), J_1),$$

where $|J_1| = k$. Suppose that the first case occurs (the argument for the others is similar). The assumption $(x,y) \in K_J$ implies that $(2x,2y) = \phi^{-1}_{(0,0)}(x,y) \in K_{J_1}$. Hence

$$\vec{f}_{k+1}(x,y) = \mathbf{T}\vec{f}_k(x,y) = T^{(0,0)}\vec{f}_k(2x,2y) = T^{(0,0)}T_{J_1}\vec{v} = T_{((0,0),J_1)}\vec{v},$$

which gives (i).

For (ii) we use the formula $\vec{f} = \mathbf{T}\vec{f}$. It is clear that it can be rewritten in the form $\vec{f}(x,y) = T_J\vec{f}(\phi_J^{-1}(x,y))$ for $(x,y) \in K_J$. Integration over K_J gives (6).

The convergence in the L^p-norm is obtained from the Banach–Steinhaus Theorem in the following way. Let

$$X = L^p(K, \mathbb{R}^N \times \mathbb{R}^N),$$

$$D = \Big\{\vec{h} \in X : \text{ there exists } n \geq 0 \text{ such that } \vec{h}_{i,j} = \sum_{|J|=n} a^J_{i,j}\chi_{K_J} \text{ for } 0 \leq i,j \leq N-1\Big\},$$

and for each $n \geq 1$ define the operator O_n on X by

$$(O_n\vec{h})_{i,j} = h_{K_J+(i,j)} \quad \text{ where } |J| = n,\ \vec{h} \in X.$$

Recall that D is dense in X. It is clear that for each $\vec{h} \in D$ there exists $N_0 \geq 1$ such that

$$(7) \qquad O_n\vec{h} = \vec{h} \quad \text{ for each } n \geq N_0.$$

Computing $\|\vec{h}\|^p_{L^p}$ we see that

$$\|\vec{h}\|^p_{L^p} = \sum_{0\le i,j\le N-1} \sum_{|J|=n} \int_{K_J} |\vec{h}_{i,j}(x,y)|^p \, dx\, dy. \tag{8}$$

Analogously

$$\|O_n\vec{h}\|^p_{L^p} = \frac{1}{4^n} \sum_{0\le i,j\le N-1} \sum_{|J|=n} |h_{K_J+(i,j)}|^p. \tag{9}$$

For any fixed n and $|J| = n$ using the Fubini Theorem and Jensen inequality we obtain

$$\frac{1}{4^n}|h_{K_J+(i,j)}|^p \le \int_{K_J} |\vec{h}_{i,j}(x,y)|^p \, dx\, dy \quad \text{where } 0 \le i,j \le N-1.$$

Then we infer from (8) and (9) that $\|O_n\vec{h}\|^p_{L^p} \le \|\vec{h}\|^p_{L^p}$. Now (7) and the Banach–Steinhaus Theorem yield the convergence of $\vec{f}_n$ to $\vec{f}$ in the L^p-norm.

LEMMA 2.5. *Let $\vec{w}$ be an eigenvector of T corresponding to the eigenvalue* 4. *Let $\vec{f}_k$ (for $k \ge 0$) be defined as in Lemma* 2.4. *Then*

$$\int_K \vec{f}_k(x,y)\, dx\, dy = \vec{w} \quad \textit{for each } k \ge 0. \tag{10}$$

Proof (by induction). The first step is obvious. Suppose that the assertion (10) holds for some k. Then

$$\begin{aligned}
\int_K \vec{f}_{k+1}(x,y)\, dx\, dy &= \int_K T\vec{f}_k(x,y)\, dx\, dy \\
&= \int_{[0,1/2]\times[0,1/2]} T^{(0,0)}\vec{f}_k(2x,2y)\, dx\, dy \\
&\quad + \int_{[0,1/2]\times[1/2,1]} T^{(0,1)}\vec{f}_k(2x,2y-1)\, dx\, dy \\
&\quad + \int_{[1/2,1]\times[0,1/2]} T^{(1,0)}\vec{f}_k(2x-1,2y)\, dx\, dy \\
&\quad + \int_{[1/2,1]\times[1/2,1]} T^{(1,1)}\vec{f}_k(2x-1,2y-1)\, dx\, dy \\
&= \tfrac{1}{4}(T^{(0,0)} + T^{(0,1)} + T^{(1,0)} + T^{(1,1)}) \int_K \vec{f}_k(x,y)\, dx\, dy \\
&= \tfrac{1}{4}(T^{(0,0)} + T^{(0,1)} + T^{(1,0)} + T^{(1,1)})\vec{w} = \vec{w},
\end{aligned}$$

which completes the proof.

3. The main theorem. Let $\vec{w}$ be an eigenvector of T corresponding to the eigenvalue 4. Then we can write

$$(T^{(1,1)} - I)\vec{w} = -((T^{(0,0)} - I)\vec{w} + (T^{(0,1)} - I)\vec{w} + (T^{(1,1)} - I))\vec{w}. \tag{11}$$

Using the notations $\vec{w}^{(i,j)} = (T^{(i,j)} - I)\vec{w}$ for $i, j \in \{0,1\}$ the expression (11) can be rewritten in the form

$$\vec{w}^{(1,1)} = -(\vec{w}^{(0,0)} + \vec{w}^{(0,1)} + \vec{w}^{(1,1)}).$$

Let H be the subspace of $\mathbb{R}^N \times \mathbb{R}^N$ defined by

$$H = \operatorname{span}\{T_J w^{(0,0)}, T_J w^{(0,1)}, T_J w^{(1,0)} : J \in \Lambda\}.$$

Our main result is as follows:

THEOREM 3.1. *Let* $1 \le p < \infty$. *The following conditions are equivalent*:

(i) *There exists a non-zero* L^p*-solution of the equation* (2) *with support in* $[0,N]^2$.

(ii) *There exists an eigenvector* $\vec{w}$ *of* T *corresponding to the eigenvalue* 4 *and*

$$\lim_{n\to\infty} \frac{1}{4^n} \sum_{|J|=n} \|T_J w^{\vec{i,j}}\|^p = 0 \quad \textit{whenever } (i,j) \in \{(0,0),(0,1),(1,0)\}. \tag{12}$$

(iii) *There exists an eigenvector* $\vec{w}$ *of* T *corresponding to the eigenvalue* 4 *and for each* $c > 0$ *there exists an integer* $l \ge 1$ *such that*

$$\frac{1}{4^l} \sum_{|J|=l} \|T_J \vec{u}\|^p < c \quad \textit{for all } \vec{u} \in H \textit{ and } \|\vec{u}\| \le 1. \tag{13}$$

Proof. Let $\vec{w}$ be an eigenvector of T corresponding to the eigenvalue 4. Define, as in Lemma 2.4, $\vec{f}_0 = \vec{w}$, $\vec{f}_{k+1} = \mathbf{T}\vec{f}_k$. Let $\vec{g}_n = \vec{f}_{n+1} - \vec{f}_n$. Then

$$\vec{f}_{n+1} = \vec{f}_0 + \vec{g}_0 + \ldots + \vec{g}_n \tag{14}$$

and

$$\vec{g}_n(x,y) = \begin{cases} T_J \vec{w}^{(0,0)}, & (x,y) \in K_{(J,(0,0))}, \\ T_J \vec{w}^{(0,1)}, & (x,y) \in K_{(J,(0,1))}, \\ T_J \vec{w}^{(1,0)}, & (x,y) \in K_{(J,(1,0))}, \\ T_J \vec{w}^{(1,1)}, & (x,y) \in K_{(J,(1,1))}. \end{cases} \tag{15}$$

Note that

$$\begin{aligned} \|\vec{g}_n\|_{L^p}^p &= \int_K \|\vec{g}_n(x,y)\|^p \, dx\, dy = \sum_{|J|=n+1} \int_{K_J} \|\vec{g}_n(x,y)\|^p \, dx\, dy \\ &= \sum_{|J|=n} \Big(\int_{K_{(J,(0,0))}} + \int_{K_{(J,(0,1))}} + \int_{K_{(J,(1,0))}} + \int_{K_{(J,(1,1))}} \Big) \|\vec{g}_n(x,y)\|^p \, dx\, dy \\ &= \frac{1}{4^n} \sum_{|J|=n} (\|T_J \vec{w}^{(0,0)}\|^p + \|T_J \vec{w}^{(0,1)}\|^p + \|T_J \vec{w}^{(1,0)}\|^p + \|T_J \vec{w}^{(1,1)}\|^p). \end{aligned} \tag{16}$$

(i)⇒(ii). Let $\vec{w}$ be the average matrix of $\vec{f}$ on unit squares, where f is the non-trivial L^p-solution of (2). Then by Lemma 2.4, $\vec{f}_n$ converges to $\vec{f}$

in L^p-norm (we know that $\vec{w}$ is an eigenvector of T corresponding to the eigenvalue 4), which together with (14) implies that $\|\vec{g}_n\|_{L^p}^p \to 0$ as $n \to \infty$. Hence we obtain (12).

(ii)⇒(iii). Let d be the dimension of H. For $d = 0$ we have the assertion at once. Suppose that $d \geq 1$. Then there exists a basis of H consisting of the vectors of the form $T_{J_k^l}\vec{w}^{(i,j)}$ where $(i,j) \in \{(0,0),(0,1),(1,0)\}$, $1 \leq l \leq d$, $|J_k^l| = k^l$ and $J_k^l \in \Lambda$.

For $\vec{u} = T_{J_k^l}\vec{w}^{(i,j)}$ we obtain

$$\frac{1}{4^n}\sum_{|J|=n}\|T_J\vec{u}\|^p \leq 4^{k^l}\frac{1}{4^{n+k^l}}\sum_{|J|=n+k^l}\|T_J\vec{w}^{(i,j)}\|^p \to 0 \quad \text{as } n \to \infty,$$

and hence for each $c > 0$, l, k^l there exists n_l such that

$$\frac{1}{4^{n_l}}\sum_{|J|=n_l+k^l}\|T_J\vec{w}^{(i,j)}\|^p < \frac{c}{2^{(d-1)(p-1)}}.$$

Let $L = \max_{1\leq l\leq d}\{n_l + k^l\}$. Let $\|\cdot\|_1$ be a norm in $\mathbb{R}^N \times \mathbb{R}^N$ such that for $H \ni \vec{u} = \sum_{l=1}^d a_l T_{J_k^l}\vec{w}^{(i,j)}$ we have $\|\vec{u}\|_1^p = \sum_{l=1}^d |a_l|^p$. Hence for $n \geq L$ and $\|\vec{u}\|_1 \leq 1$ we obtain

$$\begin{aligned}\frac{1}{4^n}\sum_{|J|=n}\|T_J\vec{u}\|_1^p &= \frac{1}{4^n}\sum_{|J|=n}\Big\|\sum_{l=1}^d a_l T_J T_{J_k^l}\vec{w}^{(i,j)}\Big\|_1^p \\ &\leq 2^{(d-1)(p-1)}\sum_{l=1}^d |a_l|^p\frac{1}{4^n}\sum_{|J|=n}\|T_J T_{J_k^l}\vec{w}^{(i,j)}\|_1^p \\ &\leq 2^{(d-1)(p-1)}\sum_{l=1}^d |a_l|^p\frac{1}{4^n}\sum_{|J|=n+k^l}\|T_J\vec{w}^{(i,j)}\|_1^p \\ &< 2^{(d-1)(p-1)}\sum_{l=1}^d |a_l|^p\frac{c}{2^{(d-1)(p-1)}} = c\|u\|_1^p \leq c.\end{aligned}$$

(iii)⇒(i). Let $\vec{w}$ be an eigenvector of T corresponding to the eigenvalue 4, and $0 < c < 1$. Consider l such that

$$\frac{1}{4^l}\sum_{|J|=l}\|T_J\vec{u}\|^p < c\|\vec{u}\|^p \quad \text{for each } \vec{u} \in H. \tag{17}$$

Let $i,j \in \{0,1\}$. Applying (17) we obtain

$$\frac{1}{4^l}\sum_{|J|=l}\|T_J T_{J_1}\vec{w}^{(i,j)}\|^p < c\|T_{J_1}\vec{w}^{(i,j)}\|^p$$

and consequently

$$\frac{1}{4^{l+n}}\sum_{|J|=l+n}\|T_J\vec{w}^{(i,j)}\|^p=\frac{1}{4^{l+n}}\sum_{|J|=l}\sum_{|J_1|=n}\|T_JT_{J_1}\vec{w}^{(i,j)}\|^p$$

$$<\frac{c}{4^n}\sum_{|J|=n}\|T_{J_1}\vec{w}^{(i,j)}\|^p,$$

which yields $\|\vec{g}_{n+l}\|^p_{L^p}<c\|\vec{g}_n\|^p_{L^p}$ for each $l\ge 0$ by (15), (17). This means that for each fixed n the sequence $\{\|\vec{g}_{n+kl}\|^p\}_{k=0}^{\infty}$ is convergent, and so is $\vec{f}_n$ by (14). From Lemma 2.5, $\vec{f}=\lim_{n\to\infty}\vec{f}_n$ is non-trivial and $\vec{f}=\mathbf{T}\vec{f}$. Hence from Lemma 2.1 the function f is a solution of the equation (2).

The following can be easily observed:

R e m a r k 3.2. In the condition (12) we can use any three elements of the set $\{(0,0),(0,1),(1,0),(1,1)\}$ instead of $(0,0),(0,1),(1,0)$.

The proof of Theorem 3.1 also yields

R e m a r k 3.3. The condition (13) can be replaced by

$$\frac{1}{4^l}\sum_{|J|=l}\|T_J\vec{u}_i\|^p<c\quad\text{ where }\{u_1,\dots,u_k\}\text{ is a basis of }H.$$

LEMMA 3.4. *Let* $1\le p<\infty$. *Assume that one of the conditions of Theorem* 3.1 *holds. Then for any eigenvector* $\vec{w}$ *of the operator* T *corresponding to the eigenvalue* 4 *we have* $\vec{w}\notin H$ *and* $\dim H<N^2-1$.

P r o o f. Suppose that (ii) of Theorem 3.1 holds and $\vec{w}\in H$. Then by the Jensen inequality we have

$$\|\vec{w}\|^p=\left\|\frac{1}{4^n}(T^{(0,0)}+T^{(0,1)}+T^{(1,0)}+T^{(1,1)})^n\vec{w}\right\|^p=\left(\frac{1}{4^n}\Big\|\sum_{|J|=n}T_J\vec{w}\Big\|\right)^p$$

$$\le\left(\frac{1}{4^n}\sum_{|J|=n}\|T_J\vec{w}\|\right)^p\le\frac{1}{4^n}\sum_{|J|=n}\|T_J\vec{w}\|^p\to 0\quad\text{ as }n\to\infty,$$

which finishes the proof.

4. Final remarks. In contrast to the one-dimensional case, even for small N, Theorem 3.1 does not give simple conditions on the coefficients c_k for which the scaling function belongs to L^p. However, p can be approximated in the following way.

Let f be a non-trivial compactly supported L^p-solution of (2). Define

$$f^x(y)=\int_{\mathbb{R}}f(x,y)\,dx,\qquad f^y(x)=\int_{\mathbb{R}}f(x,y)\,dy.$$

These are solutions of the one-dimensional equations

$$f^x(y) = \sum_{j=0}^{N} c_j^x f^x(2y-j) \quad \text{where} \quad c_j^x = \sum_{i=0}^{N} c_{(i,j)}, \tag{18}$$

$$f^y(x) = \sum_{i=0}^{N} c_i^y f^y(2x-i) \quad \text{where} \quad c_i^y = \sum_{j=0}^{N} c_{(i,j)}. \tag{19}$$

By applying Theorem 2.6 of [9] to (18), (19) one can estimate the greatest values p_x, p_y of q for which f^x, f^y belong to L^q. Let p be the greatest value of q such that the solution f of (2) belongs to L^q. Then $p \le \min(p^x, p^y)$.

Acknowledgements. The author is grateful to Professor Andrzej Łada for many valuable comments.

References

[1] M. A. Berger and Y. Wang, *Multidimensional two-scale dilation equations*, in: *Wavelets—A Tutorial in Theory and Applications*, C. K. Chui (ed.), Wavelets 3, Academic Press, 1992, 295–323.

[2] D. Colella and C. Heil, *The characterization of continuous, four-coefficient scaling functions and wavelets*, IEEE Trans. Inform. Theory 30 (1992), 876–881.

[3] —, —, *Characterization of scaling functions*, *I. Continuous solutions*, J. Math. Anal. Appl. 15 (1994), 496–518.

[4] —, —, *Dilation eqautions and the smoothness of compactly supported wavelets*, in: *Wavelets*: *Mathematics and Applications*, J. J. Benedetto, M. W. Frazier (eds.), Stud. Adv. Math., CRC Press., 1994, 163–201.

[5] I. Daubechies and J. Lagarias, *Two-scale difference equation I. Existence and global regularity of solutions*, SIAM J. Math. Anal. 22 (1991), 1388–1410.

[6] —, —, *Two-scale difference equation II. Local regularity*, *infinite products of matrices*, *and fractals*, ibid. 23 (1992), 1031–1079.

[7] T. Eirola, *Sobolev characterization of solution of dilation equations*, ibid. 23 (1992), 1015–1030.

[8] K. S. Lau and M. F. Ma, *The regularity of L^p-scaling functions*, preprint.

[9] K. S. Lau and J. Wang, *Characterization of L^p-solutions for the two-scale dilation equations*, SIAM J. Math. Anal. 26 (1995), 1018–1048.

Jarosław Kotowicz
Institute of Mathematics
Warsaw University, Białystok Branch
Akademicka 2
15-267 Białystok, Poland
E-mail: kotowicz@math.uw.bialystok.pl

Received on 28.05.1996;
revised version on 7.11.1996