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T. RYCHLIK (Toruń)

EVALUATING IMPROVEMENTS OF RECORDS

Abstract. We evaluate the extreme differences between the consecutive expected record values appearing in an arbitrary i.i.d. sample in the standard deviation units. We also discuss the relevant estimates for parent distributions coming from restricted families and other scale units.

1. Introduction and auxiliary results. We consider a sequence X_i, $i \ge 1$, of independent random variables with a common distribution function F and a finite variance σ^2. The sequences of record times and values are defined by

$$\begin{aligned} &L_0 = 1, \\ &L_n = \inf\{i > L_{n-1} : X_i > X_{L_{n-1}}\}, \qquad && n \ge 1, \\ &R_n = X_{L_n}, && n \ge 0, \end{aligned}$$

respectively. It is of practical interest to evaluate future records by means of previous ones. E.g., Ahsanullah [1] and Dunsmore [3] presented some point and interval predictions for parent sequences of the location-scale exponential models. Our purpose is to determine the sharp upper bounds on $\mathrm{E}_F(R_n - R_{n-1})/\sigma$, $n \ge 2$, for general F with a finite second moment. Dividing by the standard deviation is justified by the scale equivariance of records. Otherwise we could arbitrarily increase the difference of records simply multiplying original variables by large constants.

In this section we have compiled some basic facts useful in the proof of our basic result. That will be presented in Section 2. In Section 3 we derive more tight bounds when F belongs to restricted families of distributions with monotone failure probability and rate. Also, some other scale parameters will be discussed.

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Referring the reader to Nevzorov [7] for a thorough review of the theory of records, we recall some basic facts only. The distribution function of the nth record value is

$$F_{R_n}(x) = 1 - [1 - F(x)] \sum_{i=0}^{n} [-\ln(1 - F(x))]^i / i!.$$

For brevity, we shall further write

$$q_i(x) = [-\ln(1 - x)]^i / i!, \qquad 0 < x < 1,\ i \ge 0. \tag{1}$$

If F is absolutely continuous with a density function f, then so is F_{R_n} and the respective density is

$$f_{R_n}(x) = q_n(F(x)) f(x). \tag{2}$$

Changing the variables, one can get

$$\mathrm{E} R_n = \int_0^1 Q_F(x) q_n(x)\, dx, \tag{3}$$

where $Q_F(x) = \sup\{t : F(t) \le x\}$ is the quantile function of F and q_n is the density function of the nth record in the standard uniform sequence (cf. (2)). Applying (3) and the Schwarz inequality, Nagaraja [6] obtained

$$\begin{aligned} \mathrm{E}(R_n - \mu)/\sigma &= \frac{1}{\sigma} \int_0^1 (Q_F(x) - \mu)(q_n(x) - 1)\, dx \\ &\le \frac{1}{\sigma} \Big[\int_0^1 (Q_F(x) - \mu)^2\, dx \int_0^1 (q_n(x) - 1)^2\, dx \Big]^{1/2} \\ &= \Big[\int_0^1 \Big[\binom{2n}{n} q_{2n}(x) - 2 q_n(x) + 1 \Big]\, dx \Big]^{1/2} \\ &= \Big[\binom{2n}{n} - 1 \Big]^{1/2} = A(n) \quad \text{say,} \end{aligned} \tag{4}$$

where μ denotes the expectation of X_1. The bound (4) is the best possible and becomes equality iff $Q_F - \mu$ and $q_n - 1$ are proportional, i.e. for an affine transformation of Weibull variables with the shape parameter $1/n$. Repeating the same reasoning for

$$\mathrm{E}(R_n - R_{n-1})/\sigma = \frac{1}{\sigma} \int_0^1 Q_F(x)[q_n(x) - q_{n-1}(x)]\, dx \tag{5}$$

does not yield the optimal bound, because $q_n - q_{n-1}$ is not nondecreasing and so cannot be proportional to any quantile function. To cope with (5) we use a more delicate tool proposed in Gajek and Rychlik [4, Proposition 1]. If a statistical functional is represented as an inner product of a given function (here $\varphi_n = q_n - q_{n-1}$) with the quantile function from a convex cone, then the supremum of the functional is attained by the quantiles proportional to the projection of the function in the inner product norm onto the cone of quantiles. The supremum amounts to the norm of the projection. For general F, the functional (5) is maximized over the family of all quantile functions which coincides with the convex cone of nondecreasing (right continuous) functions on $[0,1]$. There is a general method of constructing projections onto nondecreasing functions, due to Moriguti [5]. We recall a simplified version of Moriguti's Theorem 1.

LEMMA 1. *Let $\varphi : [a,b] \to \mathbb{R}$ have a finite Lebesgue integral. For $\Phi(x) = \int_a^x \varphi(t)\,dt$, let $\overline{\Phi}$ denote the greatest convex minorant of Φ, and write $P\varphi$ for the right-hand derivative of $\overline{\Phi}$. Then*

$$\int_a^b g(t)\varphi(t)\,dt \le \int_a^b g(t)P\varphi(t)\,dt \tag{6}$$

for every nondecreasing function g, for which both the integrals exist and are finite. Equality holds in (6) *iff g is a constant in every interval where $\overline{\Phi} < \Phi$.*

Furthermore, if g and $P\varphi$ are square integrable in $[a,b]$, we can apply the Schwarz inequality to the right-hand side of (6), which yields

$$\int_a^b g(t)P\varphi(t)\,dt \le \Big[\int_a^b g^2(t)\,dt \int_a^b P\varphi^2(t)\,dt\Big]^{1/2}. \tag{7}$$

When $P\varphi \not\equiv 0$ a.e., this becomes equality iff

$$\exists A \ge 0 \quad g(t) = AP\varphi(t) \quad \text{a.e.}, \tag{8}$$

which also implies equality in (6). Combining (6) and (7), we obtain a refined Schwarz inequality for nondecreasing functions, with condition (8) of attaining equality. Accordingly, $P\varphi$ is actually the projection of φ onto the cone of nondecreasing functions in $L_2(a,b)$ (cf. e.g. Balakrishnan [2, Corollary 1.4.2]).

2. Main result

PROPOSITION 1. *For R_n, $n \ge 1$, being the record values in a sequence of independent identically distributed random variables with a finite variance σ^2, we have*

$$\mathrm{E}(R_n - R_{n-1})/\sigma \le B(n), \tag{9}$$

where

$$B(n) = \left\{ \binom{2n-2}{n-1}(1-y)\Big[\sum_{i=0}^{2n-1}\frac{(ny)^i}{i!} + \Big(1-\frac{1}{n}\Big)\frac{(ny)^{2n-1}}{(2n-1)!}\Big]\right\}^{1/2} \tag{10}$$

and $y \in (0,1)$ *satisfies the equation*

$$-\ln(1-y) = ny. \tag{11}$$

Equality is attained in (9) *iff the parent distribution has the quantile function*

$$Q_F(x) = \mu + \frac{\sigma}{B(n)}\varphi_n(\max\{x,y\}), \quad \mu \in \mathbb{R}. \tag{12}$$

Proof. Applying the modified Schwarz inequality to (5), we obtain

$$\begin{aligned}\mathrm{E}(R_n - R_{n-1})/\sigma &= \frac{1}{\sigma}\int_0^1 (Q_F(x)-\mu)\varphi_n(x)\,dx \\ &\le \frac{1}{\sigma}\Big[\int_0^1 (Q_F(x)-\mu)^2\,dx \int_0^1 P\varphi_n^2(x)\,dx\Big]^{1/2} \\ &= \Big[\int_0^1 P\varphi_n^2(x)\,dx\Big]^{1/2}.\end{aligned}$$

It suffices to prove that $P\varphi_n(x) = \varphi_n(\max\{x,y\})$ and its norm is $B(n)$, since then, by Lemma 1,

$$\mathrm{E}(R_n - R_{n-1})/\sigma = \frac{1}{B(n)}\int_0^1 P\varphi_n(x)\varphi_n(x)\,dx = \frac{1}{B(n)}\int_0^1 P\varphi_n^2(x)\,dx = B(n).$$

We first note that

$$\begin{aligned}\int_0^x q_n(t)\,dt &= \frac{(-1)^n}{n!}\int_{1-x}^1 \ln^n t\,dt \\ &= \frac{(-1)^n}{n!}\Big[-(1-x)\ln^n(1-x) - n\int_{1-x}^1 \ln^{n-1} t\,dt\Big] \\ &= -(1-x)q_n(x) + \int_0^x q_{n-1}(t)\,dt\end{aligned}$$

and so

$$\Phi_n(x) = \int_0^x \varphi_n(t)\,dt = -(1-x)q_n(x). \tag{13}$$

By a standard algebra we verify that $\Phi_n(0) = \Phi_n(1) = 0$, and Φ_n is concave decreasing, convex decreasing, and convex increasing in $[0, 1-e^{-n+1}]$, $[1-e^{-n+1}, 1-e^{-n}]$, and $[1-e^{-n}, 1]$, respectively. We can easily see that the greatest convex minorant $\overline{\Phi}_n$ of the primary function Φ_n is linear in $[0,y]$ for some $y \in [1-e^{-n+1}, 1-e^{-n}]$, and coincides with Φ_n elsewhere. The linear part vanishes at 0, and is tangent to Φ_n at the right end-point, which allows us to determine y from the equation

$$\varphi_n(y)y = \Phi_n(y). \tag{14}$$

By (13), this can be replaced by $yq_{n-1}(y) = q_n(y)$, which is equivalent to (11). Consequently, $P\varphi_n(x) = \overline{\Phi}'_n(x) = \varphi_n(y)$ for $x \le y$ and $\varphi_n(x)$ otherwise.

It remains to verify that $\int_0^1 P\varphi_n^2(x)\,dx = B^2(n)$. By (13),

$$\int_x^1 \varphi_n(t)\,dt = (1-x)q_n(x), \tag{15}$$

and therefore

$$\int_x^1 q_n(t)\,dt = (1-x)\sum_{i=0}^{n} q_i(x). \tag{16}$$

Since, moreover,

$$q_n^2 = \binom{2n}{n}\varphi_{2n} + \binom{2n-2}{n-1}q_{2n-2}, \tag{17}$$

we have

$$\int_0^1 P\varphi_n^2(x)\,dx = \int_0^y \varphi_n^2(y)\,dx + \int_y^1 \varphi_n^2(x)\,dx \qquad \text{(by (14) and (17))}$$

$$= \frac{\Phi_n^2(y)}{y} + \binom{2n}{n}\int_y^1 \varphi_{2n}(x)\,dx + \binom{2n-2}{n-1}\int_y^1 q_{2n-2}(x)\,dx \qquad \text{(by (15) and (16))}$$

$$= \binom{2n}{n}q_{2n}(y)\frac{(1-y)^2}{y} + \binom{2n}{n}q_{2n}(y)(1-y)$$

$$+ \binom{2n-2}{n-1}(1-y)\sum_{i=0}^{2n-2} q_i(y) \qquad \text{(by (11) and (9))}$$

$$= (1-y)\left[\binom{2n-2}{n-1}\sum_{i=0}^{2n-2} q_i(y)\binom{2n}{n}\frac{q_{2n}(y)}{y}\right] = B^2(n). \ \blacksquare$$

R e m a r k 1. Every distribution with quantile function (12) has a smooth density function positive on a right halfline, and a jump at the left end-point of the support. Since $1 - e^{-n+1} \le y = 1 - e^{-ny} \le 1 - e^{-n}$, at least $1 - 1/n$ of the whole probability mass is concentrated at the jump point. In fact, the actual mass of the absolutely continuous part is far less than $1/n$, which is illustrated in Table 1 for $n \le 10$.

Remark 2. As $n \to \infty$,

$$B(n) = \binom{2n-2}{n-1}^{1/2}(1+o(1)) = A(n-1)(1+o(1)), \tag{18}$$

which means that asymptotically the extreme expected difference of nth and $(n-1)$st records increases at the same rate as the extreme expectation of the $(n-1)$st record. To show that the first equality in (18) holds, we observe first that $(ny)^{2n-1}/(2n-1)! = o((ny)^{n-1}/(n-1)!)$ for $n-1 \le ny \le n$, and hence we can asymptotically neglect the last term in (10). Since $1-y = e^{-ny}$ and

$$e^{ny}\left(1 - \frac{(ny)^{2n}}{(2n)!}\right) \le \sum_{i=0}^{2n-1} \frac{(ny)^i}{i!} \le e^{ny},$$

we obtain the desired claim. The Stirling formula yields $B(n) \sim A(n-1) \sim 2^{n-1}(n\pi)^{-1/4}$. A numerical comparison of both bounds in Table 1 shows that the approximation (18) is accurate for small n.

TABLE 1

n	y	$B(n)$	$A(n-1)$
2	0.796812	1.407545	1.000000
3	0.940480	2.439827	2.236067
4	0.980173	4.461987	4.358899
5	0.993023	8.356668	8.306624
6	0.997484	15.864860	15.842980
7	0.999082	30.387854	30.380915
8	0.999664	58.573677	58.574739
9	0.999876	113.436106	113.441615
10	0.999946	220.488951	220.497166

3. Refinements and extensions. There is a standard way of determining bounds on the inner product functionals $F \mapsto \int_0^1 Q_F(x)\varphi(x)\,dx$ over symmetric families of distributions. It consists in folding the quantile function about 1/2 and maximizing the integral $\int_{1/2}^1 Q_F(x)\widetilde{\varphi}(x)\,dx$ with the symmetrized function $\widetilde{\varphi}(x) = \varphi(x) - \varphi(1-x)$. Thus, e.g., by the Schwarz inequality,

$$\mathrm{E}(R_n - \mu)/\sigma \le \left[\frac{1}{2}\int_{1/2}^{1} \widetilde{q}_n^{\,2}(x)\,dx\right]^{1/2} \tag{19}$$

is the best bound for the symmetric distributions, since $\widetilde{q}_n$ is actually increasing for $x > 1/2$ (cf. Nagaraja [6]). Since $\widetilde{\varphi}_n$ is not so, it remains to refer to Moriguti's projection, which finally gives

$$\mathrm{E}(R_n - R_{n-1})/\sigma \le \left[\frac{1}{2}\int_{1/2}^{1} P\widetilde{\varphi}_n^2(x)\,dx\right]^{1/2}. \tag{20}$$

Neither of the integrals in (19) and (20) has an explicit analytic form.

In practice we more frequently deal with record values for sequences with asymmetric distributions. It is natural to ask about the range of the expected record improvement over restricted classes of distributions appearing in the reliability theory. We here confine ourselves to the life distributions with monotone failure probability and rate. These can be defined by the property of being in either of convex partial ordering relations (for definition, see van Zwet [8]) with the uniform and exponential distributions, respectively. Note that the quantile function (12) is convex and so the respective distribution has a decreasing failure probability. This is also a DFR distribution, since $Q_F(1-e^{-x})$ is convex as well. Consequently, the bounds for these families coincide with the general ones (9). Otherwise we have

PROPOSITION 2. (a) *For the family of life distributions with increasing failure probability we have*

$$\mathrm{E}(R_n - R_{n-1})/\sigma \le \sqrt{3}/2^n,$$

with equality holding for the uniform distribution.

(b) *For the IFR distributions*,

$$\mathrm{E}(R_n - R_{n-1})/\sigma \le 1,$$

which becomes equality for the exponential F.

Proof. The problem lies in finding the projections of φ_n onto the convex cones of quantile functions of the respective families. In case (a), the cone consists of all nondecreasing concave functions integrating to zero on $(0,1)$. In case (b), it is convenient to change the variables and consider an equivalent problem of projecting $\psi_n(x) = \varphi_n(1-e^{-x})$, $x \ge 0$, onto the cone of quantiles composed with the exponential distribution function in the inner product norm on $\mathbb{R}_+$ with weight e^{-x} (cf. Gajek and Rychlik [4]). Every element of the cone is nondecreasing, concave and orthogonal to constants.

We claim that in either case the projection is a linear function. Observe that φ_n as well as ψ_n are first strictly decreasing and ultimately strictly convex, increasing to infinity. Any nondecreasing concave function, say g, is either nowhere greater than φ_n (ψ_n) or φ_n (ψ_n) and g cross each other at two points only. In the former case, the linear function separating the graphs of φ_n (ψ_n) and g is a better candidate for the projection than g. Otherwise we could take the linear function running through the crossing points. Determining the parameters of the linear function minimizing the

L_2-distance to a given function is an elementary problem which has a unique solution. We easily check that the best linear approximation has the same zero integral as the approximated function.

It follows that the extreme values of $\mathrm{E}(R_n - R_{n-1})/\sigma$ in cases (a) and (b) are attained iff $Q_F(x)$ and $Q_F(1-e^{-x})$ are linear, i.e. for the uniform and exponential distributions, respectively. It poses no problem to calculate the respective expectations. ∎

We now extend Proposition 1, replacing the standard deviation by other scale parameters connected with other central absolute moments of various orders. Nagaraja [6] proved that $\mathrm{E}|X_1|^p < \infty$, for some $p > 1$, ensures the finiteness of $\mathrm{E}R_n$ for all n, but this is no longer true for the first moment. One can therefore expect nontrivial bounds in σ_p-units for $p > 1$, where $\sigma_p^p = \mathrm{E}|X_1 - \mu|^p < \infty$.

PROPOSITION 3. *For* $P\varphi_n(x) = \varphi_n(\max\{x, y\})$ *defined in Proposition* 1 *and* $q = p/(p-1)$, *let* c_p *minimize*

$$c \mapsto \|P\varphi_n - c\|_q^q = \int_0^1 |P\varphi_n(x) - c|^q \, dx$$

among all reals. Then

$$\mathrm{E}(R_n - R_{n-1})/\sigma_p \le \|P\varphi_n - c_p\|_q,$$

which becomes equality for

$$Q_F = \mu + \frac{\sigma_p}{\|P\varphi_n - c_p\|_q^{q-1}} |P\varphi_n - c_p|^{q/p} \operatorname{sgn}(P\varphi_n - c_p), \qquad \mu \in \mathbb{R}. \tag{21}$$

Proof. By Lemma 1 and the Hölder inequality we have

$$\begin{aligned}\mathrm{E}(R_n - R_{n-1})/\sigma_p &\le \int_0^1 Q_F(x) P\varphi_n(x) \, dx \\ &= \int_0^1 (Q_F(x) - \mu)(P\varphi_n(x) - c_p) \, dx \le \|P\varphi_n - c_p\|_q.\end{aligned}$$

The former inequality becomes equality iff $Q_F(x)$ is constant for $x < y$ and the necessary and sufficient condition for the latter is (21). This is also sufficient for the former one. ∎

The value $c_p > \varphi_n(y)$ is uniquely determined but it cannot be explicitly expressed except in the case $p = 2$ when $c_2 = \int_0^1 P\varphi_n(x) \, dx = 0$. Finally, we present sharp support bounds.

PROPOSITION 4. *Suppose that* $\mathrm{E}X_1 = \mu$ *and* $\mu - a \le X_1 \le \mu + b$ *almost surely for some positive* a, b. *Put* $\alpha = b/(a+b)$.

If $\alpha \le y$, *given in* (11), *then*

$$\mathrm{E}(R_n - R_{n-1})/(a+b) \le \alpha(1-y)q_{n-1}(y). \tag{22}$$

The bound is attained iff X_1 *takes on merely two values* $\mu - b(1-y)/y$ *and* $\mu + b$ *with positive probabilities* y *and* $1-y$, *respectively.*

Otherwise

$$\mathrm{E}(R_n - R_{n-1})/(a+b) \le (1-\alpha)q_n(\alpha), \tag{23}$$

which is attained for the two-point distribution $\mathrm{P}(X_1 = \mu - a) = \alpha = 1 - \mathrm{P}(X_1 = \mu + b)$.

P r o o f. For $c \ge \varphi_n(y)$, we have

$$\begin{aligned} \mathrm{E}\frac{R_n - R_{n-1}}{a+b} &\le \frac{1}{a+b}\int_0^1 (Q_F(x)-\mu)(P\varphi_n(x)-c)\,dx \\ &\le -(1-\alpha)\int_{\{P\varphi_n(x)<c\}} (P\varphi_n(x)-c)\,dx \\ &\quad + \alpha \int_{\{P\varphi_n(x)>c\}} (P\varphi_n(x)-c)\,dx \\ &= -(1-\alpha)\int_0^1 (P\varphi_n(x)-c)\,dx + \int_{\varphi_n^{-1}(c)}^1 (\varphi_n(x)-c)\,dx \\ &= -\alpha c + c\varphi_n^{-1}(c) - \Phi_n(\varphi_n^{-1}(c)) = D(c), \quad \text{say.} \end{aligned} \tag{24}$$

Observe that we get equality in the first and second rows of (24) iff

$$Q_F \text{ is a constant for } x < y \tag{25}$$

and

$$Q_F(x) = \begin{cases} \mu - a & \text{if } P\varphi_n(x) < c, \\ \mu + b & \text{if } P\varphi_n(x) > c, \end{cases} \tag{26}$$

respectively. Since $D'(c) = \varphi_n^{-1}(c) - \alpha$, we see that $D(c)$ is minimized by $c = \varphi_n(\max\{y, \alpha\})$.

If $\alpha \le y$, the tightest bound in (24) is

$$\begin{aligned} D(\varphi_n(y)) &= (y-\alpha)\varphi_n(y) - \Phi_n(y) \\ &= (1-\alpha)q_n(y) - (y-\alpha)q_{n-1}(y) = \alpha(1-y)q_{n-1}(y). \end{aligned}$$

Conditions (25)–(26) imply that the bound is attained iff $Q_F(x) = \mu + b$ for $x > y$ and is a constant in $[\mu - a, \mu)$, which can be precisely determined by the expectation condition. If $\alpha > y$, we take

$$D(\varphi_n(\alpha)) = -\Phi_n(\alpha) = (1-\alpha)q_n(\alpha)$$

and (26) determines uniquely the distribution that attains the bound. ∎

Remark 3. For a given n, the bounds (22) and (23) depend merely on the parameter $\alpha = b/(a+b)$ which describes the location of the mean in the support interval. Denote the respective relation by $\widetilde{D}(\alpha)$, $\alpha \in [0,1]$. A natural question arises what is the general bound for all possible μ. If $\alpha \le y$, including the interesting particular case $a = b = \sigma_\infty$, $\widetilde{D}(\alpha)$ is a linear increasing function in α. At $\alpha = y$ it continuously changes into $-\Phi_n(\alpha)$, which further increases until $\alpha = 1 - e^{-n}$ and ultimately decreases to 0 at $\alpha = 1$. Therefore the best bound for an arbitrary μ is

$$\widetilde{D}(1 - e^{-n}) = -\Phi_n(1 - e^{-n}) = \frac{(n/e)^n}{n!}.$$

There is a mysterious coincidence that $\widetilde{D}(\alpha) = -\overline{\Phi}_n(\alpha)$, $\alpha \in [0,1]$, connecting the best support bound and the greatest convex minorant appearing in Moriguti's construction.

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Tomasz Rychlik
Institute of Mathematics
Polish Academy of Sciences
ul. Chopina 12/18
87-100 Toruń, Poland
E-mail: trychlik@impan.gov.pl

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